§1-5　§1-6　指數與對數

基本的問題

（i）　函數，合成函數，反函數

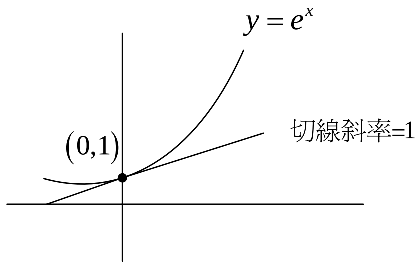
（ii）指、對數圖形及其性質

（iii）$p(t)=p_0 2^{rt}$ ，$p_0=$初始值

$$r:\begin{cases} >0 \quad \text{倍增率} = \dfrac{1}{\text{倍增期}} \\ <0 \quad \text{倍減率} = \dfrac{-1}{\text{半衰期}} \end{cases}$$

（iv）e如何來？

幾何：



代數：　$\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^n=e$

深入的問題

（i）　如何定義如下的指數 $2^{\sqrt{2}}$，$\pi^{\sqrt{2}}$ ？

（ii）能否先定義對數，再定指數？
If so,how?

1-5 Exponential Functions

Homework：2,8,9,13,18,25

指數函數

$f(x)=a^x,\ a>0$

$$\begin{cases} \text{1. 成長或衰退的非常快} \\ \text{2. Applications} \\ \text{3. } e \end{cases}$$

Example 1：Moon-Earth 的距離約為 384.400km，我們假設為 400,000km，又一張紙的厚度設為10^{-2}cm，請問摺幾次後紙張厚度超過 M-E 的距離？

Solution：

k:紙張摺的次數

$2^k \cdot 10^{-7} > 4\times 10^{-5}$

$\Rightarrow k > 2+\dfrac{12}{\log 2}$

$\Rightarrow k = 42$

Remark：當然紙張不可能摺 42 次，但這例子是告訴大家，指數的成長速度是驚人的。

Example 2：Half-life 25 years, $p_0=24$, $p_{60}=$ ？

Solution：

$$p(t)=p_0 \quad 2^{-\frac{t}{25}}$$

$$\Rightarrow p(60)=24 \quad 2^{-\frac{12}{5}}$$

Example 3：Population is doubled every 100 years, $p_0=10^5$,問

$p_{150}=p(150)=$ ？

Solution：

$$p(t)=p_0\ 2^{\frac{t}{100}} \Rightarrow p(150)=10^5\cdot 2^{1.5}$$

Example 4：e 這個數？

Solution：幾何上： $y=a^x$ ，$a>0$, 恆過 $(0,1)$ 點.

Facts：・當 $a=2$, 過 $(0,1)$ 和 $y=2^x$ 相切的切線斜率約為 0.7。

・ 當 $a=3$, 其相對的切線斜率約為 1.1。

$\Rightarrow$ 存在一實數 $a>0$, 使得其相對的切線斜率為 1.我們令此實數為 e.

此 e 為無理數約為 $e\approx 2.71828$.

§1-6 Inverse Functions and Logarithms

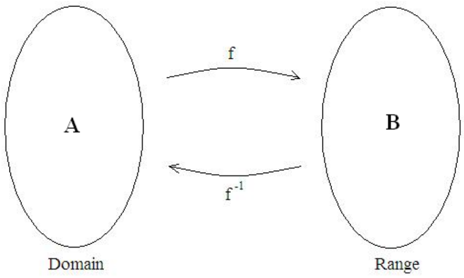
Homework：5,7,13,17,18,20,22,24,28,32,33,34,37,41,51,52,67,68,71

＊ Definition of a function f: Let a correspondence f between 2 sets A and B satisfying the following:

$\forall x \in A \Rightarrow \exists$！ (存在唯一) $y \in B$ $s.t.$(使得). $f(x) = y$. Then f is called a function from A to B.

f is 1-1 $\Leftrightarrow$ existence of f^{-1}

f^{-1}代表 f 的反函數， $f^{-1} \neq \dfrac{1}{f}$



Example 1： Let $f(x) = \dfrac{4x-1}{2x+3} = y$. Find the domain and range of f and the inverse of f.

Solution：Domain $= \left\{ x \in R : \ x \neq -\dfrac{3}{2} \right\}$

Let $\dfrac{4x-1}{2x+3} = y \Rightarrow x = \dfrac{1+3y}{4-2y} = f^{-1}(y)$

$\Rightarrow$ Range $= \{ y \in R : \ y \neq 2 \}$

$\Rightarrow f^{-1}(x) = \dfrac{1+3x}{4-2x}$

Example 2： $m = f(v) = \dfrac{m_0}{\sqrt{1-\dfrac{v^2}{c^2}}}$. Find $f^{-1}(m)$.

Here m_0 is mass at rest, and m is mass with speed v and c is the speed of the light.

Solution：

$$\begin{aligned} m &= \frac{m_0}{\sqrt{1-\dfrac{v^2}{c^2}}} \\ \Rightarrow v &= c\sqrt{1-\frac{m_0^2}{m^2}} \\ &= f^{-1}(m) \end{aligned}$$

Example 3： 解 $e^{5-3x} = 10$.

Solution：

$$5 - 3x = \ln 10$$

$$\Rightarrow \qquad x = \frac{5-\ln 10}{3}$$

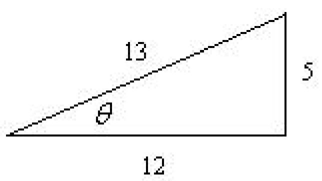
Example 4： $\tan^{-1}\left(\tan\dfrac{4\pi}{3}\right) = \dfrac{\pi}{3}$

Example 5： $\tan^{-1}\tan\sqrt{2}\pi = \left(\sqrt{2}-1\right)\pi$

Example 6： $\sin\left(\tan^{-1}x\right) = \dfrac{x}{\sqrt{1+x^2}}$

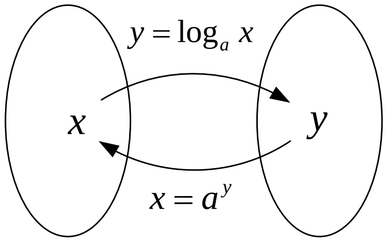
Example 7： $\sin\left(2\sin^{-1}\dfrac{5}{13}\right) = \sin 2\theta = 2\cos\theta\sin\theta = \dfrac{120}{169}$

Solution：



Properties of log.

對數函數：(指數函數的反函數)



(i) $\log_a a^x = x$, $x \in R$; $a^{\log a^x} = x$, $x > 0.$

(ii) $\log_a(xy) = \log_a x + \log_a y$, $x, y > 0.$

(iii) $\log_a \frac{x}{y} = \log_a x - \log_a y$, $x, y > 0$

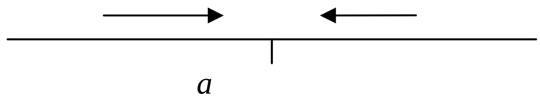
(iv) $\log_a b = \frac{\log_c b}{\log_c a}$, $a, b > 0.$

(v) $\log_e x = \ln x$

§2-2—2-3 Limit

Homework：§2-2　6,9,12,19,23,25,32,38
§2-3　2,11,17,21,29,39,46,49,58,59

$\lim_{x\to a} f(x)=L$ ，　$\lim_{x\to a^+} f(x)=L$ (右極限)，　$\lim_{x\to a^-} f(x)=L$ (左極限)



i.　$x\to a$, x 愈來愈靠近 a ,但從來不是 a .

ii.　‧極限存在 $\Rightarrow$ 極限唯一 $\Rightarrow$ 左極限＝右極限.
　　‧左、右極限分別存在且相等 $\Rightarrow$ 極限存在.

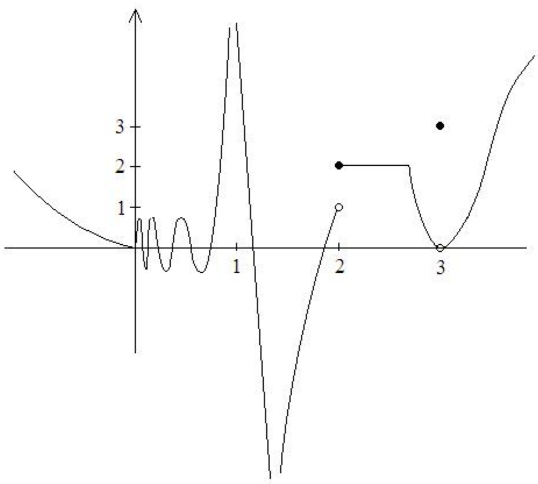
iii.　‧ L 不一定是 $f(a)$.

iv.　If

$$f:\begin{cases}\text{1. Polynomials}\\ \text{2. Rational functions, } a\in \text{Domain}(f)\\ \text{3. The graph of } f \text{ has no break,}\end{cases}$$

$\Rightarrow L=f(a)$.

Example 1：



(1)　那些 a 使得上述左、右極限，分別存在。

Solution：(i)　$R-\{1\}$ (也即只有　$a=1$ 時， 左極限不存在).

(ii) $R-\{0,1\}$ (也即在 $a=0$ or 1 時 ， 右極限不存在).

(2) 那些 a 使得上述極限存在。

Solution： $R-\{0,1,2\}$

(3) 那些 a 使得 $L=f(a)$.

Solution： $R-\{0,1,2,3\}$

註:

$$\lim_{x_n\to 0}\sin\frac{1}{x_n}=\lim_{x_n\to 0}\sin\theta=\sin\theta.$$

Here $x_n=\dfrac{1}{2n\pi+\theta}$ ， $\theta\in R.$

此表示當 x 由右邊趨近 0 時，若趨近路徑不相同時，極限不同。

Example 1： $\lim_{x\to 1}\dfrac{x-1}{x^2-1}=?$

Solution： $\lim_{x\to 1}\dfrac{x-1}{x^2-1}=\lim_{x\to 1}\dfrac{1}{x+1}=\dfrac{1}{2}$

Example 2： $\lim_{x\to 0}\left(x^2+1\right)=?$

Solution： $\lim_{x\to 0}\left(x^2+1\right)=1$

Example 3： $\lim_{x\to 1.5}\dfrac{2x^2-3x}{\left|2x-3\right|}=?$

Solution：

$$\lim_{x\to 1.5}\frac{2x^2-3x}{\left|2x-3\right|}=\lim_{x\to 1.5}\frac{x\left(2x-3\right)}{\left|2x-3\right|}=1.5$$

$$\lim_{x\to 1.5}\frac{2x^2-3x}{\left|2x-3\right|}=\lim_{x\to 1.5}\frac{x\left(2x-3\right)}{-\left(2x-3\right)}=-1.5$$

$$\Rightarrow \lim_{x\to 1.5}\frac{2x^2-3x}{\left|2x-3\right|}\ \text{不存在}$$

Example 4： $\lim_{t \to 0} \frac{\sqrt{t^2+9}-3}{t^2} = ?$

Solution：

$$\lim_{t \to 0} \frac{\left(\sqrt{t^2+9}-3\right)\left(\sqrt{t^2+9}+3\right)}{t^2\left(\sqrt{t^2+9}+3\right)}$$

$$= \lim_{t \to 0} \frac{1}{\sqrt{t^2+9}+3}$$

$$= \frac{1}{6}$$

Example 5： $\lim_{x \to 1} \left[\frac{1}{x-1} - \frac{2}{x^2-1}\right] = ?$

Solution： $\lim_{x \to 1} \left(\frac{x-1}{x^2-1}\right) = \lim_{x \to 1} \frac{1}{x+1} = \frac{1}{2}$

Example 6： $\lim_{t \to 0} \left(\frac{1}{t\sqrt{1+t}} - \frac{1}{t}\right) = ?$

Solution：

$$\lim_{t \to 0} \frac{\left(1-\sqrt{1+t}\right)}{t\sqrt{1+t}}$$

$$= \lim_{t \to 0} \frac{\left(1-\sqrt{1+t}\right)\left(1+\sqrt{1+t}\right)}{t\sqrt{1+t}\left(1+\sqrt{1+t}\right)}$$

$$= \lim_{t \to 0} \frac{1}{\sqrt{1+t}\left(1+\sqrt{1+t}\right)}$$

$$= \frac{-1}{2}$$

Example 7： $\lim_{x \to 0} \frac{(3+x)^2-9}{x} = ?$

Solution： $\lim_{x \to 0} \frac{6x+x^2}{x} = 6$

Example 8： $\lim_{x \to 2^+} [x] = ?$ $\quad \lim_{x \to 2^-} [x] = ?$

Solution： $\lim_{x\to 2^+}[x]=2$; $\lim_{x\to 2^-}[x]=1$

Example 9： $\lim_{x\to -2}\frac{3x^2+ax+a+3}{x^2+x-2}$ exits. Find a and its limit.

Solution：

$$3(-2)^2+a(-2)+a+3=0$$
$$\Rightarrow a=15$$
$$\lim_{x\to -2}\frac{3x^2+15x+18}{x^2+x-2}$$
$$=\lim_{x\to -2}\frac{3(x+2)(x+3)}{(x+2)(x-1)}$$
$$=-1$$

Example 10： $\lim_{x\to 4} f(x)=$ exists. $f(x)=\begin{cases}\sqrt{x-4} & x>4.\\ 8-ax & x>4.\end{cases}$ Find a.

Solution： $右極限=0=左極限=8-4a$

$$\Rightarrow a=2$$

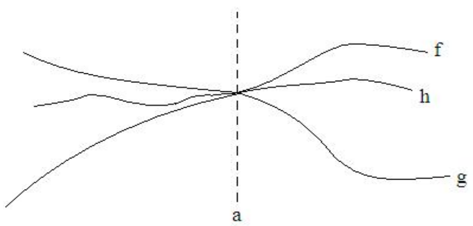
Theorem1. $\lim_{x\to a} f(x)$ exist $\Leftrightarrow$ $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ exist and equal.

Theorem2. $\lim_{x\to a} f(x)$, $\lim_{x\to a} g(x)$ exist.

$$\Rightarrow \lim_{x\to a} f(x)*g(x)=\lim_{x\to a} f(x)*\lim_{x\to a} g(x)$$

Here $*=+,-,\times.$

Theorem3. Sandwich Theorem.



If $f(x) \le h(x) \le g(x)$,except possibly at a , for x near a , and

$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = L$, then $\lim_{x \to a} g(x) = L$.

Example 11： $\lim_{x \to 0} x \sin \frac{1}{x} = ?$

Solution：

$$-|x| \le x \sin \frac{1}{x} \le |x|$$

$$\Rightarrow \lim_{x \to 0} x \sin \frac{1}{x} = 0$$

Example 12： $\lim_{x \to 2} \frac{\sqrt{6-x}-2}{\sqrt{3-x}-1} = ?$

Solution：

$$\lim_{x \to 2} \frac{\sqrt{6-x}-2}{\sqrt{3-x}-1}$$

$$= \lim_{x \to 2} \frac{(2-x)\left(\sqrt{3-x}+1\right)}{(2-x)\left(\sqrt{6-x}+2\right)}$$

$$= \frac{4}{2}$$

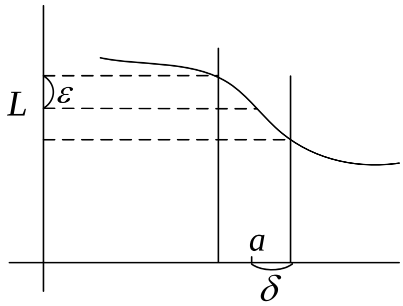
$$= 2$$

§2-4 Definition of the Limit

Homework：3,5,13,15,31

Definition： $\lim_{x\to\infty} f(x) = L \Leftrightarrow$ Given $\varepsilon > 0$,there exists $(\exists)$ a $\delta > 0$ such that $(s.t.)$

$|f(x) - L| < \varepsilon$ whenever $0 < |x - a| < \delta$.



Remarks：(i) ε是任意給，可多小就有多小。

(ii) $|x-a| > 0 \Rightarrow x \neq a.$

(iii) 給L的任一小的近旁（附近），我們需找一個a的夠小近旁，使得在此找到的a的近旁，除a之外，每個相對的f(or y)值皆介於事先任意指定的L的近旁。

Example 1： $\lim_{x\to 1}(5x-1) = 4$

Proof by definition.

Proof ：

草稿：

Wanted：

$|f(x) - L| < \varepsilon$(L 的近旁)

$|5x-1-4| < \varepsilon$

$\Rightarrow \quad |5x-5| < \varepsilon$

$\Rightarrow \quad |x-1| < \frac{\varepsilon}{5}$(a 的近旁)

Given ε_0, chose $\delta = \dfrac{\varepsilon}{5}$. We have that

if $0 < |x-1| < \dfrac{\varepsilon}{5}$, then

$|5x-5| < \varepsilon$, that is

Example 2：Prove $\lim_{x\to 3} x^2 = 9$ by definition.

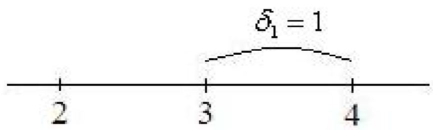
Proof：

草稿：

Wanted：

$|x^2 - 9| < \varepsilon$

$\Rightarrow |(x-3)(x+3)| < \varepsilon$



若 $2 < x < 4$，也即 $|x-3| < 1$，

則 $|(x-3)(x+3)| < 4|x-3|$

若 $|x-3|$ 也小於 $\dfrac{\varepsilon}{4}$，則 $|(x-3)(x+3)| < 4|x-3| < \varepsilon$.

Given $\varepsilon > 0$，取 $\delta = \min\left\{1, \dfrac{\varepsilon}{4}\right\} = 1$ 和 $\dfrac{\varepsilon}{4}$ 較小的那一數.

則 if $|x-3| < \delta$ then

$|x^2 - 9| = |(x-3)(x+3)| < 4|x-3| < \varepsilon.$

§2-5 Continuity

Homework：1,2,3,4,10,16,17,18,25,37,41,43(a),(b),44,45,59,60

Definition： f is continuous at $x=a$.

$\Leftrightarrow$ The graph of f has no break at $x=a$.

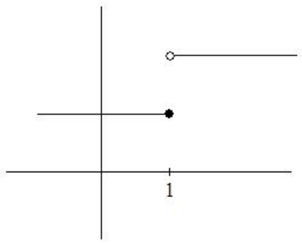
$\Leftrightarrow \lim_{x\to a} f(x) = f\left(\lim_{x\to a} x\right) = f(a)$ (極限和函數可交換).

$\Leftrightarrow$ Given $\varepsilon > 0$, $\exists$ a $\delta = \delta_\varepsilon$ s.t.

$|f(x) - f(a)| < \varepsilon$ whenever $0 < |x-a| < \delta$.

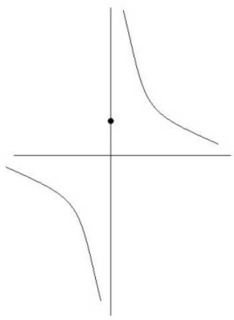
Three types of discontinuities

(i)



(jump)discontinuity at $x=1$.

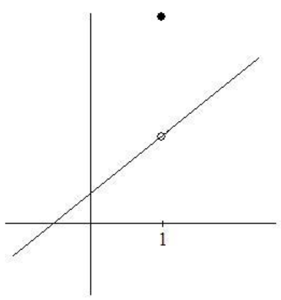
(ii)



$$f(x) = \begin{cases} \dfrac{1}{x}, & \text{if } x \neq 0 \\ 1, & \text{if } x = 0 \end{cases}$$

(infinite) discontinuity at $x=0$.

(iii)



$$f(x) = \begin{cases} \dfrac{x^2-1}{x-1}, & x \neq 1 \\ 3, & x = 1 \end{cases}$$

(removable) discontinuity at $x=1$.

Example 1： $\lim_{x\to 1} \sqrt{x} = \sqrt{\lim_{x\to 1} x} = \sqrt{1} = 1$.

Definition： f is continuous on I $\Leftrightarrow$ f is continuous at every point in I .

• Functions that are continuous on their domains: Polynomials, Expontials, Log,Trig., Iverse Trig.,Rational, Root functions.

Example 2： $\lim_{x\to 1}\sin^{-1}\left(\frac{1-\sqrt{x}}{1-x}\right)=?$

Solution： $\lim_{x\to 1}\sin^{-1}\left(1+\sqrt{x}\right)=\sin^{-1}2$.

Example 3：

$$f\left(x\right)=\begin{cases}\frac{x^2-x-2}{x-2}, & x\neq 2.\\ 1, & x=2.\end{cases}$$

(i) Is f continuous at $x=2$

(ii)If not, classify the type of discontinuity of f at $x=2$.

Solution：

(i) $\lim_{x\to 2}\frac{x^2-x-2}{x-2}=\lim_{x\to 2}\left(x+1\right)=3$ $\Rightarrow$ No.

(ii) Removable discontinuity.

Example 4：

$f_1\left(x\right)=\left[x\right]$, $f_2\left(x\right)=\frac{x-7}{\left|x-7\right|}$

(i) Find, respectively, the set of discontinuities of f_i , $i=1,2$.

(ii)Classify the type of discontinuity of.

Solution：

(i) f_1 :discontinuous at all the integers.

f_2 :discontinuous at $x=7$.

(ii)All are jump discontinuities.

Example 5： $f\left(x\right)=\begin{cases}cx^2+1, & x\leq 3\\ cx-1, & x>0\end{cases}$

Find c so that f is continuous on $\left(-\infty,\infty\right)$.

Solution ：

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-}\left(cx^2+1\right) = 9c+1$$
$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+}\left(cx-1\right) = 3c-1$$
$$\Rightarrow 9c+1 = 3c-1$$
$$\Rightarrow c = -\frac{1}{3}$$

Example 6： $f(x)=\begin{cases} 0 & x \text{ is rational.} \\ 1 & x \text{ is irrational.} \end{cases}$

Find the set of discontinuity of f.

Solution ：

因為有理數和無理數在實數線上皆稠密(dense)， 因此 f 在所有實數點上皆不連續。

Example 7： $f(x)=\begin{cases} 0 & x \text{ is rational.} \\ x & x \text{ is irrational.} \end{cases}$

Find the set of discontinuity of f.

Solution ： $\lim_{x\to 0} f(x) = 0 = f(0)$. $\Rightarrow$ 在 0 點 f is continuous.

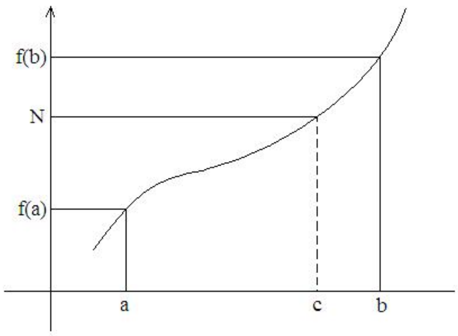
In fact, it is the only continuity.

$\Rightarrow$ Discontinuity of $f = R - \{0\}$.

The Intermediate Value Theorem (中間值定理)(IVT)

- f is cont. on $[a,b]$.
- $f(a) < N < f(b)$ or $f(b) < N < f(a)$

$\Rightarrow \exists\, c \in (a,b)$ s.t. $f(c) = N$



Example 8：If $f(x) = x^3 - x^2 + x$, show that $\exists$ a c s.t. $f(c) = 10$.

Solution ： $f(0) = 0$, $f(3) = 21 > 10$. f is continuous on R.

$\overset{\text{(IVT)}}{\Longrightarrow}$ $\exists$ a c $s.t.$ $f(c) = 10$

§2-6 Limit at Infinity; Horizontal Asymtotes

Homework：4,7,17,22,23,29,34,35(c),41

(I) Vertical Asymptotes (垂直漸近線) (V.A.).
(II) Horizontal Asymptotes (水平漸近線) (H.A.).

Definition：

(i) If either of the following 4 limits holds.

$$\lim_{x \to c^{\pm}} f(x) = \pm\infty ,$$

then $x = c$ is called a V.A. of f.

(ii) $\lim_{x \to +\infty} f(x) = d$ or $\lim_{x \to -\infty} f(x) = d$,

then $y = d$ is called a H.A. of f.

Example 1： $\lim_{x \to \infty} \dfrac{3x^2 - x - 2}{5x^2 + 4x + 3} = ?$ H.A.=? V.A.=?

Solution： $\lim_{x \to \infty} \dfrac{3x^2 - x - 2}{5x^2 + 4x + 3} = \lim_{x \to \infty} \dfrac{3x^2}{5x^2} = \dfrac{3}{5}$

H.A. $y = \dfrac{3}{5}$

V.A. No $\left(\because 5x^2 + 4x + 3 > 0 \text{ for all } x \right)$.

Example 2：(i) $f(x) = \dfrac{\sqrt{2x^2 + 1}}{3x - 5}$, (ii) $f(x) = \dfrac{\sqrt{9x^6 - x}}{x^3 + 1}$ H.A.=? V.A.=?

Solution： (i) H.A.: $y = \dfrac{\sqrt{2}}{3}$ or $y = -\dfrac{\sqrt{2}}{3}$

V.A.: $x = \dfrac{5}{3}$

(ii) H.A.: $y = 3$ or $y = -3$

(iii) V.A.: $x = -1$.

Example 3： $\lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right)=?$

Solution ：

$$\begin{aligned}
&\lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right)\\
&=\lim_{x\to\infty}\frac{\left(\sqrt{x^2+1}+x\right)}{\left(\sqrt{x^2+1}+x\right)}\left(\sqrt{x^2+1}-x\right)\\
&=\lim_{x\to\infty}\frac{1}{\sqrt{x^2+1}+x}\\
&=0
\end{aligned}$$

註：1. 這個例子 $y=x$ 稱為 $y=\sqrt{x^2+1}$ 的 slant（斜）asymptote.

2. 一般來說，若 $f(x)\neq ax+b$ when $|x|$ is "large". 此處 $a\neq 0$. 且 $\lim_{x\to\infty}\left(f(x)-ax-b\right)=0$ 或 $\lim_{x\to-\infty}\left(f(x)-ax-b\right)=0$ 則 $y=ax+b$ 爲 $y=f(x)$ 的斜漸近線。

§2-8 Derivatives

Homework：4,7,15,17,19,29,35,36

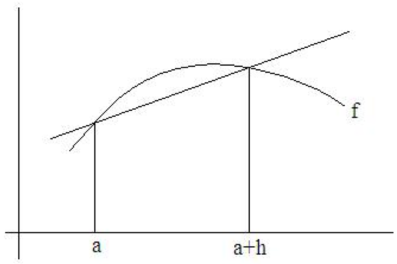
Definition： f has a derivative at $x=a$ (or f is differentiable at $x=a$) provided that $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$ exists or equivalently $\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$.
Its limit will be denoted by $f'(a)$.

Definition：The derivative of $y=f(x)$ is defined to be

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$ whenever the limit exists.

Remark： Geometrically,

(i) $\frac{f(a+b)-f(a)}{b}=$ the slope of the scant.



(ii) $f'(a)=$ the slope of the tangent of $y=f(x)$ at $x=a$.

Generally speaking： $f'(a)=$ the rate of change of f at a.

(i) 若 $f(t)=$ positron at time t.

$\Rightarrow f'(a)=$ instantaneous velocity at time $t=a$.

(ii) $f(x)=$ cost to produce x amount of fabric.

$\Rightarrow f'(a)=$ marginal cost when the product level $x=a$.

Example 1：$f(x)=x^2-8x+9$. Compute $f'(a)$ by definition.

Solution：

$$\begin{aligned}&\lim_{x\to a}\frac{f(x)-f(a)}{x-a}\\&=\lim_{x\to a}\frac{x^2-8x+9-a^2+8a-9}{x-a}\\&=\lim_{x\to a}(x+a-8)\\&=2a-8\end{aligned}$$

Example 2：Find an equation of the tangent to $y=x^2-8x+9$ at $(3,-6)$.

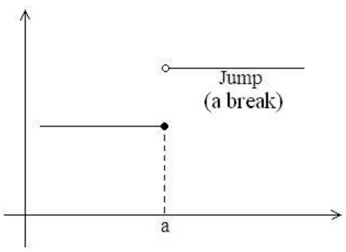
Solution： Slope $f'(3)=-2$

Point $(3,-6)$

$\Rightarrow y+6=-2(x-3)$

When does the derivative of f at $x=2$ not exist?

(i)

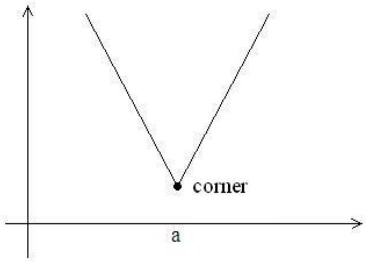


f has break at $x=a$.

例如：$f(x)=[x]$.

f has no derivatives at all the integers.

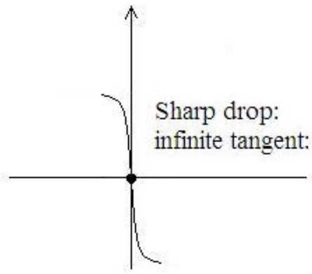
(ii)



例：$f(x)=|x|$

f has continuous at $x=0$,
but is not differentiable at $x=0$.

(iii)



例： $f(x)=x^{\frac{1}{3}}$

• Roughing speaking, if f has a break, or a corner or a sharp turn at $x=a$, then f has no derivative there.

Theorem： f has derivative at $x=a$

$\Rightarrow f$ is continuous at $x=a$.

Example 3： $\lim\limits_{x\to 0}\dfrac{3^x-1}{x}=f'(0)$. Find f.

Solution： $f(x)=3^x$.

Example 4： $f(x)=\begin{cases} x\sin\dfrac{1}{x}, & \text{if } x\neq 0 \\ 0, & \text{if } x=0 \end{cases}$

Discuss the differentiability and continuity of f at $x=0$.

Solution：

(i) f is continuous at $x=0$ by Sandwich Theorem.

(ii) $\lim\limits_{x\to 0}\dfrac{f(x)-0}{x-0}=\lim\limits_{x\to 0}\sin\dfrac{1}{x}=$不存在.

$\Rightarrow f$ is not differentiable at $x=0$ either.

Example 5： $f(x)=\begin{cases} x^2\sin\dfrac{1}{x}, & \text{if } x\neq 0 \\ 0, & \text{if } x=0 \end{cases}$

Solution：

$$f'(0)=\lim_{x\to 0}\frac{f(x)-0}{x-0}=\lim_{x\to 0}\sin\frac{1}{x}=0$$

$\Rightarrow f$ is differentiable at $x=0$ (and hence continuous at $x=0$.)

Example 6： $c=f(v)$: v:speed $\left({miles}/{hr} \right)$

c:gas consumption $\left({gallons}/{hr} \right)$

(i) What is the meaning of the derivative $f'(v)$?

What are its units?

(ii) Write a sentence that explains the meaning of the equation

$f'(20)=-0.05$.

Solution：

(i) $f'(v)=$the consumption rate of a car at the speed v.

Its unit $=\dfrac{{gallons}/{hr}}{{miles}/{hr}}={gallons}/{mile}$.

(ii) The fuel consumption is decreasing by $0.05\ {gallons}/{mile}$ as the car's reaches $20\ {miles}/{hr}$. That is ,if you increase your speed at $21\ {miles}/{hr}$, you could expect to decrease your fuel consumption by about $0.05\ {gallons}/{mile}$.

Example 7： Let $f(x)=\sqrt{ax+b}$, $f(0)=1$ and $f'(0)=1$. Find a and b

Solution：

$f(0)=1=\sqrt{b} \Rightarrow b=1$

$$
\begin{aligned}
1 &= \lim_{x \to 0} \frac{\sqrt{ax+1}-1}{x} \\
&= \lim_{x \to 0} \frac{ax\left(\sqrt{ax+1}+1\right)}{x} \\
&= 2a \\
\Rightarrow a &= \frac{1}{2}
\end{aligned}
$$

§2-9 The Derivative As a Function

Homework：5,9,11,23,27

Notations： $f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = Df(x) = D_x f(x)$.

Example 1： $f(x) = x^3 - x$. Compute $f'(x)$ by definition.

Solution：

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0}\frac{(x+h)^3-(x+h)-x^3+x}{h} \\
&= 3x-1
\end{aligned}
$$

Example 2： $f(x) = \sqrt{x-1}$. Compute $f'(x)$ by definition.

Solution：

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{\sqrt{x+h-1}-\sqrt{x-1}}{h} \\
&= \lim_{h\to 0}\frac{(x+h-1)-(x-1)}{h\left(\sqrt{x+h-1}\right)+\sqrt{x-1}} \\
&= \lim_{h\to 0}\frac{1}{h\left(\sqrt{x+h-1}\right)+\sqrt{x-1}} \\
&= \frac{1}{2\sqrt{x-1}}.
\end{aligned}
$$

Example 3： $f(x) = x|x|$. Does f has a derivative at $x=0$?

Solution： $f'(0) = \lim_{x\to 0}\frac{x|x|}{x} = \lim_{x\to 0}|x| = 0$

§3-1 Derivatives of Polynomials and Exponential Functions

Homework：2 (b),(c),13,22,23,28,39,46,52,54,55,56,57,60,61,63

多項式和指數的微分：

(I) $\frac{d}{dx}\left(x^n\right)=\left(x^n\right)'=nx^{n-1}, \quad n\in N$.

Proof： $f'(x)=\lim_{h\to 0}\frac{(x+h)^n-x^n}{h}$

$$=\lim_{h\to 0}\frac{h\left(nx^{n-1}+C_2^n x^{n-2}h+........+C_{n-1}^n xh^{n-2}+h^{n-1}\right)}{h}$$

$=nx^{n-1}$.

Remark： $\frac{d}{dx}\left(x^r\right)=rx^{r-1}$, $r\in R$,需等到學對數的微分方可證明.

(II) $(af+g)'=af'+g'$, $a\in R$.

(微分的動作是線性(Linear)).

(III) $f(x)=a^x$

$$f'(x)=\lim_{h\to 0}\frac{a^{x+h}-a^x}{h}=\lim_{h\to 0}\frac{a^x\left(a^h-1\right)}{h}$$

$$=a^x\lim_{h\to 0}\frac{a^h-1}{h}=a^x f'(0). \qquad (3.1\text{-}1)$$

When $a=2$, $f'(0)\approx 0.69$.

When $a=3$, $f'(0)\approx 1.10$.

$\Rightarrow\exists$ a number, say $e, 2<e<3$ such that

if $f(x)=e^x$, then $f'(0)=1$

(IV) **Definition：** e is the number such that

$$\lim_{h\to 0}\frac{e^h-1}{h}=1.$$

$e \approx 2.7$ is an irrational number.

由$(3.1\text{-}1)$得 $(e^x)' = e^x$.

Example 1： $(x^3 + 2x^2 - 3x + 1)' = 3x^2 + 4x - 3$.

Example 2： $y = ae^x + \dfrac{b}{x}$

$$\Rightarrow y' = ae^x - \frac{b}{x^2} .$$

Example 3：Find the points on the curve

$y = x^3 - x^2 - x + 1$ ， where the tangent is horizontal .

Solution： $y' = 3x^2 - 2x - 1 = (x-1)(3x+1) = 0$

$$\Rightarrow x = 1 \quad or \quad x = -\frac{1}{3}$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$y = 0 \qquad\qquad y = \frac{32}{27} .$$

Answer： $(1, 0)$ and $(-\frac{1}{3}, \frac{32}{27})$.

Example 4：Find $y = ax^2 + bx$ whose tangent at $(1, 1)$ has equation

$y = 3x - 2$.

Solution： Passing $(1, 1)$, $f'(1) = 3 \Rightarrow \begin{cases} a + b = 1 \\ 2a + b = 3 \end{cases} \Rightarrow a = 2 , b = -1$.

Example5：Find a cubic function $f(x) = ax^3 + bx^2 + cx + d$ whose graph has

horizontal tangents at $(-2, 6), (2, 0)$.

Solution: $f'(x) = 3a^2 + 2bx + c$

Passing $(-2, 6) \Rightarrow -8a + 4b - 2c + d = 6$

Passing $(2\,,0) \Rightarrow 8a+4b+2c+d=0$

$$f'(2)=0 \Rightarrow 12a+4b+c=0$$

$$f'(-2)=0 \quad \Rightarrow 12a-4b+c=0$$

$$\Rightarrow a=\frac{3}{16}\,,\, b=0\,,\, c=-\frac{9}{4}\,,\, d=3\,.$$

Example 6： $\lim\limits_{x\to 1}\dfrac{x^{2007}-1}{x-1}=?$

Solution： $\lim\limits_{x\to 1}\dfrac{x^{2007}-1}{x-1}=f'(1)$, where $f(x)=x^{2007}$

$\Rightarrow f'(1)=2007\,.$

Example 7： Let $f(x)=\begin{cases} x^2 & if\ \ x\le 2 \\ mx+b & if\ \ x>2 \end{cases}$

Find the values of m and b that make f differentiable everywhere .

Solution： $f'(x)=\begin{cases} 2x & if\ \ x<2 \\ m & if\ \ x>2 \end{cases}$

Continuous at $x=2 \Rightarrow f\left(2^-\right)=f\left(2^+\right) \Rightarrow 4=2m+b$.

Differentiable at $x=2 \Rightarrow f'\left(2^-\right)=f'\left(2^+\right) \Rightarrow 4=m$.

$\Rightarrow b=-4\,.$

Example 8： Let $g(x)=\begin{cases} -1-2x & x<-1 \\ x^2 & -1\le x<1 \\ x & x\ge 1 \end{cases}$.

(i) Find the set of points at which g is continuous.

(ii) Find the set of points at which g is differentiable.

Solution： (i) $g(-1^-)=1=g(-1^+)$

$g(1^-)=1=g(1^+)$

$\Rightarrow$ The set of continuity of f is R.

(ii) $$g'(x) = \begin{cases} -2 & x < -1 \\ 2x & -1 < x < 1 \\ 1 & x > 1 \end{cases}$$

$$g'(-1^-) = -2 = g'(-1^+)$$

$$g'(1^-) = 2 \neq 1 = g'(1^+)$$

$\Rightarrow$ The set of differentiability of f is $R - \{1\}$.

§3-2 The Product and Quotient Rules

Homework: 2,5,11,17,25,32,34,35,38(d),42,43

(I) Product Rule

$$(fg)' = f'g + fg'.$$

(II) Quotient Rule

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}.$$

Proof of (I)：

已知： f and g are differentiable.

求證：(I) holds.

$$(fg)'(x) = \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0}\frac{g(x+h)\left[f(x+h) - f(x)\right] + f(x)\left[g(x+h) - g(x)\right]}{h}$$

（做了加一項減一項回來的動作）

Since f and g are differentiable, and, hence continuous, the limit above exists and equals to

$$g(x)f'(x) + f(x)g'(x).$$

Remark (III)：$(fgh)' = f'gh + fg'h + fgh'$

Question：Why formulas such as (I) and (III) are "reasonable"？

(考慮當 f , g, h 是多項式時的特例，爲何(I)and (III) 是合理的)

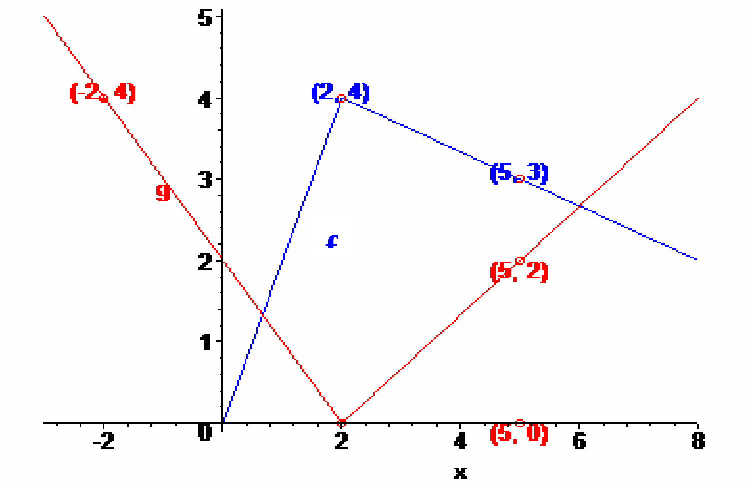
Example 1： $f(x) = \dfrac{x - 3x\sqrt{x}}{\sqrt{x}}$, Find $f'(x)$.

Solution： $f(x) = x^{\frac{1}{2}} - 3x \Rightarrow f'(x) = \dfrac{1}{2}x^{-\frac{1}{2}} - 3.$

Example 2： $h(2) = 4$, $h'(2) = -3$ Find $\dfrac{d}{dx}\left(\dfrac{h(x)}{x}\right)\Big|_{x=2}$.

Solution： $\frac{d}{dx}\left(\frac{h(x)}{x}\right)\Big|_{x=2} = \frac{h'(x)\,x - h(x)}{x^2}\Big|_{x=2} = -\frac{5}{2}$.

Example 3：



Let $u(x) = f(x)g(x)$, $v(x) = f(x)/g(x)$.

(a) Find $u'(1)$. (b) Find $v'(5)$.

Solution： $u'(1) = f'(1)\,g(1) + f(1)\,g'(1)$

$= 2\cdot 1 + 2\cdot(-1) = 0$

$$v'(5) = \frac{f'(5)\,g(5) - f(5)\,g'(5)}{[g(5)]^2}$$

$$= \frac{\left(-\frac{1}{3}\right)\cdot 2 - 3\cdot\frac{2}{3}}{2^2} = -\frac{2}{3}.$$

§3-4 Derivatives of Trigonometric Functions

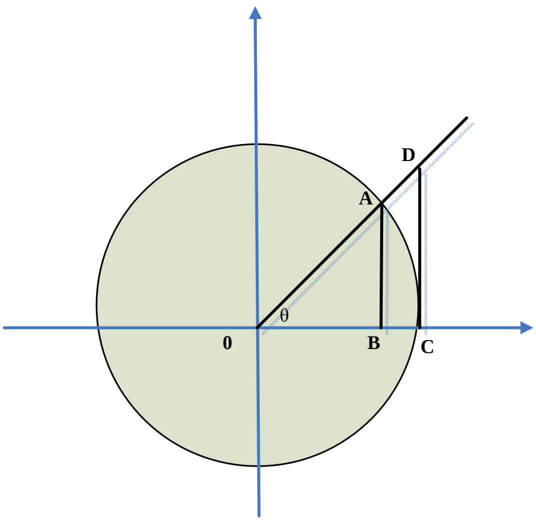
Homework.：　3,11,14,17,22,30,35,37,38,39,44,46,47

欲得三角函數的微分，需先得知(I)式中的極限值.

$$\text{(I)}\quad \text{(i)}\quad \lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1\ .$$

$$\text{(ii)}\quad \lim_{\theta\to 0}\frac{\cos\theta-1}{\theta}=0\ .$$

Proof of (I-i)：



radius=OA=1

$\overline{AB}<\overset{\frown}{AC}<\overline{CD}$

$\Downarrow \quad \Downarrow \quad \Downarrow$

$\Rightarrow \sin\theta<\theta<\tan\theta \qquad (\theta>0)$

$\Rightarrow 1<\dfrac{\theta}{\sin\theta}<\dfrac{1}{\cos\theta}$

$\Rightarrow \lim\limits_{\theta\to 0^+}\dfrac{\theta}{\sin\theta}=1$.

Similarly, $\lim\limits_{\theta\to 0^-}\dfrac{\theta}{\sin\theta}=1 \quad\Rightarrow\quad \lim\limits_{\theta\to 0}\dfrac{\theta}{\sin\theta}=1$.

註：(i)　Why $\overset{\frown}{AC}(=\theta)<\overline{CD}$?

扇形 OAC 的面積<三角形 OCD $\Rightarrow \frac{1}{2}\theta < \frac{1}{2}\overline{CD} < 0 \Rightarrow \widehat{AC} < \overline{CD}$.

(ii) 利用(I－i)和半角公式，可得(I－ii)。

(II) (i) $(\sin x)' = \cos x$ $\qquad (\cos x)' = -\sin x$

$(\tan x)' = \sec^2 x$ $\qquad$ (iv) $(\cot x)' = -\csc^2 x$

$(\sec x)' = \sec x \tan x$ $\qquad (\csc x)' = -\csc x \cot x$

Proof of (II-i)：

$$
\begin{aligned}
(\sin x)' &= \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} \\
&= \lim_{h\to 0}\frac{\sin x\cos h + \cos x \sin h - \sin x}{h} \\
&= \sin x \lim_{h\to 0}\frac{\cos h - 1}{h} + \cos x \lim_{h\to 0}\frac{\sin h}{h} \\
&= \cos x .
\end{aligned}
$$

註：嚴格的寫法，需先說明第三等式兩個極限存在，因此第二等式等於第三等式。

Proof of (II-iv) ：

$$(\cot x)' = \left(\frac{\cos x}{\sin x}\right)' = \frac{-\sin^2 x - \cos^2 x}{\sin x^2} = -\csc^2 x .$$

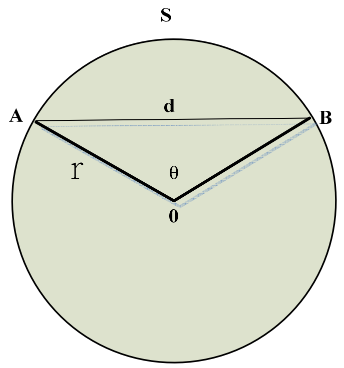
Example 1： $\lim_{x\to 0}\frac{\sin 3x}{x} = ?$

Solution ： $\lim_{x\to 0}\frac{\sin 3x}{x} = \lim_{x\to 0}\frac{3x}{x} = 3$.

Example 2： $\lim_{t\to 0}\frac{\sin^2 3t}{t^2} = ?$

Solution ： $\lim_{t\to 0}\frac{\sin^2 3t}{t^2} = \lim_{t\to 0}\frac{(3t)^2}{t^2} = 9$.

Example 3：



$s = \overset{\frown}{AB}$

求 $\lim_{\theta \to 0^+} \frac{s}{d} = ?$

Solution ： $\lim_{\theta \to 0^+} \frac{s}{d} = \lim_{\theta \to 0^+} \frac{r\theta}{2r \sin \frac{\theta}{2}} = \lim_{\theta \to 0^+} \frac{\theta}{2 \sin \frac{\theta}{2}}$

$$= \lim_{\theta \to 0^+} \frac{\theta}{2 \cdot \frac{\theta}{2}} = 1 .$$

§3-5 The Chain Rule

Homework: 3,13,21,23,31,37,39,41,53,57,64,80

The chain rule：

(I) (i) $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

(ii) $\left\{ f\left(g\left(x\right)\right) \right\}' = f'(g(x))g'(x)$.

Key to apply the chain rule: Identify u .

註：The chain rule 的証明較 tricky, 可參考課本 P223 的証明。

Example 1： $y = e^{\sin y}$

(i) Let $u = \sin x$,

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = e^u \cdot \cos x = e^{\sin x} \cos x .$$

(ii) $f\left(x\right) = e^x$, $g\left(x\right) = \sin x \Rightarrow f\left(g\left(x\right)\right) = e^{\sin x}$

$$\left\{ f\left(g\left(x\right)\right) \right\}' = f'\left(g\left(x\right)\right) g'\left(x\right) = e^{\sin x} \cos x .$$

Example 2： $y = \sqrt{x + \sqrt{x}}$, Find y' .

Solution ： Let $u = x + \sqrt{x}$.

$$\frac{dy}{dx} = \frac{1}{2} u^{-\frac{1}{2}} \left(1 + \frac{1}{2} x^{-\frac{1}{2}} \right)$$

$$= \frac{1}{2} \left(x + \sqrt{x} \right)^{-\frac{1}{2}} \left(1 + \frac{1}{2} x^{-\frac{1}{2}} \right).$$

Example 3： $y = \tan\left(\sin e^{x^2} \right)$ ，求 y' .

(Need to apply the chain rule three times)

Solution ： Let $u = \sin e^{x^2}$, $v = e^{x^2}$, $s = x^2$.

$$\frac{dy}{dx} = \sec^2 u \frac{du}{dx} = \sec^2 u \frac{du}{dv}\frac{dv}{dx}$$
$$= \sec^2 u \cos v \frac{dv}{dx}$$
$$= \sec^2 u \cos v \frac{dv}{ds}\frac{ds}{dx}$$
$$= (\sec^2 u)(\cos v)(e^S)\, 2x .$$

Example 4： $y = \sqrt{\cot\left(e^{\sin x}\right)}$.

Solution ： $\frac{dy}{dx} = \frac{1}{2}\left(\cot e^{\sin x}\right)^{-\frac{1}{2}}\left(-\csc^2 e^{\sin x}\right)e^{\sin x}\cos x$.

Example 5： $y = e^{x\ln a}\left(= e^{\ln a^x} = a^x\right),\ \ a > 0$.

Solution ： $\frac{dy}{dx} = \ln a\, e^{x\ln a} = \left(\ln a\right)a^x$

公式：$\left(a^x\right)' = \left(\ln a\right)a^x,\ \ a > 0$.

Example 6： $y = |x| = \sqrt{x^2}$.

Solution ： $\frac{dy}{dx} = \frac{1}{2}\left(x^2\right)^{-\frac{1}{2}}\left(2x\right) = \frac{x}{|x|}\quad , x \neq 0$.

$$\Rightarrow \frac{dy}{dx} = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}.$$

註：y 在 x=0 的導數不存在。

Example 7： $y = |\sin x|$.

Solution ： $\frac{dy}{dx} = \frac{\sin x}{|\sin x|}\cos x = \begin{cases} \cos x, & \sin x > 0 \\ -\cos x, & \sin x < 0 \end{cases}.$

§3.6 Implicit Differentiation

Homework:5,9,15,21,25,36,41,43,54,67

Question： 一個二元方程式是否隱藏函數的關係?

例如： (i) $x+y=1$ 能否寫成

$x+(f(x))=1$, where $y=f(x)$?

(ii) $x^2+y^2=1$ 能否寫成

$x^2+(f(x))^2=1$, where $y=f(x)$?

(iii) $x^3+y^3=6xy$ 能否寫成

$x^3+(f(x))^3=6x(f(x))$, where $y=f(x)$?

(iv) 給一個二元方程式 $g(x,y)=0$ 能否將此方程式寫成

$g(x, f(x))=0$?

很明顯 (i) 的答案是肯定的，(ii)、(iii)、(iv) 的答案是較複雜。數學系大二的高微課中的隱函數的定理 (Implicit Function Theory) 會給出較完整的答案。粗略的講，除了少數的點之外，其它點的局部附近上述問題的答案是肯定的。在初微的課程中，我們因此都假設: The given equation determines y (respectively, x) implicitly as a differentiable function of x (respectively, y) so that the method of implicit differentiation can be applied .

Example1： $x^2+y^2=25$，求 $\frac{dy}{dx}, \frac{dx}{dy}$ and equation of the tangent at $(3,4)$.

Solution：

$$2x+2y\left(\frac{dy}{dx}\right)=0 \quad\quad 2x\left(\frac{dx}{dy}\right)+2y=0$$

$$\Rightarrow \frac{dy}{dx}=-\frac{x}{y} . \quad\quad \frac{dx}{dy}=-\frac{y}{x} .$$

$$m=\left.\frac{dy}{dx}\right|(3,4)=-\frac{3}{4} .$$

$$(y\text{-}4)=-\frac{3}{4}(x\text{-}3) .$$

Example2： $x^3+y^3=6xy$ 求 $\frac{dy}{dx}=?$

Solution：

$$3x^2+3y^2\left(\frac{dy}{dx}\right)=6y+6x\left(\frac{dy}{dx}\right)$$

$$\frac{dy}{dx}=\frac{2y-x^2}{y^2-2x} .$$

Example3： $\sin(x+y)=y^2\cos x$ 求 $\frac{dy}{dx}=?$

Solution：

$$\cos(x+y)\left(1+\frac{dy}{dx}\right)=2y\left(\frac{dy}{dx}\right)\cos x-y^2\sin x$$

$$\frac{dy}{dx}=\frac{\cos(x+y)+y^2\sin x}{2y\cos x-\cos(x+y)} .$$

Example4： $f\left(f^{-1}(x)\right)=x$.

(i) Find $\left(f^{-1}\right)'(x)$.

(ii) if $f(4)=5$, $f'(4)=\frac{2}{3}$, 求 $\left(f^{-1}\right)'(5)=?$

Solution：(i) $f'\left(f^{-1}(x)\right)\left(f^{-1}\right)'(x)=1$

$$\Rightarrow \left(f^{-1}\right)'(x)=\frac{1}{f'\left(f^{-1}(x)\right)} .$$

(ii) $\left(f^{-1}\right)'(5)=\dfrac{1}{f'(4)}=\dfrac{3}{2}$.

反三角函數的微分

(i) $\left(\sin^{-1}x\right)'=\dfrac{1}{\sqrt{1-x^2}}$　　(ii) $\left(\cos^{-1}x\right)'=\dfrac{-1}{\sqrt{1-x^2}}$

(iii) $\left(\tan^{-1}x\right)'=\dfrac{1}{1+x^2}$　　(iv) $\left(\cot^{-1}x\right)'=\dfrac{-1}{1+x^2}$

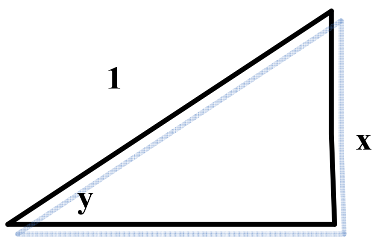
(v) $\left(\sec^{-1}x\right)'=\dfrac{1}{x\sqrt{x^2-1}}$　　(vi) $\left(\csc^{-1}x\right)'=\dfrac{-1}{x\sqrt{x^2-1}}$

($\csc^{-1}x, \sec^{-1}x \in$ III 象限 when x＜0)

1. $y=\sin^{-1}x$ $\Rightarrow$ $x=\sin y$

$$1=\cos y\left(\frac{dy}{dx}\right)$$

$$\frac{dy}{dx}=\frac{1}{\cos y}=\frac{1}{\sqrt{1-x^2}}.$$



2. $y=\cos^{-1}x$

(Method 1)：

$\Rightarrow$ $x=\cos y$

$$1=-\sin y\left(\frac{dy}{dx}\right)$$

$$\frac{dy}{dx}=-\frac{1}{\sin y}=\frac{-1}{\sqrt{1-x^2}}.$$

(Method 2)：

$$\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}$$

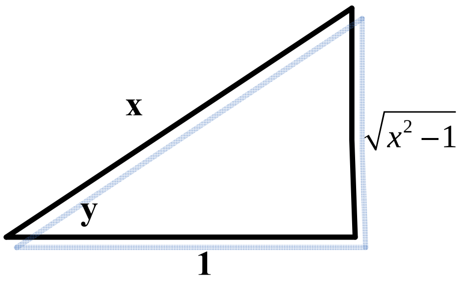
$$\Rightarrow \left(\sin^{-1} x\right)' + \left(\cos^{-1} x\right)' = 0$$

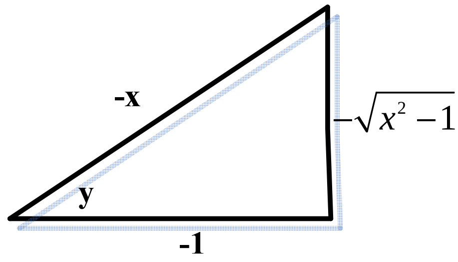
$$\Rightarrow \left(\cos^{-1} x\right)' = -\left(\sin^{-1} x\right)' = \frac{-1}{\sqrt{1-x^2}} .$$

3. $y = \sec^{-1} x \quad \Rightarrow \quad x = \sec y$

$$1 = \sec y \tan y \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{\sec y \tan y} = \frac{1}{x\sqrt{x^2-1}}$$





(i) x > 0 , $0 \le y < \frac{\pi}{2}$. (ii) x < 0 $(\Rightarrow -x > 0)$, $\pi \le y < \frac{3\pi}{2}$.

Note that if $\sec^{-1} x \in$ second quadrant when x < 0 ,

Then $\frac{dy}{dx} = \begin{cases} \frac{1}{x\sqrt{x^2-1}} & x>0 \\ \frac{1}{-x\sqrt{x^2-1}} & x<0 \end{cases}$.

That is why in this book, $\sec^{-1} x$ is chosen to be in the third quadrant whenever x < 0.

3.7 Higher Derivatives

Homework：1,3,7,15,20,30,35,40,57,59,67

Notation： $\dfrac{d^n y}{dx^n} = f^{(n)} = D^n f$.

Example 1： $D^{2007} \cos x = \sin x$.

$$\begin{aligned}
n &= 0, \quad \cos x \\
n &= 1, \quad -\sin x \\
n &= 2, \quad -\cos x \\
n &= 3, \quad \sin x \\
n &= 4, \quad \cos x.
\end{aligned}$$

Example 2： $f(t) =$ position function.

$$\begin{aligned}
f'(t) &= v(t) \\
f''(t) &= a(t) \\
f'''(t) &= j(t) = \text{jerk.}
\end{aligned}$$

Example 3： $x^4 + y^4 = 16$, 求 y'' ？

Solution： $4x^3 + 4y^3 y' = 0 \quad y' = -\dfrac{x^3}{y^3}$

$$y'' = \frac{-y^3 \cdot \left(3x^2\right) + x^3 \left(3y^2 y'\right)}{y^6}$$

$$= \frac{-48x^2}{y^7}.$$

Example 4： Let $y = h(x) = f\left(g(x)\right)$ Find $\dfrac{d^2 y}{dx^2} = h''(x)$.

($y = h(x) = f(u), u = g(x)$, Find $\dfrac{d^2 y}{dx^2}$)

Solution： $h'(x) = f'(g(x))g'(x)$

$$h''(x) = f''(g(x))(g'(x))^2 + f'(g(x))g''(x)$$

Example 5： $y = f(u),\ u = g(x).$ Find $\dfrac{d^2y}{dx^2}$

Solution： $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$

$$\Rightarrow \frac{d^2y}{dx^2} = \left(\frac{d}{dx}\left(\frac{dy}{du}\right)\right)\frac{du}{dx} + \frac{dy}{du}\frac{d^2u}{dx^2}$$

$$= \frac{dz}{dx}\frac{du}{dx} + \frac{dy}{du}\frac{d^2u}{dx^2} \qquad (\text{Let } z = \frac{dy}{du})$$

$$= \frac{dz}{du}\frac{du}{dx}\frac{du}{dx} + \frac{dy}{du}\frac{d^2u}{dx^2}$$

$$= \frac{d^2y}{du^2}\left(\frac{du}{dx}\right)^2 + \frac{dy}{du}\frac{d^2y}{dx^2}$$

Example 6： $f(x) = xg(x^2)$ Find $f',\ f''.$

Solution： $f'(x) = g(x^2) + xg'(x^2)(2x)$

$$= g(x^2) + 2x^2g'(x^2).$$

$$f''(x) = 2xg'(x^2) + 4xg'(x^2) + 4x^3g''(x^2).$$

Remark：Examples 5 and 6 是同一個問題，不同符號的表達方式 .

§3.8 Derivatives of Logarithmic Functions

Homework：3,9,17,21,28,35,39,41,43,48,50

3.8 節的一些公式.

(I) $\frac{d}{dx}\left(\log_a x\right)=\frac{1}{x\ln a}$.

$\frac{d}{dx}\ln x=\frac{1}{x}$.

$\frac{d}{dx}\ln|x|=\frac{1}{x}$.

(II) $\left(x^r\right)'=rx^{r-1}$, $r\in R$.

(III) $e=\lim_{x\to 0}\left(1+x\right)^{\frac{1}{x}}=\lim_{x\to\infty}\left(1+\frac{1}{n}\right)^n$.

$e^x=\lim_{x\to\infty}\left(1+\frac{x}{n}\right)^n$, x >0.

Why？

(I) $y=\log_a x$

$\Rightarrow a^y=x\Rightarrow a^y\left(\ln a\right)y'=1\Rightarrow y'=\frac{1}{\ln a}\frac{1}{a^y}=\frac{1}{\ln a}\frac{1}{x}$.

if $x<0$, $\ln|x|=\ln\left(-x\right)\Rightarrow\frac{d}{dx}\ln|x|=\frac{d}{dx}\ln\left(-x\right)=\frac{-1}{-x}=\frac{1}{x}$.

if $x>0$, $\ln|x|=\ln x\Rightarrow\frac{d}{dx}\ln|x|=\frac{d}{dx}\ln x=\frac{1}{x}$.

(II) $y=x^r\Rightarrow\ln|y|=\ln|x|^r=r\ln|x|$

$\Rightarrow\frac{y'}{y}=\frac{r}{x}\Rightarrow y'=r\frac{y}{x}=rx^{r-1}$.

(III) Let $f(x)=\ln x \Rightarrow f'(x)=\dfrac{1}{x} \Rightarrow f'(1)=1.$

$$1=f'(1)=\lim_{x\to 0}\frac{f(1+x)-f(1)}{x}=\lim_{x\to 0}\frac{\ln(1+x)}{x}=\lim_{x\to 0}\frac{1}{x}\ln(1+x)$$

$$=\lim_{x\to 0}\ln(1+x)^{\frac{1}{x}}=\ln\lim_{x\to 0}(1+x)^{\frac{1}{x}}$$

$$\Rightarrow \lim_{x\to 0}(1+x)^{\frac{1}{x}}=e.$$

(最後一個等式用到連續函數和極限可交換的性質)

Example 1： $y=\log\left(x^3+1\right)$

$$\Rightarrow y'=\frac{3x^2}{(\ln 10)\left(x^3+1\right)}.$$

Example 2： $y=\dfrac{x^{\frac{3}{4}}\sqrt{x^2+1}}{(3x+2)^5}$

$$\Rightarrow \ln y=\frac{3}{4}\ln x+\frac{1}{2}\ln\left(x^2+1\right)-5\ln(3x+2)$$

$$\Rightarrow \frac{y'}{y}=\frac{3}{4x}+\frac{x}{x^2+1}-\frac{15}{3x+2}$$

$$\Rightarrow y'=\left(\frac{3}{4x}+\frac{x}{x^2+1}-\frac{15}{3x+2}\right)y.$$

Example 3： $y=x^x$, *Find* $\dfrac{dy}{dx}$. (若 $y=x^2$, 則 $y'=2x$. 若 $y=2^x$, 則 $y'=2^x\ln 2$)

Solution： $\ln y=x\ln x$

$$\Rightarrow \frac{y'}{y}=\ln x+1 \Rightarrow y'=x^x(1+\ln x).$$

Example 4： $y=x^{\cos x}.$

$$\Rightarrow \ln y = \cos x \ln x$$

$$\Rightarrow \frac{y'}{y} = -\sin x \ln x + \frac{\cos x}{x}$$

$$y' = \left(-\sin x \ln x + \frac{\cos x}{x}\right) x^{\cos x}.$$

Example 5： $\frac{d^9}{dx^9}\left(x^8 \ln x\right)$

$$= \frac{d^8}{dx^8}\left(8x^7 \ln x + x^7\right) = D^8\left(8x^7 \ln x\right)$$

$$= D^7\left(8\times 7x^6 \ln x + 8x^6\right) = D^7\left(8\times 7x^6 \ln x\right)$$

$$= D\left(8!\, x^0 \ln x\right) = \frac{8!}{x}.$$

§3-10 Related Rates

Homework：3,5,7,8,9,12,14,17,19,25,31,35,38

Related Rates：若問題出現如下敘述：

討論某些量 x 對時間 t 的變化率 (rate of change)的應用問題.

at what rate ....x.......?
How fast(slow).............x.......? } 則是問 $\frac{dx}{dt}=?$ (即 x 對時間 t 的變化率) .

Steps to solve related rates problems.

1. 引進〝符號〞將題目 (文字) 轉成〝圖〞.

2. 找一方程式〝連結〞此圖.

3. $\frac{d}{dt}$ (此方程式).

Example1：

Air is being pumped into a spherical balloon so that its volume increases at a rate of 100 cm^3/s. How fast is the radius of the balloon increasing when the diameter is 50 cm?

Solution： 已知 $\frac{dV}{dt}=100$ ，求 $\left.\frac{dr}{dt}\right|_{r=25}$.

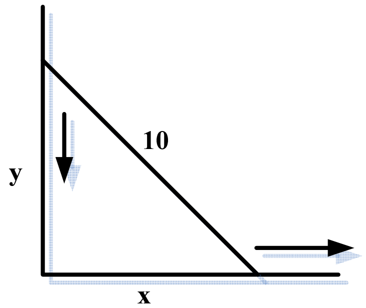
$$V=\frac{4}{3}\pi r^3$$

$$\frac{dV}{dt}=4\pi r^2\frac{dr}{dt}\Rightarrow\frac{dr}{dt}=\frac{1}{25\pi}\ (\mathrm{cm}^3/\mathrm{s}).$$

Example2：

A ladder 10 ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 ft/s. How fast is the top of the ladder sliding down the wall when the ladder is 6 ft from the wall?

Solution：



已知: $\frac{dx}{dt}=1\,(\text{ft/s})$

求 : $\frac{dy}{dt}\Big|_{x=6}$.

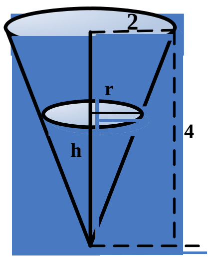
$$x^2+y^2=100 \quad \Rightarrow 2x\left(\frac{dx}{dt}\right)+2y\left(\frac{dy}{dt}\right)=0.\ \text{當} x=6 \Rightarrow y=8\,.$$

$$\Rightarrow \frac{dy}{dt}=-\frac{3}{4}\,(\text{ft/s})\,.$$

Example3：

A water tank has the shape of an inverted circular cone with base radius 2 m and height 4 m. If water is being pumped into the tank at a rate of $2\,\text{m}^3/\text{min}$, find the rate at which the water level is rising when the water is 3 m deep.

Solution：



已知 : $\frac{dV}{dt}=2\,(\text{m}^3/\text{min})$.

求 : $\frac{dh}{dt}\Big|_{h=3}=?$

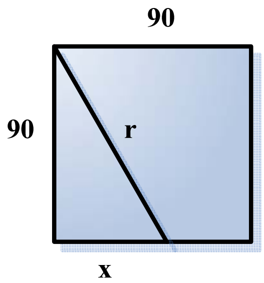
$$V=\frac{1}{3}\pi r^2 h=\frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h=\frac{\pi}{12}h^3$$

$$\Rightarrow \frac{dV}{dt}=\frac{\pi}{4}h^2\frac{dh}{dt}\Rightarrow \frac{dh}{dt}=\frac{8}{9\pi}(\text{m}^3/\text{min}).$$

Example4：

A baseball diamond is a square with side 90 ft. A batter hits the ball and runs toward first base with a speed of 24 ft/s. At what rate is his distance from third base increasing when he is half way to first base?

Solution：



已知: $\frac{dx}{dt} = 24$ (ft/s).

求 ： $\left.\frac{dr}{dt}\right|_{x=45} = ?$

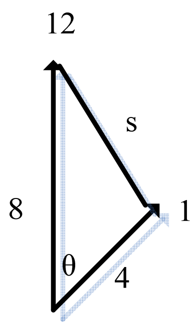
$$r^2 = 90^2 + x^2 \quad \Rightarrow \quad 2r\frac{dr}{dt} = 2x\frac{dx}{dt}$$

$$\Rightarrow \quad \frac{dr}{dt} = \frac{x}{r}\frac{dx}{dt} = \frac{24}{\sqrt{5}} = \frac{24\sqrt{5}}{5}. \text{ (}\because \text{當 } x = 45,\ r = 45\sqrt{5}.\text{)}$$

Example5：

The minute hand on a watch is 8 mm long and the hour hand is 4 mm long. How fast is the distance between the tips of the hands changing at one o'clock?

Solution：



已知: $\frac{d\theta}{dt} = \frac{\pi}{6} - 2\pi = -\frac{11}{6}\pi \left(rad/h \right)$

分針的角速度=2π

時針的角速度=$\frac{\pi}{6}$

求 ： $\left.\frac{ds}{dt}\right|_{\theta=\frac{\pi}{6}} = ?$

$$s^2 = 64 + 16 - 64\cos\theta$$

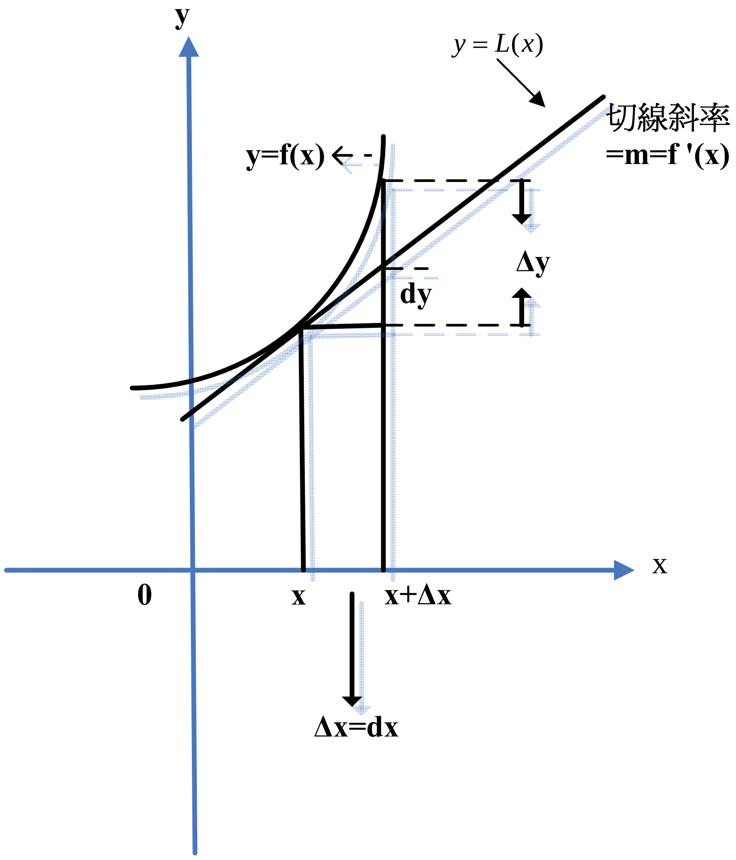
$$\Rightarrow 2s\frac{ds}{dt} = 64\sin\theta\frac{d\theta}{dt} \Rightarrow \frac{ds}{dt} = \frac{-88\pi}{3\sqrt{80 - 32\sqrt{3}}} \ (rad/h).$$

$$(When\ \theta = \frac{\pi}{6},\ s = \sqrt{80 - 32\sqrt{3}}).$$

§3-11 Linear Approximations and Differentials

Homework：7,17,23,31,39,43

給 $f(x)$,利用過 a 點的切線方程式來逼近當 x 靠近 a 的 $f(x)$.



- $\Delta x = dx =$ initial error
- Approximate error = $dy = f'(x)\,dx$
- Exact error $= \Delta y = f(x+\Delta x) - f(x)$
- dx and dy are called differentials

公式

1. $\Delta y \approx dy$ when $\Delta x = dx$ is small. 在實際問題，我們常能以dy(較易算)來逼近Δy(較難算).

2. $L(x) = y = f(a) + f'(a)(x-a)$ is the linear approximation or tangent line approximation of f at a.

- $y = L(x)$是過$(a, f(a))$和$f(x)$相切的切線方程式.
- $\lim\limits_{x \to a}(f(x) - L(x)) = 0$ $(\Rightarrow$ 誤差$= |f(x) - L(x)|$趨近於0,當$x \to a)$.
- $\lim\limits_{x \to a}\dfrac{f(x) - L(x)}{x - a} = 0$ $(\Rightarrow$ 誤差趨近於0的速度比$x \to a$趨近於0的速度還快$)$.

Example1：

(i) Find the linearization of

$f(x) = \sqrt{x+3} \quad at \quad a = 1$·

(ii) Use it to approximate $\sqrt{3.98}$ and $\sqrt{4.05}$.

Solution：

$$f'(x) = \frac{1}{2}(x+3)^{-\frac{1}{2}} \Rightarrow f'(1) = \frac{1}{4}$$

$$L(x) = 2 + \frac{1}{4}(x-1)$$

$$\Rightarrow \sqrt{3.98} \approx 2 + \frac{1}{4}(0.98 - 1) = 1.995$$

$$\sqrt{4.05} \approx 2 + \frac{1}{4}(1.05 - 1) = 2.0125.$$

Example2：

The radius of a sphere was measured and found to be 21 cm with a possible error in measurement of at most 0.05 cm. What is the maximum error in using the value of radius to compute the volume of the sphere? (Use differential to estimate such maximum error.)

Solution：

$$V = \frac{4}{3}\pi r^3$$

$$\Rightarrow \frac{dV}{dr} = 4\pi r^2 \Rightarrow dV = 4\pi r^2 dr$$

$$= 4\pi(21)^2(0.05)$$

$$\approx 277.$$

Example3：

Approximate $\ln(1.05)$.

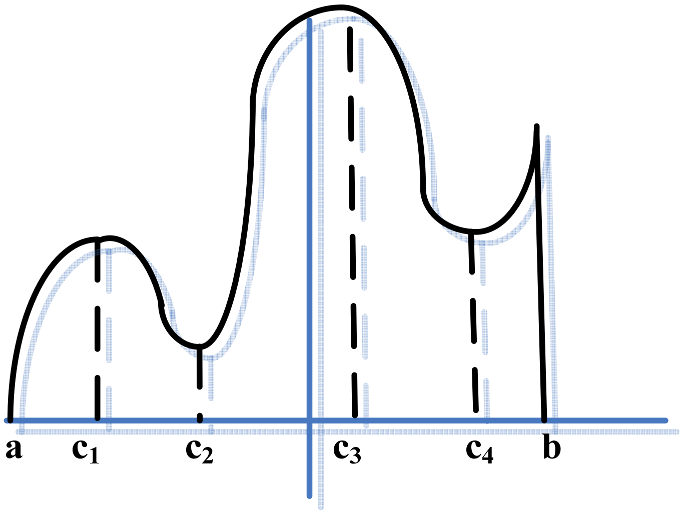
Solution：

$$\begin{aligned} &y = \ln x \\ &\Rightarrow dy = \frac{dx}{x} = \frac{0.05}{1} = 0.05 \\ &\Rightarrow \ln(1.05) \approx \ln 1 + 0.05 = 0.05. \end{aligned}$$

§4-1 Maximum and Minimum Values

Homework：5,20,26,37,41,45,52,53,59,63,70,74,75

- Absolute (Global) extreme
- Local (Relative) extreme



$f:[a,b]\to R$

Absolute Maximum： $f(c_3)$ ； f has an absolute max. at $x=c_3$.

Absolute Minimum： $f(a)$.

Local min： $f(a), f(c_2), f(c_4)$.

Local max： $f(b), f(c_1), f(c_3)$.

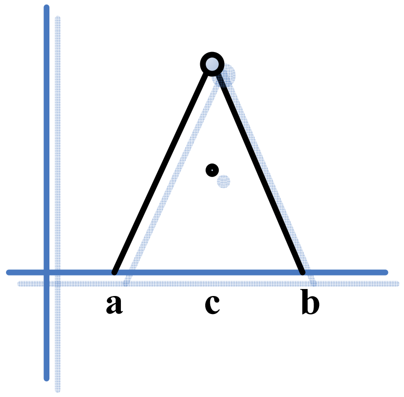
Note that $f(c_4)>f(c_1)$.

Theorem：

f is continuous on $[a,b] \Rightarrow f$ attains it's absolute extrema.

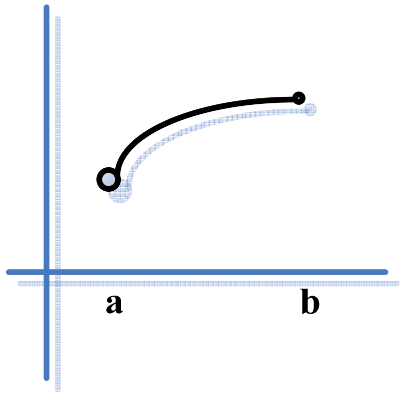
- Assumptions of the theorem need to be satisfied.
  Otherwise the assertion of the theorem fails.

Example：



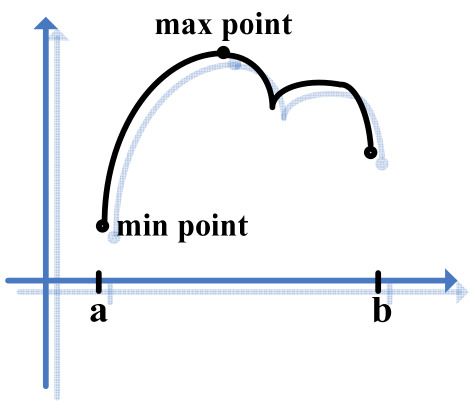
- not continuous at $x = c$
- no absolute max.

Example：



- not continuous at (a,b)
- no absolute min.

Example：



Theorem：

f has a local extremum at $x=c$.

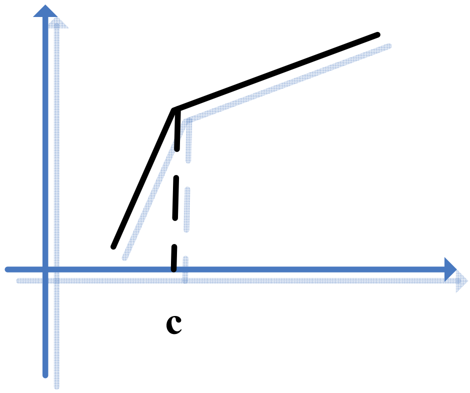
$\Rightarrow f'(c)=0$ or $f'(c)$ does not exist.

- The converse of the theorem is not necessarily true.

Example：

$f(x)=x^3, f'(0)=0, f(0)$ is neither a local min nor a local max.

Example：



- $f'(c)$ does not exist
- $f(c)$ is neither a local min nor a local max.

Definition：

c: a <u>critical point</u> (c.p.) of f

$\Leftrightarrow f'(c)=0$ or $f'(c)$ does not exist.

Remark：

Extreme take place at critical points.

- A critical point is not necessarily a local extremum point.

<u>Steps</u> to find absolute extrema for a continuous function defined on $[a,b]$.

(1) Find all c.p's.

(2) Compare the value of f at all c.p.'s and endpoint points.

x	c.p's	a	b
y			

Example 1： $f(x)=x^3+3x^2+1, \quad -\frac{1}{2}\le x\le 4.$

Solution： $f'(x)=3x^2+6x=3x(x+2)$

x	$-\frac{1}{2}$	0	4
y	$\frac{13}{8}$	1	113

Answer： $\begin{cases}\text{maximum}=113\\ \text{minimum}=1.\end{cases}$

Example 2： Let $a,b>0$.

$f(x)=x^a(1-x)^b, 0\le x\le 1.$

Solution： $f'(x)=ax^{a-1}(1-x)^b-bx^a(1-x)^{b-1}$

$$=x^{a-1}(1-x)^{b-1}\left[a(1-x)-bx\right]$$

$$=x^{a-1}(1-x)^{b-1}\left[a-(a+b)x\right]$$

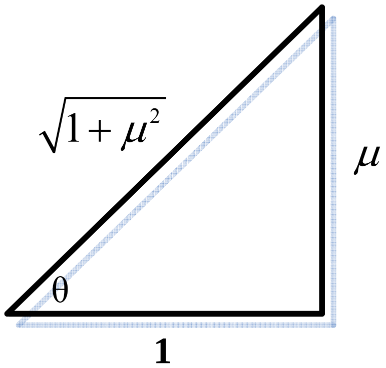
x	0	$\frac{a}{a+b}$	1
y	0	$\left(\frac{a}{a+b}\right)^a\left(\frac{b}{a+b}\right)^b$	0

Answer： $\begin{cases}\text{maximum}=\left(\frac{a}{a+b}\right)^a\left(\frac{b}{a+b}\right)^b=\frac{a^ab^b}{(a+b)^{a+b}}\\ \text{minimum}=0.\end{cases}$

Example 3： $F(\theta)=\dfrac{\mu\omega}{\mu\sin\theta+\cos\theta} \quad 0\le\theta\le\dfrac{\pi}{2},\ \mu,\omega>0$

Solution： $F'(\theta)=\dfrac{-\mu\omega(\mu\cos\theta-\sin\theta)}{(\mu\sin\theta+\cos\theta)^2}=0$

$$\Rightarrow \mu=\tan\theta$$



θ	0	$\dfrac{\pi}{2}$	$\tan^{-1}\mu$
$F(0)$	$\mu\omega$	ω	$\dfrac{\mu\omega}{\sqrt{1+\mu^2}}$

$\Rightarrow F$ is minimized when $\tan\theta=\mu$.

4-2 The Mean Value Theorem

Homework：1,3,5,11,13,15,17,23,25,27,29,35

Rolle's Theorem

Assumptions：
(1) f is continuous on $[a,b]$
(2) f is differentiable on (a,b)
} 簡稱 f is smooth on $[a,b]$
(3) $f(a)=f(b)$ (等高)

$$\Rightarrow \exists c\in(a,b)\ s.t.\quad f'(c)=0$$

- Smooth (平滑) + 等高

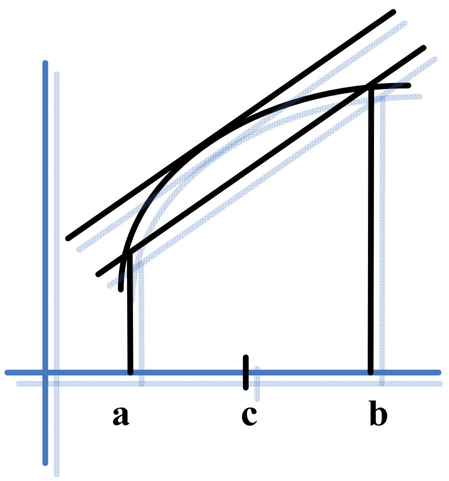
$\Rightarrow$
1. 數學: 有切線斜率為 0 的點(即有峰或谷)
2. 物理: 有速度為 0 的時間

Mean Value Theorem (MVT)

(1) + (2) (即 f is smooth on $[a,b]$)

$\Rightarrow \exists c\in(a,b)$ s.t.

$$f(b)-f(a)=f'(c)(b-a)$$

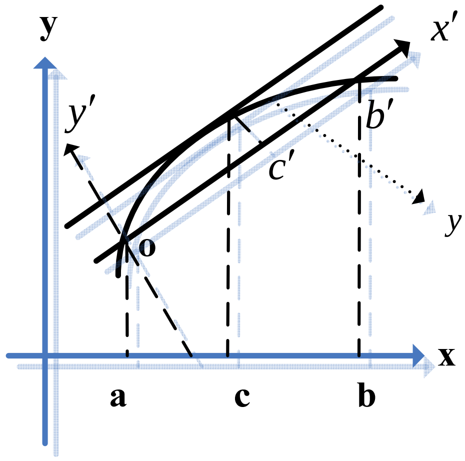


$\dfrac{f(b)-f(a)}{b-a}$=割線斜率

$f'(c)$=切線斜率

若 f 在 $[a,b]$ 是平滑的 $\Rightarrow$
- 數學上: 割線斜率=某一切線斜率
- 物理上: 平均速度=某一瞬間速度

Proof of MVT：



$y = f(x) \Leftrightarrow y' = g(x')$

Proof：在 $x' - y'$ 新座標系，將 Rolle 用在 $y' = g(x')$

$$\Rightarrow \exists c' \in (0, b') \ s.t. \quad g'(c') \,//\, x'\text{軸}$$

$$\Leftrightarrow \exists c \in (a, b) \ s.t. \quad f'(c) \,//\, x'\text{軸}$$

$$\Leftrightarrow f'(c) = \frac{f(b) - f(a)}{b - a}$$

Example 1： $f(x) = x^3 - x,\ [a,b] = [0,2]$, Find c.

Solution： $\frac{6-0}{2-0} = 3c^2 - 1 \quad \Rightarrow c = \frac{2\sqrt{3}}{3}$.

Example 2： f：平滑 $f(0) = -3,\ f'(x) \le 5,\ 0 \le x \le 2$.

How large can $f(2)$ possibly be?

Solution： $\frac{f(2)+3}{2-0} = f'(c) \le 5 \quad \Rightarrow f(2) \le 7 \Rightarrow f(2)$最大的可能是7.

Example 3： f：平滑 $f(0) = -3,\ f'(x) \ge 2, 0 \le x \le 4$.

How small can $f(4)$ possibly be?

Solution： $\frac{f(4)+3}{4-0} = f'(c) \ge 2 \quad \Rightarrow f(4) \ge 5 \Rightarrow f(4)$最小的可能是5.

Example 4： Does there exist a function f s.t.

$$f(0)=-1, f(2)=4, \text{and } f'(x)\le 2 \text{ for all } x?$$

Solution： $\frac{4+1}{2-0}=\frac{5}{2}=f'(c)\Rightarrow$ 不可能.

Example 5： Prove that $\sqrt{1+x}<1+\frac{1}{2}x$ if $x>0$.

Solution： Let $f(x)=(1+x)^{\frac{1}{2}}\Rightarrow f'(x)=\frac{1}{2}(1+x)^{-\frac{1}{2}}$

$$\overset{MVT}{\Rightarrow} f(x)-f(0)=f'(c)(x-0), c\in(0,x)$$
$$\Rightarrow \sqrt{1+x}-1=\frac{1}{2}(1+c)^{-\frac{1}{2}}x.$$
$$\Rightarrow \sqrt{1+x}=1+\frac{1}{2}(1+c)^{-\frac{1}{2}}x<1+\frac{1}{2}x.$$

$(\because \ (1+c)^{-\frac{1}{2}}<1)$

Example 6： Prove that $|\cos a-\cos b|\le|a-b|$ for all a and b.

Proof：Let $f(x)=\cos x$, assume without loss of generality, that $a\ge b$

$$\overset{MVT}{\Rightarrow} \cos a-\cos b=f'(c)(a-b)=(\sin c)(a-b)$$
$$\Rightarrow |\cos a-\cos b|\le|a-b|.$$
$$(\because |\sin c|\le 1).$$

Example 7： $f'(x)=0$ on $(a,b)\Rightarrow f$ is constant on (a,b).

Proof：Let $x_1,x_2\in(a,b), x_1<x_2$

$$\Rightarrow f(x_2)-f(x_1)=f'(c)(x_2-x_1)=0, \quad c\in(x_1,x_2)$$
$$\Rightarrow f(x_2)=f(x_1)$$

$\Rightarrow f$ has the same value at any two numbers $x_1,x_2\in(a,b)$

$\Rightarrow f$ is constant on (a,b).

Example 8： $f'(x) = g'(x)$ for all $x \in (a,b)$

$$\Rightarrow f(x) = g(x) + \text{c} \text{ for all } x \in (a,b).$$

Here c is a constant.

Proof： A direct consequence of Ex7.

§4-3 + §4-5: 1st ＆ 2nd Derivatives and Curve Sketching

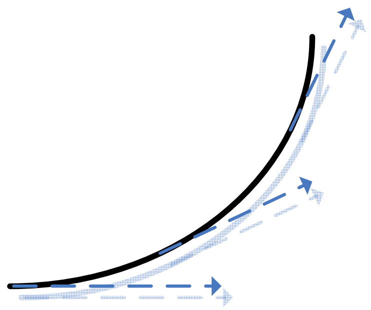
Homework：§4-3　1,5,15,17,19,41,44,47,49,63,68,75

§4-5　3,17,45,61

1. (i) $f'(x)>0$ on $(a,b) \Rightarrow f$ is increasing on (a,b).

   (ii) $f'(x)<0$ on $(a,b) \Rightarrow f$ is decreasing on (a,b)

2. (i)

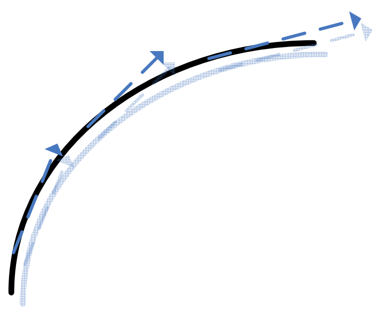


concave up (C.U.) on $(a,b) \Leftrightarrow$ 圖形在切線上方.

上方 $\Leftarrow f'$ is increasing on $(a,b) \Leftarrow f''>0$ on (a,b).

(註: 完整的証明可由 2 階 Taylor's 展開式來證明。)

(ii)



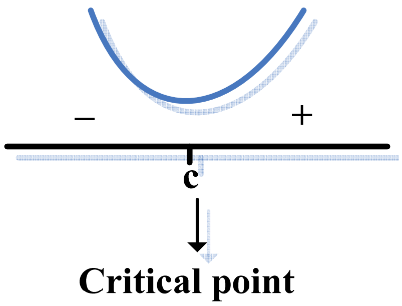
concave down (C.D.) on $(a,b) \Leftrightarrow$ 圖形在切線下方.

下方 $\Leftarrow f'$ is decreasing on $(a,b) \Leftarrow f''<0$ on (a,b).

3. Inflection point: the point at which concavity is reversed.

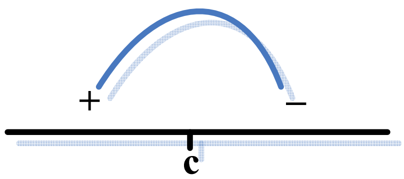
4. 1st Derivative Test

(i)



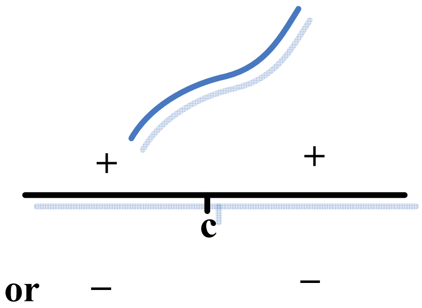
$f' \Rightarrow f(c)$: local min.

(ii)



$f' \Rightarrow f(c)$: local max.

(iii)



$f(c)$: neither a min nor a max.

5. 2nd Derivative Test

c : critical point

$f''(c)>0$	$f(c)$ ：local min (∵在 c 的附近 f C.U.)

$f''(c)<0$	$f(c)$ ：local max (∵在 c 的附近 f C.D.)

$f''(c)=0$	The test fails

Example： $f(x)=x^3, f''(0)=0.$ $f(0)$ is neither a max. nor a min.

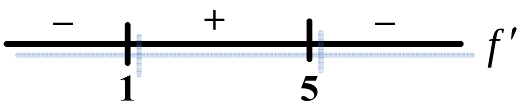
6. Curve Sketching

(i) Intercepts (ii) Symmetry (iii) Asymptotes
(iv) Local extrema (v) Inflection points

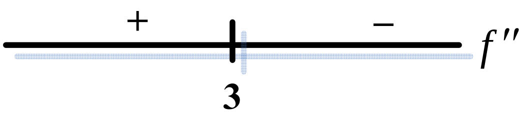
Example 1： $f(x) = 2 - 15x + 9x^2 - x^3$

Solution： $f(x) = 2 - 15x + 9x^2 - x^3 = -(x-2)(x^2 - 7x + 1)$

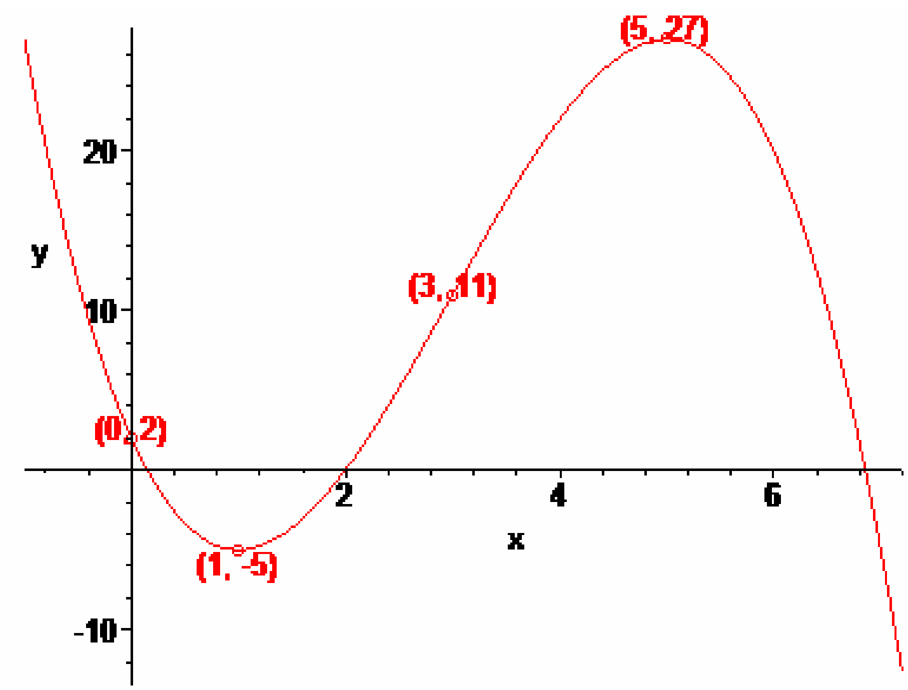
$$f'(x) = -15 + 18x - 3x^2 = -3(x^2 - 6x + 5) = -3(x-5)(x-1).$$



$$f''(x) = 18 - 6x = 6(3 - x)$$



x	0	1	3	5
y	2	-5	11	27



Example 2： $f\left(x\right)=x^{\frac{2}{3}}\left(6-x\right)^{\frac{1}{3}}$

Solution： $f'\left(x\right)=\frac{2}{3}x^{-\frac{1}{3}}\left(6-x\right)^{\frac{1}{3}}-\frac{1}{3}x^{\frac{2}{3}}\left(6-x\right)^{-\frac{2}{3}}$

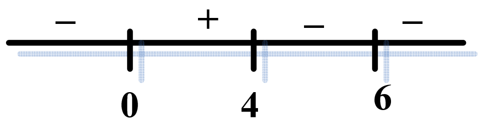
$$=\frac{1}{3}x^{-\frac{1}{3}}\left(6-x\right)^{-\frac{2}{3}}\left(2\left(6-x\right)-x\right)$$

$$=x^{-\frac{1}{3}}\left(6-x\right)^{-\frac{2}{3}}\left(4-x\right).$$

$$f''\left(x\right)=-\frac{1}{3}x^{-\frac{4}{3}}\left(6-x\right)^{-\frac{2}{3}}\left(4-x\right)+\frac{2}{3}x^{-\frac{1}{3}}\left(6-x\right)^{-\frac{5}{3}}\left(4-x\right)-x^{-\frac{1}{3}}\left(6-x\right)^{-\frac{2}{3}}$$

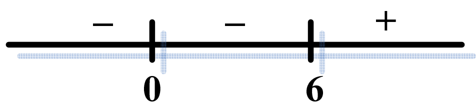
$$=\frac{1}{3}x^{-\frac{4}{3}}\left(6-x\right)^{-\frac{5}{3}}\left(-\left(6-x\right)\left(4-x\right)+2x\left(4-x\right)-3x\left(6-x\right)\right)$$

$$=\frac{-8}{x^{\frac{4}{3}}\left(6-x\right)^{\frac{5}{3}}}.$$

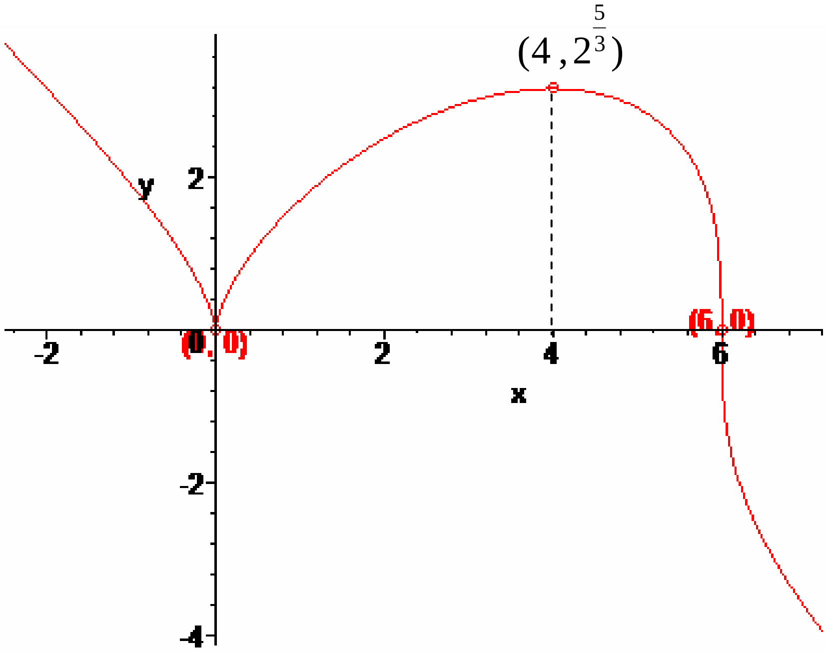


f' $\Rightarrow$ $f(0)=0$ is a local min.

$f(4)=2^{\frac{5}{3}}$ is a local max.



f'' $\Rightarrow$ $(6,0)$ is an inflection point.



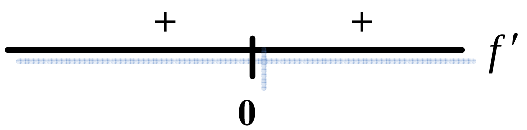
Example 3： $f(x) = xe^{-x}$.

Example 4： $f(x) = \dfrac{x^3}{x^2+1}$.

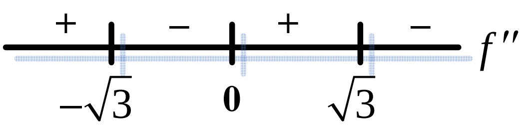
Solution： $y = f(x) = \dfrac{x^3}{x^2+1} = x - \dfrac{x}{x^2+1}$

Slant Asymtotic. $y = x$

$$y = x - \frac{x}{x^2+1}$$



$$f'(x) = \frac{x^2(x^2+3)}{(x^2+1)^2}$$

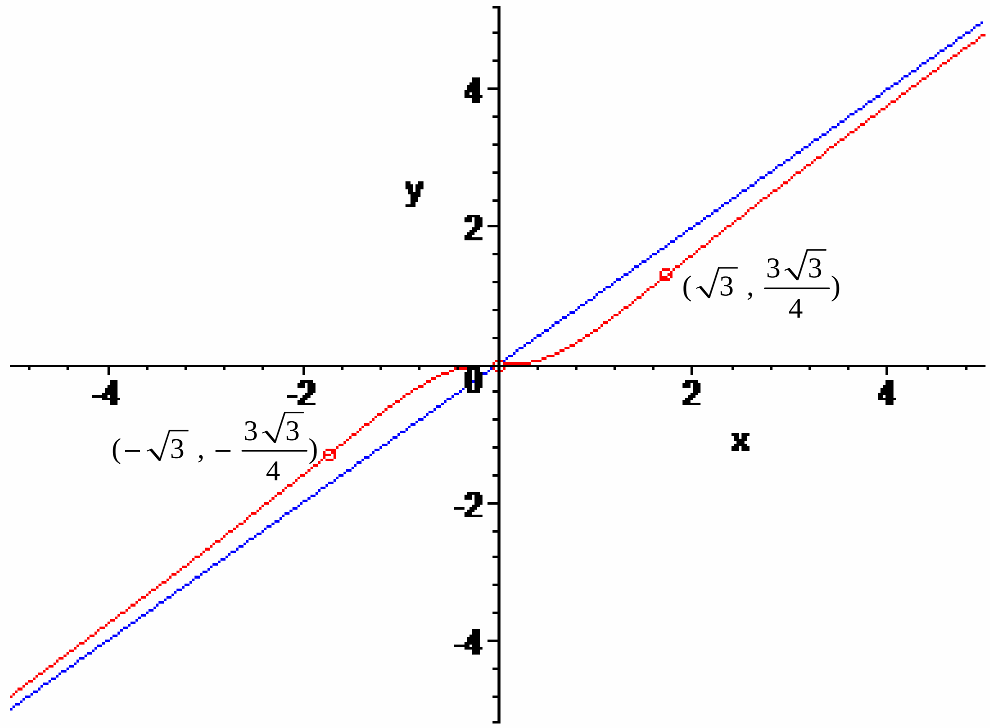


$$f''(x) = \frac{-2x(x^2-3)(x^2+1)}{(x^2+1)^4}$$

x	0	$\sqrt{3}$	$-\sqrt{3}$

y	0	$\frac{3\sqrt{3}}{4}$	$-\frac{3\sqrt{3}}{4}$

Symmetry：原點(0,0)



§4-4 Indeterminate Forms and L'Hospital's Rule

Homework：5,9,11,15,17,26,47,49,50,51,59,73,74

(I) Indeterminate forms: (不確定型)

$$\frac{0}{0}, \frac{\infty}{\infty} = \infty \cdot (0), 1^{\infty}, 0^{0}, \infty^{0}, \infty - \infty$$

Determinate forms

$$\infty + \infty = \infty, (\infty)(\infty) = \infty, \frac{3}{5}$$

(II) L'Hospital's Rule

(i) $\frac{f}{g}$: indeterminate form

(ii) $\frac{f'}{g'}$: determinate form

$$\Rightarrow \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

解題常遇的狀況.

(iii) $\frac{f'}{g'}$:indeterminate$\Rightarrow$same procedure can be continued.(見 Examples 2,6,8)

(iv) $1^{\infty}, 0^{0}, \infty^{0} \Rightarrow$ Take $\ell n \Rightarrow \frac{f}{g}$ (見 Examples 4,5)

Example 1： $\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\cos x}{1} = 1$.

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} \frac{2x}{1} = 2.$$

Example 2： $\lim_{x \to 1} \frac{\ln x}{x - 1} = \lim_{x \to 1} \frac{1}{x} = 1$.

Example 3： $\lim_{x \to \infty} \frac{e^x}{x^2} = \lim_{x \to \infty} \frac{e^x}{2x} = \lim_{x \to \infty} \frac{e^x}{2} = \infty$.

Generally speaking：

$x^x \ll x! \ll e^x \ll x^n \ll \ln x$ when x large

Example 4： $\lim\limits_{x\to 0^+}(1+\sin 4x)^{\cot x}$.

Solution： $y=(1+\sin 4x)^{\cot x}$

$$\ln y = \cot x \ln(1+\sin 4x)$$

$$= \frac{\ln(1+\sin 4x)}{\tan x}$$

$$\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \frac{\ln(1+\sin 4x)}{\tan x}$$

$$= \lim_{x\to 0^+} \frac{4\cos 4x}{(\sec^2 x)(1+\sin 4x)} = 4$$

$$\Rightarrow \lim_{x\to 0^+} y = e^4.$$

Example 5： $\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^x$.

Solution： $y=\left(1+\frac{1}{x}\right)^x$

$$\ln y = x\ln\left(1+\frac{1}{x}\right) = \frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}$$

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}} = \lim_{x\to\infty} \frac{1}{1+\frac{1}{x}} = 1$$

$$\Rightarrow \lim_{x\to\infty} y = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e.$$

Example 6： $\lim\limits_{x\to 0}\frac{\tan x - x}{x^3} = \lim\limits_{x\to 0}\frac{\sec^2 x - 1}{3x^2}$

$$= \lim_{x\to 0}\frac{2\sec x\tan x}{6x} = \frac{1}{3}.$$

Example 7： If f' is continuous, $f(2)=0, f'(2)=7,$ evaluate

$$\lim_{x\to 0}\frac{f(2+3x)+f(2+5x)}{x}.$$

Solution： $\lim_{x\to 0}\frac{f(2+3x)+f(2+5x)}{x}$

$$=\lim_{x\to 0}\frac{3f'(2+3x)+5f'(2+5x)}{1}$$

$$=3f'(2)+5f'(2)=56.$$

Example 8：For what values of a and b is the following equation true?

$$\lim_{x\to 0}\left(\frac{\sin 2x}{x^3}+a+\frac{b}{x^2}\right)=0.$$

Solution： $\lim_{x\to 0}\frac{\sin 2x+ax^3+bx}{x^3}$

$$= \lim_{x\to 0}\frac{2cos2x+3ax^2+b}{3x^2} \quad \left(=\frac{f}{g}\right)$$

$\frac{f}{g}$ must be indeterminate form

$$\Rightarrow 2+b=0 \Rightarrow b=-2$$

$$=\lim_{x\to 0}\frac{-4\sin 2x+6ax}{6x}$$

$$=\lim_{x\to 0}\frac{(8-6a)x}{6x}=\lim_{x\to 0}\frac{8-6a}{6}=0$$

$$\Rightarrow a=\frac{4}{3}.$$

§4-6

Where is wrong with the following calculation?

$$\lim_{x\to0}\frac{\tan x - x}{x^3} \overset{(1)}{=} \lim_{x\to0}\frac{\frac{\sin x}{\cos x} - x}{x^3} \overset{(2)}{=} \lim_{x\to0}\frac{\sin x - x\cos x}{x^3\cos x}$$

$$\overset{(3)}{=} \lim_{x\to0}\frac{x\left(\frac{\sin x}{x} - \cos x\right)}{x^3\cos x} \overset{(4)}{=} \lim_{x\to0}\frac{1-\cos x}{x^2\cos x}$$

$$\overset{(5)}{=} \lim_{x\to0}\frac{(1-\cos x)(1+\cos x)}{x^2\cos x(1+\cos x)} \overset{(6)}{=} \lim_{x\to0}\frac{\sin^2 x}{x^2\cos x(1+\cos x)}$$

$$\overset{(7)}{=} \lim_{x\to0}\frac{1}{\cos x(1+\cos x)}$$

$$\overset{(8)}{=} \frac{1}{2}$$

從 (3) 到 (4) 出了問題。

令 $\frac{x}{x^3\cos x} = f(x)$, $\frac{\sin x}{x} = g(x)$, $\frac{x\cos x}{x^3\cos x} = \frac{1}{x^2} = h(x)$

則 (3) 可寫成 $\lim_{x\to0}\left(f(x)g(x) - h(x)\right)$.

仔細看看從 (3) 如何變成 (4)。

$$\lim_{x\to0}\left(f(x)g(x)-h(x)\right) \overset{(i)}{=} \lim_{x\to0}\left(f(x)g(x)\right) - \lim_{x\to0}h(x)$$

$$\overset{(ii)}{=} \lim_{x\to0}f(x)\lim_{x\to0}g(x) - \lim_{x\to0}h(x)$$

$$\overset{(iii)}{=} \lim_{x\to0}f(x) - \lim_{x\to0}h(x)$$

$$\overset{(iv)}{=} \lim_{x\to0}\left(f(x)-h(x)\right)$$

$$\overset{(v)}{=} \lim_{x\to0}\frac{1-\cos x}{x^2\cos x}$$

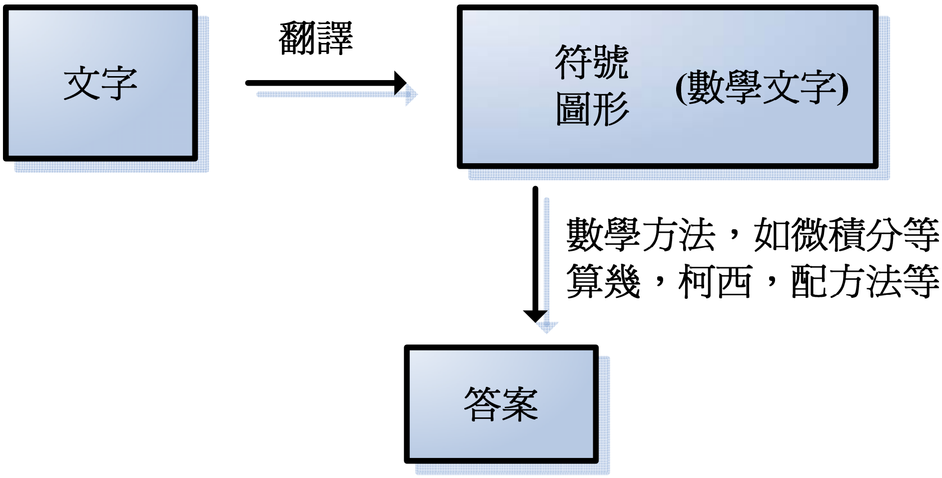
請注意 $\lim_{x\to0}f(x)$，$\lim_{x\to0}h(x)$ 皆不存在，因此上述運算皆不合法。

正確的算法可由 L'Hospital's Rule 得

$$\begin{aligned}\lim_{x\to 0}\frac{\tan x - x}{x^3} &= \lim_{x\to 0}\frac{\sec^2 x - 1}{3x^2}\\ &= \lim_{x\to 0}\frac{2\sec^2 x\tan x}{6x}\\ &= \lim_{x\to 0}\frac{\sec^2 x}{3}\left(\frac{\sin x}{x}\right)\\ &= \lim_{x\to 0}\frac{\sec^2 x}{3} = \frac{1}{3}\ .\end{aligned}$$

§4-7 Optimization

Homework：4,10,17,24,29,33,35,51,57.



微積分方法 ： 求最大值與最小值，可以用下列兩種方法來解。

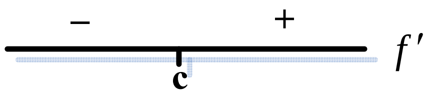
(i) f is continuous on $[a,b]$.

比較

x	a	critical points	B
y			

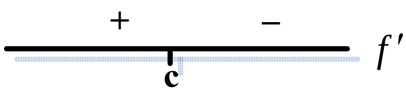
(ii) f has exactly one critical point c in the interval I .

If



$\Rightarrow f(c)$ is an absolute minimum.

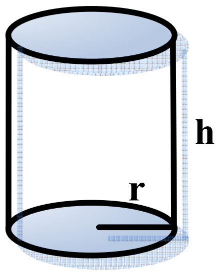
If：



$\Rightarrow f(c)$ is an absolute maximum.

Example 1：A cylindrical can is to be made to hold 1 L of oil. Find the dimensions that will minimize the cost of the metal to manufacture the can?

Solution：



已知： $\pi r^2 h = 1000$

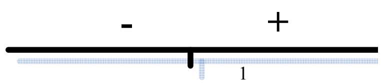
$$\text{Minimize} \quad A = 2\pi r^2 + 2\pi r h$$

$$= 2\pi r^2 + \frac{2000}{r} \quad , \quad 0 < r < \infty$$

$$\frac{dA}{dr} = 4\pi r - \frac{2000}{r^2} = \frac{4\left(\pi r^3 - 500\right)}{r^2}$$

$$\Rightarrow r = \left(\frac{500}{\pi}\right)^{\frac{1}{3}}$$

$$A\left[\left(\frac{500}{n}\right)^{\frac{1}{3}}\right] \text{is a absolute minimun.}$$

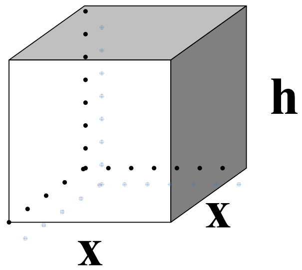


$$\left(\frac{500}{\pi}\right)^{\frac{1}{3}} \qquad \Rightarrow$$

$$\Rightarrow h = \frac{1000}{\pi\left(\frac{500}{\pi}\right)^{\frac{2}{3}}} = 2\sqrt[3]{\frac{500}{\pi}} = 2r.$$

Example 2： A box with a square base and open top must have a volume of 32,000 cm^3. Find the dimensions of the box that minimize the amount of material used.

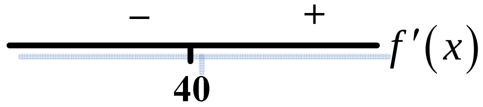
Solution：



已知： $x^2h = 32000$

$$\text{Minimize} \quad x^2 + 4xh = x^2 + \frac{128000}{x} =: f(x)$$

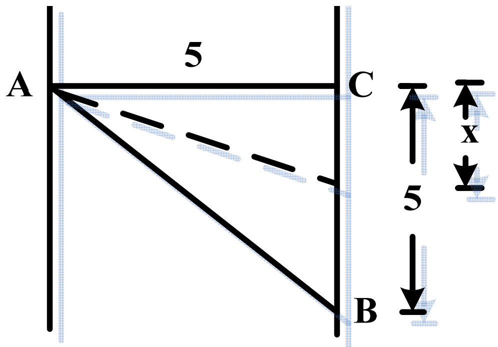
$$f'(x) = 2x - \frac{128000}{x^2} = \frac{2(x^3 - 64000)}{x^2} \quad , \quad 0 < x < \infty$$



$$\Rightarrow x = 40 \qquad h = 20.$$

Example 3： A man launches his boat from point A on a bank of a straight river, 5 km wide, and wants to reach point B, 5 km downstream on the opposite bank, as quick as possible. If he can row 6 km/h and run 8 km/h, where should he land to reach B as soon as possible?

Solution：



Minimize

$$T(x)=\frac{\sqrt{25+x^2}}{6}+\frac{5-x}{8},\quad 0\le x\le 5$$

$$T'(x)=\frac{x}{6\sqrt{25+x^2}}-\frac{1}{8}$$

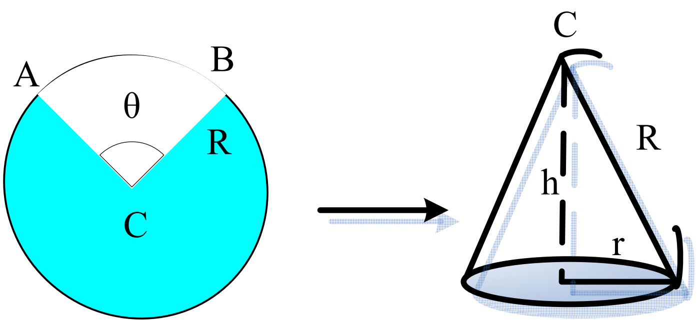
$$T'(x)=0 \Rightarrow \frac{x}{6\sqrt{25+x^2}}=\frac{1}{8}\Rightarrow 16x^2=9x^2+225$$

$$\Rightarrow x=\frac{15}{\sqrt{7}}>5$$

x	0	5
y	$\frac{5}{6}+\frac{5}{8}$	$\frac{5}{6}\sqrt{2}$

He should row directly to *B*.

Example 4：A cone-shaped drinking cup is made from radius R by cutting out a sector and joining the edges *CA* and *CB*. Find the maximum capacity of such a cup.

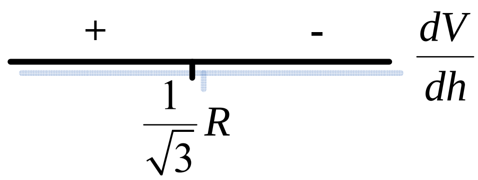


Proof： $V(h)=\frac{1}{3}\pi r^2 h=\frac{1}{3}\pi\left(R^2-h^2\right)h$

$$= \frac{\pi}{3}R^2 h - \frac{\pi}{3}h^3, \quad 0 < h < R$$

$$\frac{dV}{dh} = \frac{\pi}{3}\left(R^2 - 3h^2\right) = 0$$

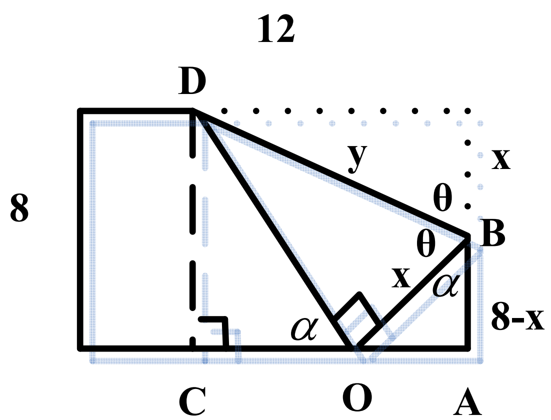
$$h = \frac{1}{\sqrt{3}}R$$



$$\Rightarrow V\left(\frac{1}{\sqrt{3}}R\right) = \frac{2\pi}{9\sqrt{3}}R^3 \quad \text{is the maximum.}$$

想想看: 為達最大容量，角度 θ 要取多少?

Example 5： The upper right-hand corner of a piece of paper, 12 inch by 8 inch, as in the figure, is folded over to the bottom edge. How would you fold it, so as to minimize the length of the fold ? In other words, how would you choose x to minimize y?



Solution： $\Delta OCD \sim \Delta BAO$

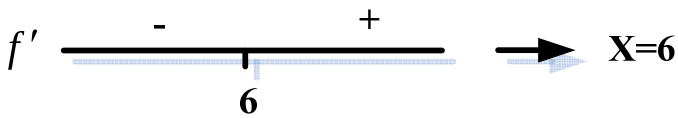
$$\Rightarrow \frac{8}{\sqrt{y^2 - x^2}} = \frac{4\sqrt{x-4}}{x}$$

$$\Rightarrow \left(y^2 - x^2\right)(x-4) = 4x^2$$

想想看: why $4 < x \le 8$?

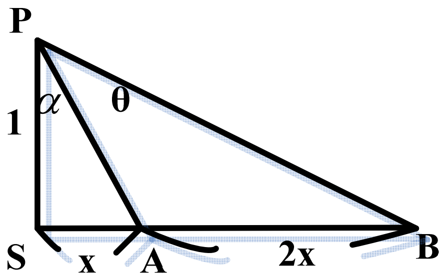
$$\Rightarrow y^2 = \frac{4x^2}{x-4} + x^2 = \frac{x^3}{x-4} =: f(x), \; 4 < x \le 8.$$

$$\Rightarrow f'(x)=\frac{2x^2(x-6)}{(x-4)^2}=0 \quad \Rightarrow \quad x=6$$



Example 6： An observer stands at a point P, one unit away from a track. Two runners stand at the point S in the figure and run along the track. One runner runs three times as fast as other. Find the maximum value of the observer's angle of sight between runners?

Solution：

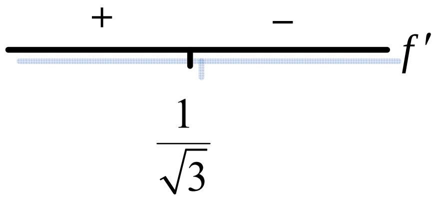


runner *B* runs three times as fast as runner *A*

$$\tan\theta=\tan(\theta+\alpha-\alpha)=\frac{\tan(\theta+\alpha)-\tan\alpha}{1+\tan(\theta+\alpha)\tan\alpha}$$

$$=\frac{3x-x}{1+3x^2}=\frac{2x}{1+3x^2}=: f(x), \quad 0\le x<\infty$$

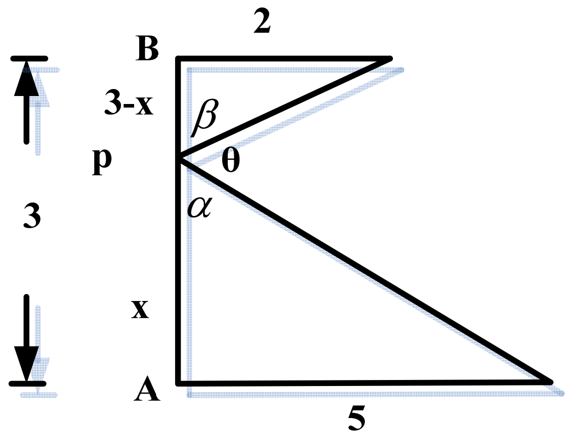
$$f'(x)=\frac{2(1-3x^2)}{(1+3x^2)^2}=0\Rightarrow x=\frac{1}{\sqrt{3}} \quad (x\ge 0)$$



$$\Rightarrow \tan\theta=\frac{2\frac{1}{\sqrt{3}}}{1+3\left(\frac{1}{\sqrt{3}}\right)^2}=\frac{1}{\sqrt{3}}.$$

$\Rightarrow \theta = \dfrac{\pi}{6}$ is the max angle of sight.

Example 7： Where should the point p be chosen on the line segment AB so as to maximize the angle θ ?



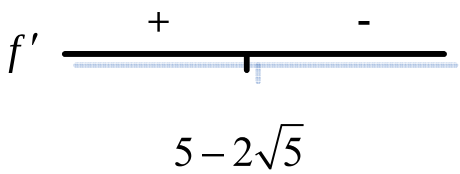
Solution： $\tan\theta = \tan\left(\pi - (\alpha+\beta)\right) = -\tan(\alpha+\beta)$

$$= -\left(\frac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta}\right) = \frac{\dfrac{5}{x}+\dfrac{2}{3-x}}{\dfrac{5}{x}\cdot\dfrac{2}{3-x}-1}$$

$$= \frac{3(5-x)}{x^2-3x+10} =: f(x), \quad 0 \le x \le 5$$

$$f'(x) = \frac{3\left(x^2-10x+5\right)}{\left(x^2-3x+10\right)^2}$$

$x = 5-2\sqrt{5}$ ($5+2\sqrt{5} > 3$ 不合)



$\Rightarrow x = 5-2\sqrt{5}$

will maximize the angle θ

§4-10 Antiderivatives

Homework：7,11,27,35,45,59,74.

Definition：If $F'(x)=f(x)$ for all $x\in I$, then F is called an antiderivative of f on I.

Example 1： $f(x)=x,\quad F(x)=\dfrac{x^2}{2}+1,\quad F(x)=\dfrac{x^2}{2}-1$

Theorem：If F is an antiderivative of f on *I*, then the most general antiderivative of f on I is $F(x)+C$, where C is an arbitrary constant.

Example 2： $f'(u)=\dfrac{u^4+3\sqrt{u}}{u^2}$,求 $f(u)$.

Solution： $f'(u)=u^2+3u^{-\frac{3}{2}} \Rightarrow f(u)=\dfrac{u^3}{3}-6u^{-\frac{1}{2}}+C$.

Example 3： $f'(x)=\sqrt{x}(6+5x),\ f(1)=10$,求 $f(x)$.

Solution： $f'(x)=6x^{\frac{1}{2}}+5x^{\frac{3}{2}}$

$$\Rightarrow f(x)=4x^{\frac{3}{2}}+2x^{\frac{5}{2}}+C$$
$$\Rightarrow f(1)=4+2+C=10$$
$$\Rightarrow C=4$$
$$\Rightarrow f(x)=4x^{\frac{3}{2}}+2x^{\frac{5}{2}}+4.$$

Example 4： A car traveling at 50 mile/hr when the brakes are fully applied, producing a constant deceleration of 22 ft/s2. What is the distance covered before the car comes to a stop ?

Solution： $a = \text{-}22$ ft/s^2 1 mile＝5280 ft

$$V(t) = -22t + C, \quad V(0) = 50\ miles/h = \frac{220}{3}\ ft/\sec.$$

$$\Rightarrow V(t) = -22t + \frac{220}{3} \quad 令 V(t) = 0 \Rightarrow t = \frac{10}{3}$$

$$\Rightarrow S(t) = -11t^2 + \frac{220}{3}t + C_1$$

$$d = S\left(\frac{10}{3}\right) - S(0) = \frac{1100}{9}(\text{ft/sec}).$$

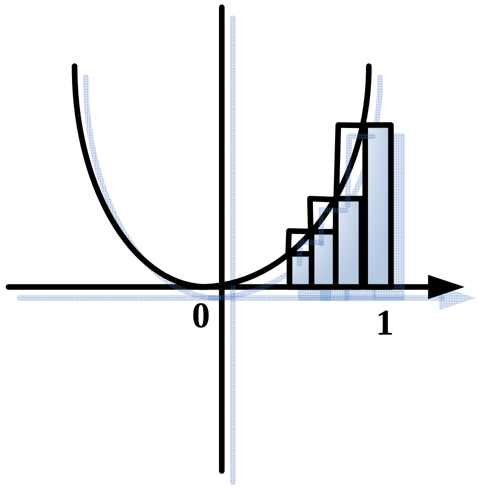
§5-1 Areas and Distances

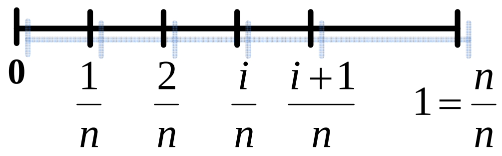
Homework：17,20,21,26.

How to find the area of a region whose boundary does not consist of just line segments ?

Example 1：The area under the curve $f(x)=x^2$ on $[0,1]$.

想法：將此區域分成許多的小長方形，利用夾擊。





把 $[0,1]$ 分成 n 等份。假設 $S=$ exact area

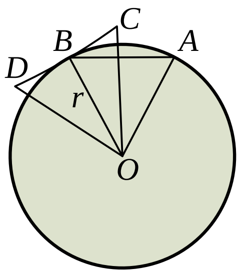
因為 f 在 $[0,1]$ 是 ↗ ，所以 $\sum_{i=0}^{n-1}\frac{1}{n}f\left(\frac{i}{n}\right)<S<\sum_{i=0}^{n-1}\frac{1}{n}f\left(\frac{i+1}{n}\right)$.

但 $\sum_{i=0}^{n-1}\frac{1}{n}f\left(\frac{i}{n}\right)=\sum_{i=0}^{n-1}\frac{i^2}{n^3}=\frac{(n-1)(n)(2n-1)}{6n^3}\to\frac{1}{3}\ as\ n\to\infty.$

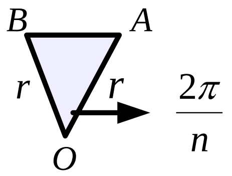
$$\sum_{i=0}^{n-1}\frac{1}{n}f\left(\frac{i+1}{n}\right)=\sum_{i=0}^{n-1}\frac{(i+1)^2}{n^3}=\sum_{i=1}^{n}\frac{i^2}{n^3}=\frac{n(n+1)(2n+1)}{6n^3}\to\frac{1}{3}\ as\ n\to\infty.$$

$$\Rightarrow S=\frac{1}{3}$$

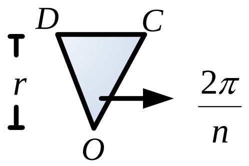
Example 2：



想法：將圓形分為許多小三角形，且利用夾擊。



下界：內接正 n 多邊形(S_n)



上界：外切正 n 多邊形(T_n)

$$S_n = \frac{n}{2} r^2 \sin\frac{2\pi}{n} \to \pi r^2 \quad as\ n \to \infty$$

$$T_n = n r^2 \tan\frac{\pi}{n} \to \pi r^2 \quad as\ n \to \infty$$

$$\Rightarrow A = \pi r^2$$

Summary：irregular area $= \sum$ regular areas

$= \sum$ 小長方形

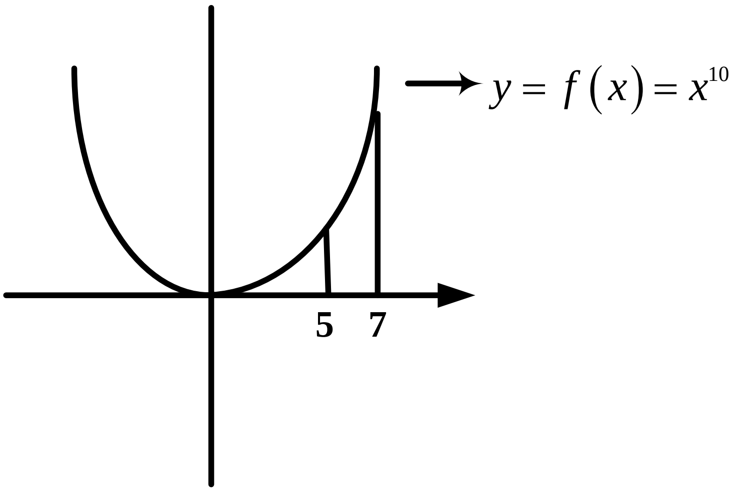
$= \sum$ 小三角形.

Example 3：

Determine a region whose area is equal to the given limit as follows：

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{2}{n}\left(5+\frac{2i}{n}\right)^{10}.$$

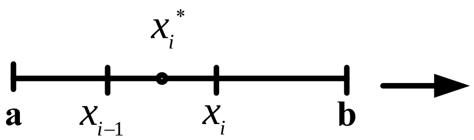
Solution：



§5-2 The Definite Integral

Homework：1,9,19,23,29,33,35,37,41,47,50,53,67,68.

介紹定積分 $\int_a^b f(x)dx$ 的定義，幾何意義和性質。



n等分, $[x_{i-1}, x_i]$=the ith subinterval.

- Riemanm sum = $\sum_{i=1}^{n} f\left(x_i^*\right)\Delta x$, $\Delta x = \dfrac{b-a}{n}$

  $\overset{\text{若 } f(x)\geq 0}{=}$ $\sum$小長方形面積

- Definite integral of f from a to b

  $= \int_a^b f(x)\,dx := \lim\limits_{n\to\infty}$ Riemanm sum

  $= \lim\limits_{n\to\infty} \sum_{\tau=1}^{n} f\left(x_i^*\right)\Delta x \overset{f(x)\geq 0}{=}$ the area

  under the curve $y = f(x)$ from a to b.

- $f(x)$ is called the integrand ; a is the lower limit ; b is the upper limit.

  Theorem：If f is continuous on $[a,b]$.

  $\Rightarrow \int_a^b f(x)\,dx$ is well-defined.

  That is, the limit of the Riemanm sum is independent of the choice of x_i^*.

- Well-definedness of $\int_a^b f(x)\,dx$. (註:不等分的子區間也對)
- $\sum$: 離散加法　　$\int$: 連續加法

Example 1：Evaluate the Riemanm sum for $f(x) = x^2$ with $n = 6$,

x_i^* = left endpoints, $a = 0$, $b = 3$.

Solution： $\frac{1}{2}\left[0^2+\left(\frac{1}{2}\right)^2+1^2+\left(\frac{3}{2}\right)^2+2^2+\left(\frac{5}{2}\right)^2\right]=\frac{55}{8}$.

Example 2： Compute $\int_0^3 x^3-6x\,dx$ by definition.

Solution：

$$S_n=\sum_{i=1}^{n}\frac{3}{n}f\left(\frac{3i}{n}\right)=\sum_{i=1}^{n}\frac{3}{n}\left(\frac{27i^3}{n^3}-\frac{18i}{n}\right)$$

$$=\frac{81}{n^4}\sum_{i=1}^{n}i^3-\frac{54}{n^2}\sum_{i=1}^{n}i=\frac{81n^2(n+1)^2}{4n^4}-\frac{27n(n+1)}{n^2}$$

$$=\frac{81}{4}-27=\frac{-27}{4}.$$

Example 3： Evaluate $\int_0^1\sqrt{1-x^2}\,dx$ by interpreting in terms of area.

Sol：$\int_0^1\sqrt{1-x^2}\,dx=$ 四分之一的單位圓面積$=\frac{\pi}{4}$.

Example 4： $\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{1}{1+\left(\frac{i}{n}\right)^2}=\int_{\square}^{\square}\square\,dx$

(Fill in the blanks, that is, identify f, a and b.)

Solution : $a=0$, $b=0$, $f(x)=\frac{1}{1+x^2}$.

(註：這種問題解答不是唯一)

Properties of the Integral

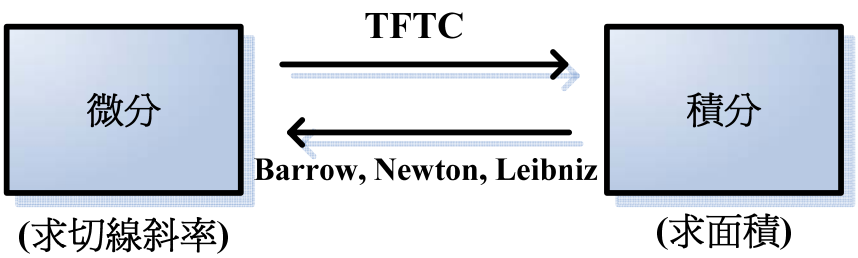
- $\int_a^b f(x)dx=-\int_b^a f(x)dx$.
- $\int_a^a f(x)dx=0$.
- $\int_a^b[\alpha f(x)+\beta g(x)]dx=\alpha\int_a^b f(x)dx+\beta\int_a^b g(x)dx$.

  (積分和微分一樣都是線性運算)
- $\int_a^c f(x)dx+\int_c^b f(x)dx=\int_a^b f(x)dx$, for any $c\in R$.
- $f(x)\ge 0$ on $[a,b]\Rightarrow\int_a^b f(x)dx\ge 0$.
- $f(x)\ge g(x)$ on $[a,b]\Rightarrow\int_a^b f(x)dx\ge\int_a^b g(x)dx$.

- If $m \le f(x) \le M$ for $a \le x \le b$, then $m(b-a) \le \int_a^b f(x)dx \le M(b-a)$.

§5-3 The Fundamental Theorem of Calculus (TFTC)

Homework: 3,9,13,17,29,30,37,42,47,51,53,54,62,67



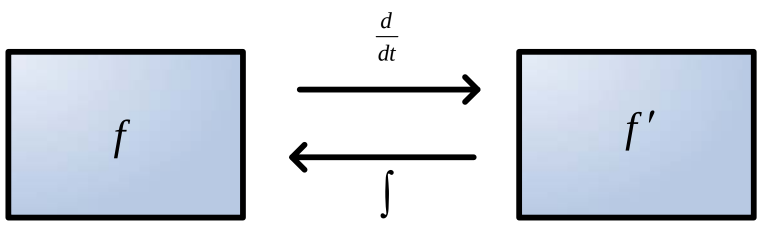
<u>TFTC</u>：Let f be continuous on $[a,b]$.

<u>part I</u>：If $g(x)=\int_a^x f(t)\,dt$,

$$\Rightarrow g'(x)=f(x).$$

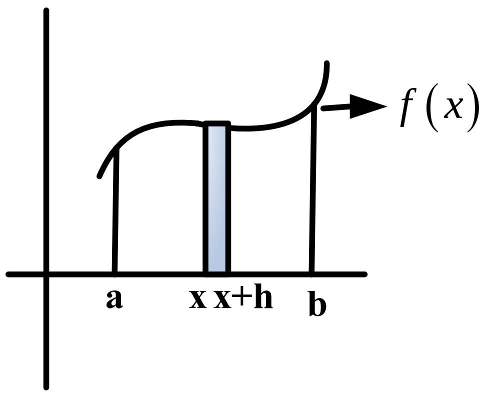
<u>part II</u>：$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b)-F(a)$,

where F is any antiderivative of f.



Geometric Intuition.

<u>part I</u>：



$$g(x+h)-g(x)\approx hf(x)$$

$$\Rightarrow \frac{g(x+h)-g(x)}{h}\approx hf(x)$$

$$\Downarrow$$

$$g'(x)=f(x).$$

<u>part II</u>：

From part I $\Rightarrow g(x)$ is an antiderivative of f

$$\Rightarrow F(x) = g(x) + c$$
$$\Rightarrow F(b) - F(a) = \big(g(b) + c\big) - \big(g(a) + c\big)$$
$$= g(b) - g(a) = \int_a^b f(t)\,dt.$$

Example 1： $\int_1^4 \frac{1}{\sqrt{x}}\,dx.$

Solution： $\int_1^4 x^{-\frac{1}{2}}\,dx = 2x^{\frac{1}{2}}\Big|_1^4 = 4 - 2 = 2.$

Find $\frac{dg}{du}$ in Example2- Example4

Example 2： $g(u) = \int_3^u \frac{1}{x^2 + x}\,dx.$

Solution： $\frac{dg}{du} = \frac{1}{u^2 + u}.$

Example 3： $g(u) = \int_u^3 \frac{1}{x^2 + x^2}\,dx = -\int_3^u \frac{1}{x^2 + x^2}\,dx.$

Solution： $\Rightarrow \frac{dg}{du} = -\frac{1}{u^2 + u}.$

Example 4： $g(u) = \int_2^{\sqrt{u}} \frac{1}{x^2 + x}\,dx$

$$= F\big(\sqrt{u}\big) - F(2), \text{ where } F'(x) = f(x).$$

Solution： $\Rightarrow \frac{dg}{du} = F'\big(\sqrt{u}\big)\frac{1}{2}u^{-\frac{1}{2}} = f\big(\sqrt{u}\big)\frac{1}{2}u^{-\frac{1}{2}}$

$$= \frac{1}{2\sqrt{u}}\left(\frac{1}{u + \sqrt{u}}\right).$$

- $\frac{d}{dx}\int_a^{f(x)} g(t)\,dt = g\big(f(x)\big)f'(x).$
- $\frac{d}{dx}\int_{h(x)}^{f(x)} g(t)\,dt = g\big(f(x)\big)f'(x) - g\big(h(x)\big)h'(x).$

Example 5： $g(x) = \int_{\tan x}^{x^2} \frac{1}{\sqrt{2 + t^4}}\,dt$

Solution： $\Rightarrow g'(x) = \frac{2x}{\sqrt{2 + x^8}} - \frac{1}{\big(1 + x^2\big)\sqrt{2 + \tan^4 x}}.$

Example 6： If $F(x)=\int_1^x f(t)\,dt,\ f(t)=\int_1^{t^2}\frac{\sqrt{1+u^4}}{u}\,du,$

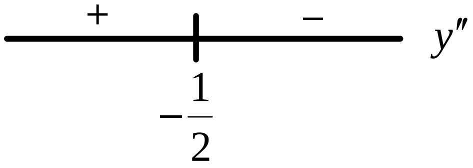
find $F''(2)$.

Solution： $F'(x)=f(x)\Rightarrow F''(x)=f'(x)=\frac{\sqrt{1+x^8}\,(2x)}{x^2}$

$\Rightarrow F''(2)=\sqrt{1+2^8}=\sqrt{257}.$

Example 7： Find the interval on which the curve $y=f(x)=\int_0^x\frac{1}{1+t+t^2}\,dt$ is concave upward.

Solution： $y'=\frac{1}{1+x+x^2}\Rightarrow y''=\frac{-(1+2x)}{\left(1+x+x^2\right)^2}$



$\Rightarrow\left(-\infty,-\frac{1}{2}\right).$

Example 8： Find $\lim_{n\to\infty}\frac{1}{n}\left(\sqrt{\frac{1}{n}}+\sqrt{\frac{2}{n}}+\cdots+\sqrt{\frac{n}{n}}\right).$

Solution：上式$=\int_0^1\sqrt{x}dx=\frac{2}{3}x^{\frac{3}{2}}\Big|_0^1=\frac{2}{3}.$

Example 9： Find a function f and a number a such that

$6+\int_a^x\frac{f(t)}{t^2}\,dt=2\sqrt{x}$, (1)

for all $x>0$.

Solution： (i) $6+\int_a^a\frac{f(t)}{t^2}\,dt=2\sqrt{a}$ (將a代入(1))

$\Rightarrow a=9$

(ii) $\frac{f\left(x\right)}{x^2} = x^{\frac{-1}{2}}$ (將$\left(1\right)$式兩邊微分)

$\Rightarrow f\left(x\right) = x^{\frac{3}{2}}$.

§5-4 Indefinite Integrals

Homework：2,9,17,27,37,39,43,44.

Indefinite Integral

$\int f(x)\,dx =$ The most general form of the antiderivatives of f

$$= F(x) + c, \quad \text{where } F'(x) = f(x).$$

Example：

(i) $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + C \quad n \neq -1$

(ii) $\int e^x\,dx = e^x + C$.

(iii) $\int \sin x\,dx = -\cos x + C$.

(iv) $\int \dfrac{1}{x}\,dx = \ln|x| + C$.

(v) $\int \dfrac{1}{1+x^2}\,dx = \tan^{-1} x + C$.

(vi) $\int \dfrac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C$.

The Net change Theorem：

The integral of a rate of change
= the net change

$\int_a^b F'(x)\,dx = F(b) - F(a).$

Example 1： $\int \dfrac{2t^2 + t^2\sqrt{t} - 1}{t^2}\,dt$.

Solution： $= \int \left(2 + t^{\frac{1}{2}} - t^{-2} \right) dt = 2t + \dfrac{2}{3} t^{\frac{3}{2}} + t^{-1} + C$.

Example 2： $\int \left(10x^4 - \sec^2 x \right) dx$.

Solution： $= 2x^5 - 2\tan x + C.$

Example 3： $\int_{-1}^{2}\left(x-2|x|\right)dx.$

Solution： $=\int_{-1}^{0}\left(x+2x\right)\,dx+\int_{0}^{2}\left(-x\right)\,dx$

$$=\frac{3}{2}x^2\Big|_{-1}^{0}-\frac{x^2}{2}\Big|_{0}^{2}=-\frac{3}{2}-2=-\frac{7}{2}\ .$$

§5-5 The Substitution Rules

Homework：3,4,12,13,17,21,23,36,37,43,51,65,75,79,82,84,85.

Two systematic methods to do integration：

1. The substitution rule (積分) $\Leftrightarrow$ chain rule (微分).
2. Integration by parts (積分) $\Leftrightarrow$ product rule (微分).

Example 1：$\int 2x\sqrt{1+x^2}\,dx$

Key：Just like doing differentiation by chain rule, we first need to identify u.

Solution：$u=1+x^2$

$$\frac{du}{dx}=2x \quad \Rightarrow \int \sqrt{u}\,du=\frac{2}{3}u^{\frac{3}{2}}+C$$

$$du=2xdx \qquad\qquad =\frac{2}{3}\left(1+x^2\right)^{\frac{3}{2}}+C.$$

Example 2：$\int \frac{x}{\sqrt{1-4x^2}}dx$

Solution： $u=1-4x^2$

$$\frac{du}{dx}=-8x \quad \Rightarrow -\frac{1}{8}\int\frac{du}{\sqrt{u}} = -\frac{1}{8}\int u^{-\frac{1}{2}}$$

$$du=-8xdx \qquad\qquad =-\frac{1}{4}u^{\frac{1}{2}}+C=-\frac{1}{4}\left(1-4x^2\right)^{\frac{1}{2}}+C$$

$$-\frac{1}{8}du=xdx.$$

Summary： $\int f\left(g(x)\right)\ g'(x)dx=\int f(u)du$

$$\underline{\quad g(x) \quad} \downarrow u \qquad \underline{\quad g'(x)dx \quad} \downarrow du \qquad u=g(x).$$

Example 3： $\int x^3 \cos\left(x^4+2\right) dx$.

Solution： $u = x^4 + 2$

$$du = 4x^3 dx \qquad \Rightarrow \qquad = \frac{1}{4}\int \cos u \, du = \frac{1}{4}\sin u + C$$

$$\frac{1}{4}du = x^3 dx \qquad\qquad = \frac{1}{4}\sin\left(x^4+2\right) + C.$$

Example 4： $\int \frac{x}{\sqrt{x-1}} dx.$

Solution： $u = x-1 \Rightarrow x = u+1$

$$du = dx \quad \Rightarrow \int \frac{(\mathrm{u}+1)\,du}{\sqrt{u}} = \int \left(u^{\frac{1}{2}} + u^{\frac{-1}{2}} \right) du$$

$$= \frac{2}{3}(x-1)^{\frac{2}{3}} + 2(x-1)^{\frac{1}{2}} + C.$$

Example 5： $\int \tan x \, dx \left(= \int \frac{\sin x}{\cos x} du \right).$

Solution： $u = \cos x$

$$du = -\sin x dx \qquad \Rightarrow -\int \frac{du}{u} = -\ln|u| + C$$

$$= -\ln|\cos x| + C$$

$$= \ln|\sec x| + C.$$

Example 6： $\int_1^e \frac{\ln x}{x} dx.$

Solution： $u = \ln x$

$$du = \frac{1}{x}dx \qquad \Rightarrow \int_0^1 u du = \frac{u^2}{2}\Big|_0^1 = \frac{1}{2}.$$

Example 7： $\int_0^4 \sqrt{2x+1} \, dx.$

Solution： $u = 2x+1$

$$du = 2dx \qquad \Rightarrow \frac{1}{2}\int_1^9 u^{\frac{1}{2}} du = \frac{1}{3}u^{\frac{3}{2}} \Big|_1^9$$

$$\frac{1}{2}du = dx \qquad = 9 - \frac{1}{3} = \frac{26}{3}.$$

Example 8： $\int \sec^3 x \tan x \, dx.$

Solution： $u = \sec x$

$$du = \sec x \tan x dx \qquad \Rightarrow \int \sec^2 x (\sec x \tan x) dx$$

$$= \int u^2 du = \frac{u^3}{3} + C$$

$$= \frac{\sec^3 x}{3} + C.$$

Example 9： $\int \frac{x}{\sqrt[4]{x+2}} dx.$

Solution： $u = x + 2 \Rightarrow x = u - 2$

$$du = dx \qquad \Rightarrow \int \frac{u-2}{u^{\frac{1}{4}}} du = \int u^{\frac{3}{4}} - 2u^{\frac{-1}{4}} du$$

$$= \frac{4}{7}u^{\frac{7}{4}} - \frac{8}{3}u^{\frac{3}{4}} + C$$

$$= \frac{4}{7}(x+2)^{\frac{7}{4}} - \frac{8}{3}(x+2)^{\frac{3}{4}} + C.$$

Example 10： $\int_0^4 f(x)\, dx = 10,$ Find $\int_0^2 f(2x)\, dx$.

Solution： $u = 2x$

$$\Rightarrow \frac{1}{2}du = dx \qquad \Rightarrow \int_0^2 f(2x)\, dx = \int_0^4 \frac{1}{2} f(u) du = \frac{1}{2}\int_0^4 f(u) du$$

$$= \frac{1}{2}\int_0^4 f(x) dx = 5.$$

Example 11： Prove that $\int_0^{\pi} xf(\sin x)\, dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)\, dx$

(Hint: $u = \pi - x$)

Solution： $u = \pi - x$

$\Rightarrow du = -dx \qquad \Rightarrow \int_0^{\pi} xf\left(\sin x\right) dx = -\int_{\pi}^{0} (\pi - u)\, f\left(\sin(\pi - u)\right) du$

$-du = dx \qquad = \int_0^{\pi} (\pi - u)\, f\left(\sin(u)\right) du$

$= \pi \int_0^{\pi} f\left(\sin(u)\right) du - \int_0^{\pi} uf\left(\sin(u)\right) du$

$= \pi \int_0^{\pi} f\left(\sin(x)\right) dx - \int_0^{\pi} xf\left(\sin(x)\right) dx$

That is,

$$\int_0^{x} xf\left(\sin(x)\right) dx = \pi \int_0^{\pi} f\left(\sin(x)\right) dx - \int_0^{\pi} xf\left(\sin(x)\right) dx$$

$$\Rightarrow \int_0^{x} xf\left(\sin(x)\right) dx = \frac{\pi}{2} \int_0^{\pi} f\left(\sin(x)\right) dx.$$

Example 12：Use example 11 to evaluate $\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\, dx$.

Solution： Let $f(y) = \frac{y}{2-y^2} \Rightarrow f(\sin x) = \frac{\sin x}{2-\sin^2 x} = \frac{\sin x}{1+\cos^2 x}$

$$\Rightarrow \int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\, dx \overset{ex11.}{=} \frac{\pi}{2} \int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\, dx$$

$u = \cos x \qquad \Rightarrow -\frac{\pi}{2} \int_1^{-1} \frac{du}{1+u^2} = \frac{\pi}{2} \int_{-1}^{1} \frac{du}{1+u^2}$

$-du = \sin x dx \qquad = \frac{\pi}{2} \tan^{-1} u \Big|_{-1}^{1} = \frac{\pi}{2}\left(\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right) = \left(\frac{\pi}{2}\right)^2.$

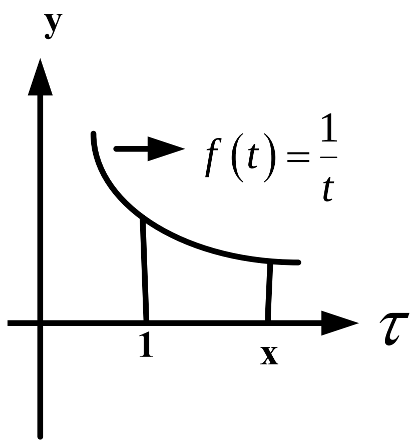
Symmetry：

- $f(x) = f(-x) \Leftrightarrow f$： even function $\Rightarrow \int_{-a}^{a} f(x) dx = 2\int_0^{a} f(x) dx.$
  (對稱 y 軸) $\Leftrightarrow f$： 偶函數
- $f(x) = f(-x) \Leftrightarrow f$： odd function $\Rightarrow \int_{-a}^{a} f(x) dx = 0.$
  (對稱原點) $\Leftrightarrow f$： 奇函數

§5-6 The Logarithm Defined as an Integral

Homework：3,4.

Definition：



定義：$\ln x = \int_1^x \frac{1}{t} dx$, $x > 0$

(I) Properties of $\ln x$.

(i) $\ln 1 = 0$, $\ln x > 0$, if $x > 1$; $\ln x < 0$ if $x < 1$.

(ii) $\frac{d}{dx} = \ln x = \frac{1}{x}$.

(iii) $\ln xy = \ln x + \ln y$.

proof of (iii)

Let $f(x) = \ln ax$, $a>0$ $\Rightarrow f(x) = \int_1^{ax} \frac{1}{t} dt$

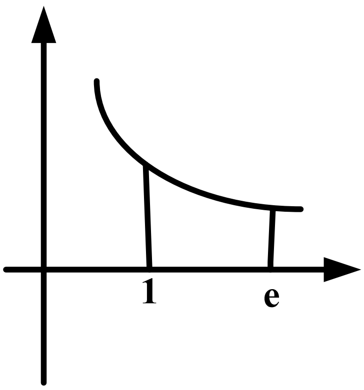
$\Rightarrow f'(x) = \frac{a}{ax} = \frac{1}{x} \Rightarrow \ln ax = \ln x + C$

$\overset{pick\ x=1}{\Rightarrow} \ln a = \ln 1 + C \qquad \Rightarrow C = \ln a$

$\Rightarrow \ln ax = \ln x + \ln a.$

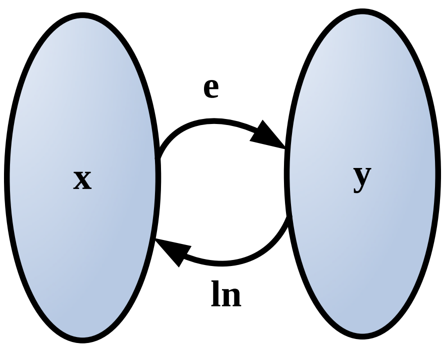
(II) Define e to be such that

$$\ln e = 1.$$



(III) Exponential Functions： $e^x = y \Leftrightarrow \ln y = x$

(Inverse function of $\ln$)



$$\Rightarrow \frac{1}{y} y' = 1 \Rightarrow y' = y \Rightarrow \frac{d}{dx} e^x = e^x.$$

(IV) (i) General Exponential Functions.

$$a^x = \left(e^{\ln a}\right)^x = e^{x\ln a} \Rightarrow \frac{d}{dx}a^x = \left(\ln a\right)a^x.$$

(ii) General Logarithm (Inverse function of $f\left(x\right) = a^x$)

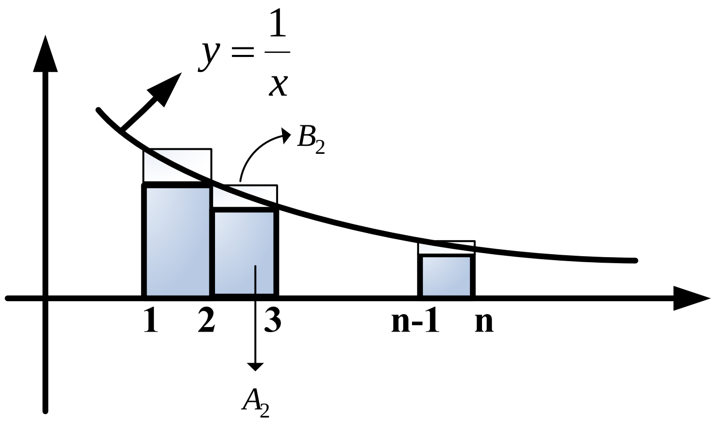
$$\log_a x = y \Leftrightarrow a^y = x \Rightarrow \left(\ln a\right)a^y y' = 1$$
$$\Rightarrow y' = a^y\left(\frac{1}{\ln a}\right) \Rightarrow \frac{d}{dx}\log_a x = \left(\frac{1}{\ln a}\right)x.$$

(V) $e = \lim_{x\to 0}\left(1+x\right)^{\frac{1}{x}}$.

Remark：有些微積分的書先講積分，再提微分。如此則可先定義 $y = \ln x$ 再定義 $y = e^x$ 再定義 $y = a^x$ 再定義 $y = \log_a x$。這樣的好處可以避免需要解釋什麼是 2^π?

Example 1：

$$\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n} < \ln n < 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n-1}.$$



因爲 $f(x) = \frac{1}{x}$ 爲遞減函數，

$\frac{1}{2}+\frac{1}{3}+..+\frac{1}{n} = \Sigma$ 小長方形 A_i 面積，且這些小長方形 A_i 皆在 $f(x) = \frac{1}{x}$ 底下.

$\frac{1}{2}+\frac{1}{3}+..+\frac{1}{n-1} = \Sigma$ 小長方形 B_i 面積，且這些小長方形 B_i 皆有部分在 $f(x) = \frac{1}{x}$ 上面.

§6-1 Areas Between Curves

Homework：1,4,7,14,19,24,29,44,45,47

Key

- $\sum$（離散的加法符號）

$\Updownarrow$

$\int$ （連續的加法符號）

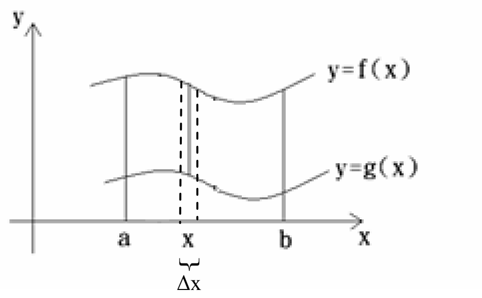
- A_n = **n** 維物體的體積

A_{n-1}=此物體 **n-1** 維截面的體積

$$\Rightarrow A_n = \int A_{n-1}dx = \int (截面)_{n-1}dx \approx \sum \underbrace{A_{n-1}}_{\text{n-1 維}} \underbrace{\Delta x}_{\text{1 維}}$$

例子：

(a)

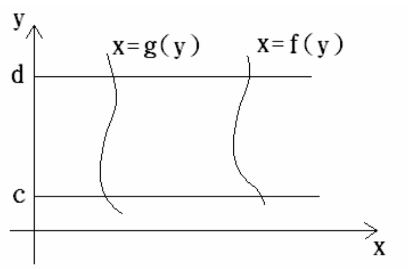


A_2 = Area between f and g from a to b

$A_1 = f(x) - g(x) = 1$維截面(線段)

$A_2 = \int_a^b \left(f(x) - g(x) \right) dx$

(b)

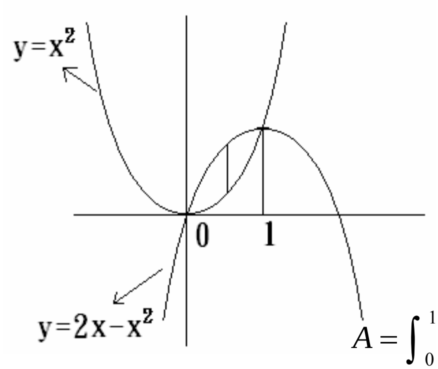


$A_2 = \int_c^d A_1 dy = \int_c^d \left(f(y) - g(y) \right) dy$

Example 1：Find the area of the region enclosed by the parabolas

$y=x^2$ and $y=2x-x^2$.

Solution：



$$\begin{cases} y=x^2 \\ y=2x-x^2 \end{cases}$$

$$\Rightarrow x^2 = 2x - x^2$$

$$\Rightarrow x(x-1) = 0$$

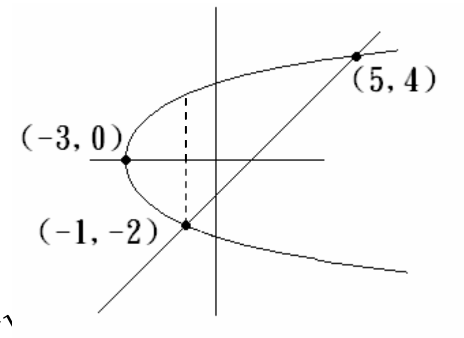
$$\Rightarrow \quad x = 0 \quad or \quad 1$$

$$A = \int_0^1 \left(2x - x^2 - x^2\right) dx = 2\int_0^1 \left(x - x^2\right) dx = 2\left(\frac{x}{2} - \frac{x^3}{3}\Big|_0^1\right)$$

$$= \frac{1}{3}.$$

Example 2：Find the area enclosed by the line $y=x-1$ and the parabola $y^2=2x+6$

Solution：



$$\begin{cases} y=x-1 \\ y^2=2x+6 \end{cases}$$

$$\Rightarrow (x-1)^2 = 2x + 6$$

$$\Rightarrow \quad x = 5 \quad or -1$$

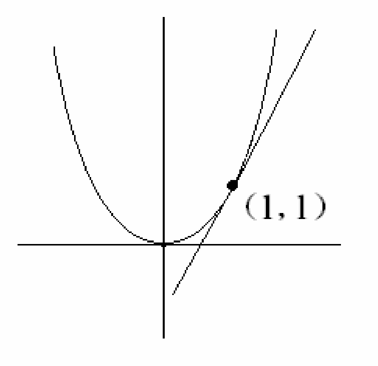
$$\Rightarrow \quad y = 4 \quad or -2$$

$$A = \int_{-3}^{-1} 2$$

$$= \int_{-2}^{4} \left[y + 1 - \frac{y^2 - 6}{2} \right] dy = 18.$$

Example 3：Find the area of the region bounded by $y=x^2$, the tangent line to this parabola at (1,1) , and the x-axis.

Solution：



$$y = x^2$$
$$\Rightarrow \quad y' = 2x$$
$$\Rightarrow y'(1) = 2$$

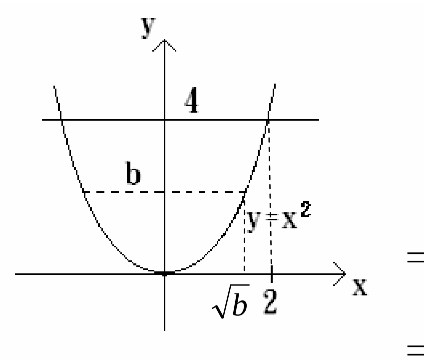
Tangent at $(1,1)$:

$$y - 1 = 2(x-1)$$
$$\Rightarrow \quad x = \frac{y+1}{2}$$

$$A = \int_0^1 \left(\frac{y+1}{2} - \sqrt{y} \right) dy = \frac{1}{12}.$$

Example 4： Find the number b such that the line y=b divides the region bounded by the curves $y=x^2$ and y=4 into two regions with equal area.

Solution：



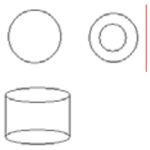
$$\frac{1}{2}\int_0^2 \left(4 - x^2\right) dx = \int_0^{\sqrt{b}} \left(b - x^2\right) dx$$
$$\Rightarrow \frac{8}{3} = \frac{2}{3} b^{\frac{3}{2}}$$
$$\Rightarrow b = 4^{\frac{2}{3}}.$$

§6-2 Volumes + §6-3 Volumes by Cylindrical Shells

Homework(§6-2)：7,11,18,19,21,22,28,35,41,47,49,50,56,64,65,67
\Homework(§6-3)：3,13,17,20,22,23,29,30,39,41

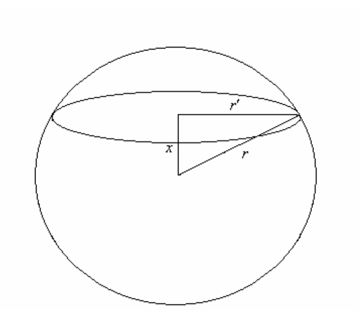
公式：

● $V_3 = \int V_2 \, dx$

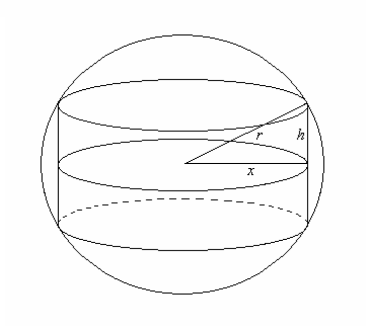
V_2(2維截面)= {  §6.2（截面為環狀）; §6.3（截面為圓柱狀）}

Example 1：Prove that the volume of the sphere with radius *r* is $\frac{4}{3}\pi r^3$.

Solution：

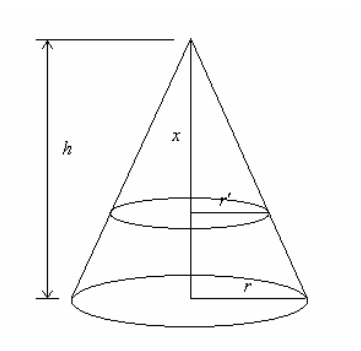


$$V = 2\int_0^r \pi r'^2 dx = 2\int_0^r \pi\left(r^2 - x^2\right)dx$$
$$= \frac{4}{3}\pi r^3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (\S 6.2)$$



$$V = 2\int_0^r 2\pi x h dx = \int_0^r 2\pi x\sqrt{r^2 - x^2}dx$$
$$= -\frac{4\pi}{3}\left(r^2 - x^2\right)^{\frac{3}{2}}\Big|_0^r = \frac{4}{3}\pi r^3 \ldots\ldots \quad (\S 6.3)$$

Example 2：Find the volume of the cone with base radius *r* and the height *h*.
Solution：

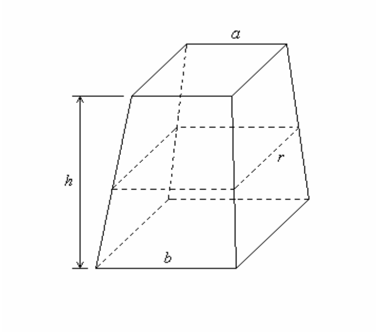


$$\frac{x}{h} = \frac{r'}{r}$$
$$\Rightarrow r' = \frac{r}{h}x.$$

$$V = \int_0^h \pi r'^2 dx = \int_0^h \frac{\pi r^2}{h^2}x^2 dx$$
$$= \frac{1}{3}\pi r^2 h.$$

Example 3：A frustum of a pyramid with square base of side b, square top of side a, and

height h.

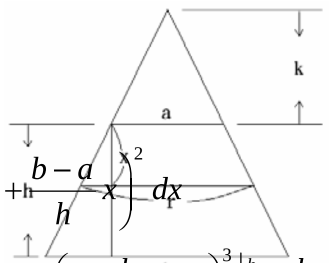


Solution：

$$\mathrm{V}=\int_0^h r^2dx$$

$$\frac{k}{h}=\frac{a}{b-a}\Rightarrow k=\frac{ah}{b-a}$$

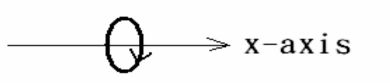
$$\frac{x+k}{k}=\frac{r}{a}\Rightarrow r=a+\left(\frac{b-a}{h}\right)x.$$



$$\Rightarrow \mathrm{V}=\int_0^h\left(a+\frac{b-a}{h}x\right)^2dx$$

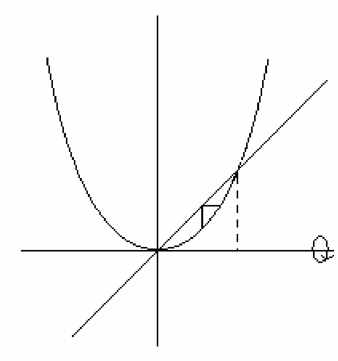
$$=\frac{h}{3(b-a)}\left(a+\frac{b-a}{h}x\right)^3\Bigg|_0^h=\frac{h}{3}\left(a^2+ab+b^2\right).$$

Example 4： $\begin{cases} y=x \\ y=x^2 \end{cases}$



（由此 2 曲線所圍區域繞著 x 軸所旋轉出來的體積）

Solution：



方法 1（截面為環狀）

$$\mathrm{V}=\int_0^1 \pi r_1^{\,2}-\pi r_2^{\,2}dx \quad r_1=x, r_2=x^2$$

$$=\int_0^1 \pi x^2-\pi x^4dx$$

$$=\frac{2\pi}{15}.$$

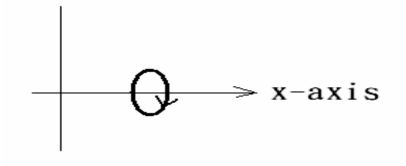
方法 2（截面為圓柱狀）

$$\mathrm{V}=\int_0^1 2\pi rhdy$$

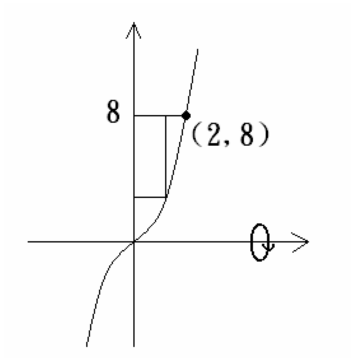
$$=\int_0^1 2\pi y\left(\sqrt{y}-y\right)dy$$

$$=\frac{2\pi}{15}.$$

Example 5： $\begin{cases} y = x^3 \\ y = 8 \\ x = 0 \end{cases}$



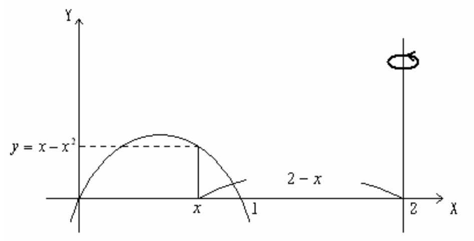
Solution：



$$\begin{aligned} V &= \int_0^2 \left(61 - x^6\right)\pi\, dx \quad \text{（截面為環狀）} \\ &= \int_0^8 2\pi y^{\frac{1}{3}} y\, dy \quad \text{（截面為圓柱狀）} \\ &= \frac{768}{7}\pi. \end{aligned}$$

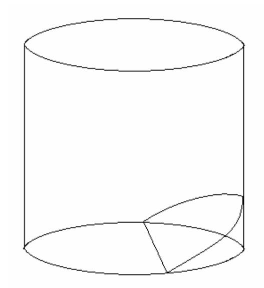
Example 6： $\begin{cases} y = x - x^2 \\ y = 0 \end{cases}$ $\quad x = 2$

Solution：



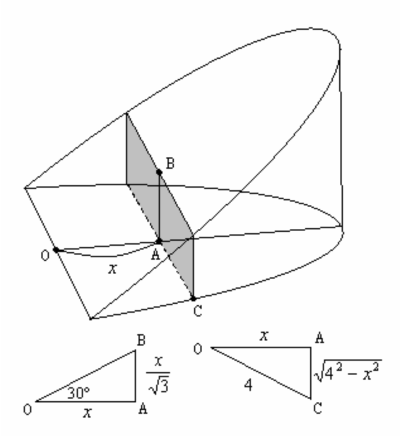
$$\begin{aligned} V &= \int_0^1 2\pi(2 - x)\left(x - x^2\right)dx \\ &= \frac{\pi}{2}. \end{aligned}$$

Example 7： A wedge is cut out of a circular cylinder of radius 4 by two planes. One plane is perpendicular to the axis of the cylinder. The other intersects the first at an angle of 30° along a diameter of the cylinder. Find the volume of the wedge.



Solution：

(Method 1)

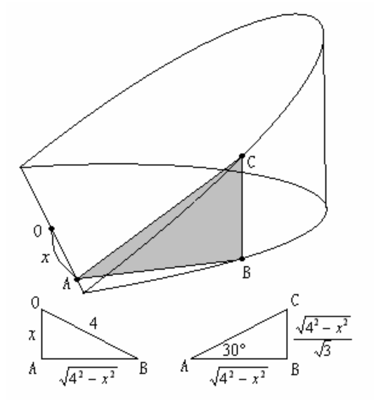


$$V=\int_0^4(\text{長方形截面})dx$$

$$=\int_0^4 \underbrace{2\sqrt{4^2-x^2}}_{\text{長}}\underbrace{\left(\frac{x}{\sqrt{3}}\right)}_{\text{寬}}dx$$

$$=\frac{128}{3\sqrt{3}}.$$

(Method 2)



$$V=2\int_0^4(\Delta\text{ABC截面})dx$$

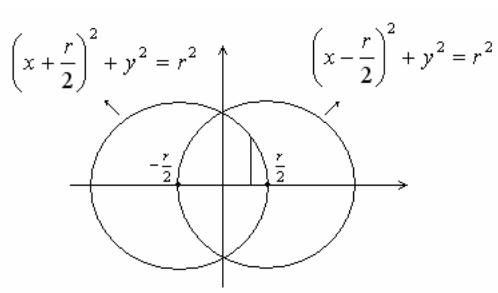
$$=\int_0^4(\text{AB}\times\text{BC})dx$$

$$=\int_0^4\frac{16-x^2}{\sqrt{3}}dx$$

$$=\frac{128}{3\sqrt{3}}.$$

Example 8：Find the volume common to two spheres, each with radius r , if the center of each sphere lies on the surface of other .

Solution：



$$V=2\int_0^{\frac{r}{2}}\pi y^2dx$$

$$=2\int_0^{\frac{r}{2}}\pi\left(r^2-\left(x+\frac{r}{2}\right)^2\right)dx$$

$$=\frac{5}{24}\pi r^3.$$

or

$$V=2\int_{-\frac{r}{2}}^{0}\pi y^2dx$$

$$=2\int_{-\frac{r}{2}}^{0}\pi\left(r^2-\left(x-\frac{r}{2}\right)^2\right)dx$$

$$=\frac{5}{24}\pi r^3.$$

§7-1 Integration by Parts

Homework.： 3,9,10,15,21,25,29,40,45,57,63

公式：Integration by parts

- $\int udv = uv - \int vdu$.
- $\int_a^b udv = uv\Big|_a^b - \int_a^b vdu$.

Integration by parts ↔ Product rule
(積分) (微分)

Why？

$$u = f(x) \quad v = g(x);$$
$$\left(f(x)g(x)\right)' = f'(x)g(x) + f(x)g'(x)$$
$$\Rightarrow \int (\underbrace{f(x)g(x)}_{uv})' dx = \int \underbrace{g(x)}_{v}\underbrace{f'(x)dx}_{du} + \int \underbrace{f(x)}_{u}\underbrace{g'(x)dx}_{du}$$
$$\Rightarrow uv = \int v\,du + \int u\,dv$$
$$\Rightarrow \int u\,dv = uv - \int v\,du.$$

Keys：

1. Identify u and dv. Easy to recover v from dv. $\int vdu$ is no complicated than $\int udv$.
2. 可能需要作不只一次的 integration by parts（見 Examples 4.5 and 6）.
3. 可能需要解方程式（見 Examples 7）.

Example 1：$\int x\sin x\,dx = -x\cos x + \int \cos x\,dx = -x\cos x + \sin x + \text{C}.$.

Solution：

$$u = x \quad dv = \sin xdx$$
$$du = dx \quad v = -\cos x$$

Example 2：$\int \tan^{-1} xdx = x\tan^{-1} x - \int \frac{x}{1+x^2}dx$.

Solution：

$$u = \tan^{-1} x \qquad dv = dx$$
$$du = \frac{dx}{1+x^2} \qquad v = x$$
$$\Rightarrow \int \tan^{-1} x\,dx = x\tan^{-1} x - \frac{1}{2}\ln\left(1+x^2\right) + \text{C}.$$

Example 3：$\int \ln(x-1)dx = (x-1)\ln(x-1) - \int 1dx = (x-1)\ln(x-1) - x + \mathrm{C}$.

Solution：

$$u = \ln(x-1) \qquad dv = dx$$

$$du = \frac{dx}{x-1} \qquad v = x-1.$$

Example 4：$\int x^2 e^x dx$.

Solution：

$$\begin{aligned} &\int x^2 e^x dx \\ &= x^2 e^x - \int 2xe^x dx \\ &= x^2 e^x - 2xe^x - 2\int e^x dx \\ &= x^2 e^x - 2xe^x + 2e^x + \mathrm{C}. \end{aligned}$$

$u = x^2$	$dv = e^x dx$
$du = 2x\,dx$	$v = e^x$
$u = 2x$	$dv = e^x dx$
$du = 2dx$	$v = e^x$

Example 5：$\int x^3 e^x dx$.

Solution： 可將上式的 Example 4 的過程作簡化但傳神的設計如左下.

$$\begin{aligned} &\int x^3 e^x dx \\ &= x^3 e^x - 3x^2 e^x + 6xe^x - 6e^x + \mathrm{C}. \end{aligned}$$

$(\,'\,)$		$(\int)$
u		v
x^3	+	e^x
$3x^2$	−	e^x
$6x$	+	e^x
6	−	e^x
0		e^x

註：以上設計必須有一多項式，如此微分夠多次會成 0（如此 integration by parts 的過程才能停止），最好是另一函數（如 Example 5 的 e^x ，和 Example $\sin x$ ）很容易找出反導數。

Example 6：$\int x^2 \sin x dx$.

Solution：

$$\begin{aligned} &\int x^2 \sin x dx \\ &= -x^2 \cos x + 2x\sin x + 2\cos x + \mathrm{C}. \end{aligned}$$

$(\,'\,)$		$(\int)$
u		v
x^2	+	$\sin x$
$2x$	−	$-\cos x$
2	+	$-\sin x$
0		$\cos x$

Example 7： $\int e^x \sin x dx$.

Solution：

$$\int e^x \sin x dx$$
$$= e^x \sin x - \int e^x \cos x dx$$
$$= e^x \sin x - e^x \cos x - \int e^x \sin x dx$$
$$\Rightarrow \int e^x \sin x dx = \frac{1}{2} e^x (\sin x - \cos x) + C.$$

$u = \sin x$	$dv = e^x dx$
$du = \cos x dx$	$v = e^x$
$u = \cos x$	$dv = e^x dx$
$du = -\sin x dx$	$v = e^x$

Example 8： $\int e^{\sqrt{x}} dx.$ (Hint：Substitution + Integration by Parts)

Solution：

$$u = \sqrt{x} \qquad du = \frac{1}{2} x^{\frac{-1}{2}} dx = \frac{dx}{2u} \Rightarrow dx = 2udu$$

$$\int \mathrm{e}^{\sqrt{\mathrm{x}}} dx$$
$$= 2\int u e^u du$$
$$= 2ue^u - 2e^u + C$$
$$= 2\sqrt{x} e^{\sqrt{x}} - 2e^{\sqrt{x}} + C.$$

$(\,'\,)$		$(\int)$
u	$+$	e^u
1	$-$	e^u
0		e^u

§7-2 Trigonometric Integrals

Homework: 3,5,13,16,19,21,29,33,41,67

題型：

(I) $\int \sin^m x \cos^n x\, dx.$

(II) $\int \tan^m x \sec^n x dx$, $\int \cot^m x \csc^n x dx.$

(III) $\int \sin mx \cos nx\, dx$, $\int \sin mx \sin nx\, dx$, $\int \cos mx \cos nx\, dx.$

(I)

i、 m or n is odd.

- $\begin{cases} u = \sin x & du = \cos x dx \\ u = \cos x & du = (-\sin x) dx \end{cases}$
- $\cos^2 x + \sin^2 x = 1$
- 選擇偶數次方的項為 u ，若 m 和 n 皆為奇數，則 u 任意數。

$$\int \sin^3 x \cos^2 x\, dx = \int \sin^2 x \cos^2 x \sin x\, dx = \int \left(1-u^2\right) u^2 \left(-du\right)$$

$$\int \cos^5 x \sin^4 x\, dx = \int \cos^4 x \sin^4 x \cos x\, dx = \int \left(1-u^2\right)^2 u^4\, du .$$

$$\int \cos^3 x \sin^3 x dx = \begin{cases} \int \cos^3 x \sin^2 x \sin x\, dx = \int u^3 \left(1-u^2\right)\left(-du\right) & \left(u = \cos x\right) \\ \int \cos^2 x \sin^3 x \cos x\, dx = \int \left(1-u^2\right) u^3 du . & \left(u = \sin x\right) \end{cases}$$

ii、 m and n are even.

利用 $\cos^2 \theta = \dfrac{1+\cos 2\theta}{2}$, $\sin^2 \theta = \dfrac{1-\cos 2\theta}{2}$ 來降階.

$$\int \sin^4 \theta\, d\theta = \int \left(\frac{1-\cos 2\theta}{2}\right)^2 d\theta = \frac{1}{4}\int \left(1 - 2\cos 2\theta + \cos^2 2\theta\right) d\theta$$

$$= \frac{\theta}{4} - \frac{1}{4}\sin 2\theta + \frac{1}{4}\int \frac{1+\cos 4\theta}{2} d\theta = \frac{3\theta}{8} - \frac{1}{4}\sin 2\theta + \frac{1}{32}\sin 4\theta + \text{C}.$$

(II)

- $\begin{cases} u = \tan x & du = \sec^2 x\, dx & (a) \\ u = \sec x & du = \sec x \tan x\, dx & (b) \end{cases}$

- $\sec^2 x = \tan^2 x + 1.$

（1）

n *is even* $\Rightarrow$ 利用 (a)

$$\int \tan^3 x \sec^4 x \, dx$$

$$= \int \tan^3 x \sec^2 x \sec^2 x \, dx$$

$$= \int u^3 \left(1+u^2\right) du.$$

（2）

m *is odd* $\Rightarrow$ 利用 (b)

$$\int \tan^3 x \sec^3 x \, dx$$

$$= \int \tan^2 x \sec^2 x \tan x \sec x \, dx$$

$$= \int \left(u^2 - 1\right) u^2 \, du.$$

（3）

m is even and n is odd. $\begin{cases} \text{沒有固定方法.} \\ \text{方法之一，將此型變成} \sin x, \cos x \text{即 } \mathbf{I} \text{ 型.} \end{cases}$

(i) $\int \tan x dx = \int \dfrac{\sin x}{\cos x} dx = \ln|\sec x| + \text{C}.$

(ii) $\int \sec x dx = \int (\sec x) \dfrac{\sec x + \tan x}{\sec x + \tan x} dx = \ln|\sec x + \tan x| + \text{C}.$

(iii) $\int \sec^{3x} dx$

$$u = \sec x \qquad dv = \sec^2 x dx$$

$$du = \sec x \tan x dx \qquad v = \tan x$$

$$= \sec x \tan x - \int \sec x \tan^2 x \, dx = \sec x \tan x - \int \sec x \left(\sec^2 x - 1\right) dx$$

$$\Rightarrow \int \sec^3 x \, dx = \frac{1}{2}\left[\sec x \tan x + \ln|\tan x + \sec x|\right] + \text{C}.$$

(III)：將乘法變加法

（1）$\sin A \cos B = \dfrac{1}{2}\left[\sin(A-B) + \sin(A+B)\right].$

（2）$\sin A \sin B = \dfrac{1}{2}\left[\cos(A-B) - \cos(A+B)\right].$

（3）$\cos A \cos B = \dfrac{1}{2}\left[\cos(A-B) + \cos(A+B)\right].$

§7-3 Trigonometric Substitution

Homework：5,9,11,17,21,27,30,34,41

題型：(利用三角代換將根號除掉)

(I) $\sqrt{a^2-x^2}\xrightarrow{x=a\sin\theta} a\cos\theta\ ,\theta\in \text{I}\cup\text{IV}.$

(II) $\sqrt{x^2-a^2}\xrightarrow{x=a\sec\theta} a\tan\theta\ ,\theta\in \text{I}\cup\text{III}.$

(III) $\sqrt{x^2+a^2}\xrightarrow{x=a\tan\theta} a\sec\theta\ ,\theta\in \text{I}\cup\text{IV}.$

Example 1： $\int\frac{\sqrt{9-x^2}}{x^2}\,dx$.

Solution：

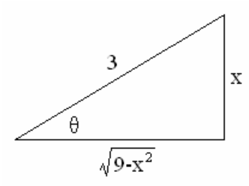
$$\int\frac{\sqrt{9-x^2}}{x^2}\,dx \qquad x=3\sin\theta \quad\Rightarrow\quad dx=3\cos\theta d\theta$$

$$=\int\frac{3\cos\theta}{9\sin^2\theta}3\cos\theta d\theta$$

$$=\int\frac{1-\sin^2\theta}{\sin^2\theta}d\theta$$

$$=-\cot\theta-\theta+C$$

$$=-\frac{\sqrt{9-x^2}}{x}-\sin^{-1}\frac{x}{3}+C.$$



Example 2： $\int\frac{dx}{x^2\sqrt{x^2+4}}$.

Solution：

$$\int\frac{dx}{x^2\sqrt{x^2+4}} \qquad x=2\tan\theta \quad\Rightarrow\quad dx=2\sec^2\theta d\theta$$

$$=\int\frac{2\sec^2\theta d\theta}{4\tan^2\theta 2\sec\theta}$$

$$=\frac{1}{4}\int\frac{\sec\theta d\theta}{\tan^2\theta}$$

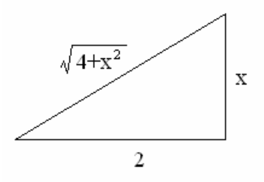
$$=\frac{1}{4}\int\frac{\cos\theta}{\sin^2\theta}d\theta$$

$$= \frac{1}{4}\int \frac{1}{u^2}\,du \qquad (u = \sin\theta)$$

$$= -\frac{1}{4}\frac{1}{u} + \mathrm{C}$$

$$= -\frac{1}{4}\frac{1}{\sin\theta} + \mathrm{C}$$

$$= -\frac{1}{4}\frac{\sqrt{x^2+4}}{x} + \mathrm{C}.$$



Example 3： $\int \frac{x}{\sqrt{x^2+4}}\,dx = \left(x^2+4\right)^{\frac{1}{2}} + C\,.$

Solution：

$$\int \frac{x}{\sqrt{x^2+4}}\,dx$$

$$= \left(x^2+4\right)^{\frac{1}{2}} + C.$$

註：若 Substitution 的方法可解決，優先使用。

Example 4： $\int_0^{\frac{3\sqrt{3}}{2}} \frac{x^3}{\left(4x^2+9\right)^{\frac{3}{2}}}\,dx.$

Solution：

$$\int_0^{\frac{3\sqrt{3}}{2}} \frac{x^3}{\left(4x^2+9\right)^{\frac{3}{2}}}\,dx \qquad x = \frac{3}{2}\tan\theta \quad \Rightarrow \quad dx = \frac{3}{2}\sec^2\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{3}} \frac{\frac{27}{8}\tan^3\theta\,\frac{3}{2}\sec^2\theta}{27\sec^3\theta}\,d\theta$$

$$= \frac{3}{16}\int_0^{\frac{\pi}{3}} \frac{\tan^3\theta}{\sec\theta}\,d\theta$$

$$= \frac{3}{16}\int_0^{\frac{\pi}{3}} \frac{\sin^3\theta}{\cos^2\theta}\,d\theta$$

$$= \frac{3}{16}\int_0^{\frac{\pi}{3}} \frac{\left(1-\cos^2\theta\right)}{\cos^2\theta}\sin\theta\, d\theta$$

$$= \frac{-3}{16}\int_{1}^{\frac{1}{2}} \frac{1-u^2}{u^2}\,du \qquad (u=\cos\theta)$$

$$= \frac{3}{16}\int_{\frac{1}{2}}^{1} \left(u^{-2}-1\right)du$$

$$= \frac{3}{16}\left(-u^{-1}-u\Big|_{\frac{1}{2}}^{1}\right)$$

$$= \frac{3}{32}.$$

Example 5： $\int \frac{x\ \ dx}{\sqrt{3-2x-x^2}}$.

Solution：

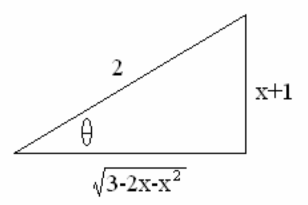
$$\int \frac{x\ \ dx}{\sqrt{3-2x-x^2}}$$

$$= \int \frac{x\ \ dx}{\sqrt{4-(x+1)^2}} \qquad x=-1+2\sin\theta \quad dx=2\cos\theta d\theta$$

$$= \int \frac{-1+2\sin\theta}{2\cos\theta}2\cos\theta d\theta$$

$$= -\theta - 2\cos\theta + C$$

$$= -\sin^{-1}\frac{x+1}{2} - \sqrt{3-2x-x^2} + C.$$



Example 6： $\int_0^{\frac{\pi}{2}} \frac{\cos t}{\sqrt{1+\sin^2 t}}\ d\ t.$

Solution：

$$\int_0^{\frac{\pi}{2}} \frac{\cos t}{\sqrt{1+\sin^2 t}}dt \qquad u=\sin t \quad du=\cos t\,dt$$

$$= \int_0^1 \frac{du}{\sqrt{1+u^2}} \qquad u=\tan\theta \quad du=\sec^2\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{4}} \sec\theta d\theta = \ln\left|\sec\theta+\tan\theta\right|\Big|_0^{\frac{\pi}{4}}$$

$$= \ln\left(\sqrt{2}+1\right).$$

Example 7： $\int \sqrt{\frac{a+cx}{x}}\ dx$

Hint：令 $x=?$ 可以將 $\sqrt{\quad}$ 除掉.

§7-4 Integration of Rational Function by Partial Fractions

Homework：9,11,23,26,29,31,37,39,45,47

How to integrate $\int \frac{P(x)}{Q(x)}\,dx$？Here $P(x)$，$Q(x)$ are polynomials？

Step1：Write $\frac{P(x)}{Q(x)}$ as $\frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$,where $\deg R < \deg Q$.

Step2：將 $Q(x)$ 因式分解成一次或二次的連乘積,即 $Q(x)$ 的因式基本型為

$(ax^2+bx+c)^n$, 和 $(ex+f)^m$, where $b^2-4ac<0$, and n and m are nonnegative integers.

Step3：

i. 含 $(ex+f)^m$ 的因式，將 $\frac{R(x)}{Q(x)}$ 寫成 partial fractions（部份分式）時，需含有下列各項

$$(ex+f)^m \to \frac{f_1}{ex+f} + \frac{f_2}{(ex+f)^2} + \ldots\ldots + \frac{f_m}{(ex+f)^m} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \ (1)$$

ii. 同理含 $(ax^2+bx+c)^n$ 的因式，部份分式需含下列各項

$$(ax^2+bx+c)^n \to \frac{b_1x+c_1}{(ax^2+bx+c)} + \frac{b_2x+c_2}{(ax^2+bx+c)^2} + \ldots\ldots + \frac{b_nx+c_n}{(ax^2+bx+c)^n} \ldots\ldots(2)$$

Why？

給一 $(m-1)$ 次多項式 $R(x)$,可將 $R(x)$ 唯一的表成（Why？）

$$R(x) = f_1(ex+f)^{m-1} + f_2(ex+f)^{m-2} + \ldots\ldots + f_{m-1}(ex+f) + f_m$$

（即將 x 轉成 $ex+f$ ）

$$\Rightarrow \frac{R(x)}{(ax+b)^m} = \frac{f_1}{ex+f} + \frac{f_2}{(ex+f)^2} + \ldots\ldots + \frac{f_{m-1}}{(ex+f)^{m-1}} + \frac{f_m}{(ex+f)^m}.$$

Step4：

i. (1)式每一項,都很容易逐項積分.

ii. (2)式每一項,都可透過配方法及 substitution 寫成

$$\frac{\alpha u+\beta}{\left(u^2+r\right)^i}=\frac{\alpha u}{\left(u^2+r\right)^i}+\frac{\beta}{\left(u^2+r\right)^i}.$$

上式的第一項可用 direct substitution.

上次的第二項可用三角 substitution.

Example 1： $\int\frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}.$

Solution：

$$\frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}\overset{長除法}{=\!=}(x+1)+\frac{4x}{(x+1)(x-1)^2}$$

$$\frac{R(x)}{Q(x)}=\frac{4x}{(x+1)(x-1)^2}=\frac{a}{x+1}+\frac{b}{(x-1)}+\frac{c}{(x-1)^2}$$

$$\Rightarrow\quad 4x=a(x-1)^2+b(x+1)(x-1)+c(x+1)$$

$$x=-1\quad\Rightarrow\quad a=-1$$

$$x=1\quad\Rightarrow\quad c=2$$

比較 2 次項的係項：$0=a+b\Rightarrow b=1$

$$\Rightarrow 原式\ =\int(x+1)dx+\int\left(\frac{-1}{x+1}\right)+\frac{1}{x-1}+\frac{2}{(x-1)^2}dx$$

$$=\frac{x^2}{2}+x-\ln|x+1|+\ln|x-1|-\frac{2}{(x-1)}+\text{C}.$$

Example 2： $\int\frac{2x^2-x+4}{x^3+4x}dx.$

Solution：

$$\frac{2x^2-x+4}{x^3+4x}=\frac{2x^2-x+4}{x\left(x^2+4\right)}=\frac{a}{x}+\frac{bx+c}{x^2+4}$$

$$2x^2-x+4=a\left(x^2+4\right)+x(bx+c)$$

$$x=0\Rightarrow a=1$$

$$\text{比較係數} \Rightarrow \begin{cases} a+b=2 \quad \Rightarrow \quad b=1 \\ c=-1 \end{cases}$$

$$\Rightarrow \text{原式} = \int \frac{1}{x} + \frac{x-1}{x^2+4} dx$$

$$= \int \left(\frac{1}{x} + \frac{x}{x^2+4} - \frac{1}{x^2+4} \right) dx$$

$$= \ln|x| + \frac{1}{2}\ln\left(x^2+4\right) - \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + \text{C}.$$

Example 3：$\int \frac{4x^2-3x+2}{4x^2-4x+3} dx$.

Solution：

$$\int \frac{4x^2-3x+2}{4x^2-4x+3} dx$$

$$= \int 1 dx + \int \frac{x+1}{(2x-1)^2+2} dx \qquad \begin{pmatrix} u = \text{-}1 \Rightarrow du = 2dx \\ x = \frac{u+1}{2} \end{pmatrix}$$

$$= x + \frac{1}{2} \int \frac{\frac{u+1}{2} - 1}{u^2+2} du$$

$$= x + \frac{1}{4} \int \frac{u-1}{u^2+2} du$$

$$= x + \frac{1}{8}\ln\left(u^2+2\right) - \frac{1}{4\sqrt{2}} \tan^{-1} \frac{x}{\sqrt{2}} + \text{C}.$$

Example 4：$\int \frac{1-x+2x^2-x^3}{x\left(x^2+1\right)^2} dx.$

Solution：

$$\frac{1-x+2x^2-x^3}{x\left(x^2+1\right)^2} = \frac{a}{x} + \frac{bx+c}{x^2+1} + \frac{dx+e}{\left(x^2+1\right)^2}$$

$$\Rightarrow \left(1-x+2x^2-x^3\right) = a\left(x^2+1\right)^2 + x(bx+c)\left(x^2+1\right) + x(dx+e)$$

$$x = 0 \quad \Rightarrow \quad a = 1$$

x^4 的係數　　$0 = a + b$　　$\Rightarrow b = -1$

x^3 的係數　　$-1 = c$

x 的係數　　$-1 = c + f$　　$\Rightarrow e = 0$

x^2 的係數　　$2 = 2a + b + d$　　$\Rightarrow d = 1$

$$原式 = \ln|x| - \frac{1}{2}\ln\left(x^2+1\right) - \tan^{-1} x - \frac{1}{2\left(x^2+1\right)} + C.$$

Example 5： $\int \frac{1}{x\sqrt{x+1}}dx..$

Solution：

$u = (x+1)^{\frac{1}{2}}$　　$u^2 = x+1$　　$\Rightarrow$　　$2udu = dx$

$$\int \frac{1}{x\sqrt{x+1}}dx$$

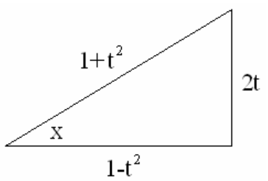
$$= \int \frac{2udu}{u\left(u^2-1\right)}.$$

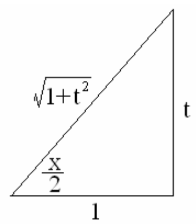
$$= \int \left(\frac{1}{u-1} - \frac{1}{u+1}\right)du$$

$$= \ln\left|\frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}\right| + C.$$

Example 6： $\int \frac{dx}{3\sin - 4\cos x}.$

Solution：





$$t = \tan\frac{x}{2} \Rightarrow \tan x = \tan\left(\frac{x}{2}+\frac{x}{2}\right) = \frac{2t}{1-t^2}$$

$$\Rightarrow \sin x = \frac{2t}{1+t^2}, \quad \cos x = \frac{1-t^2}{1+t^2}$$

$$dt = \frac{1}{2}\sec^2\frac{x}{2}dx = \frac{1}{2}\left(1+t^2\right)dx$$

$$\Rightarrow dx = \frac{2}{1+t^2}dt$$

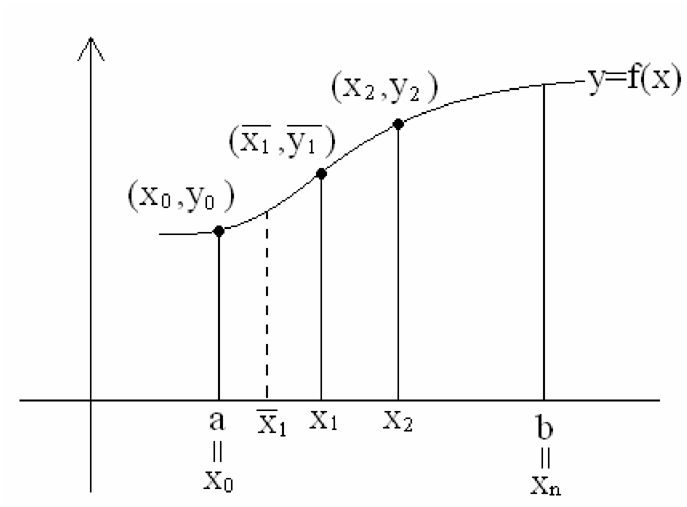
$$原式 = \int \frac{P(t)}{Q(t)}dt = \int \frac{1}{2t^2+3t-2}dt$$

即任一 rational function of $\sin x$ and $\cos x$

可改成 rational function of $t, t = \tan\frac{x}{2}.$

§7-7 Approximate Integration

Homework：7,19



$$\overline{x_i} = \frac{x_{i-1} + x_i}{2} \qquad \Delta x = \frac{b-a}{n}$$

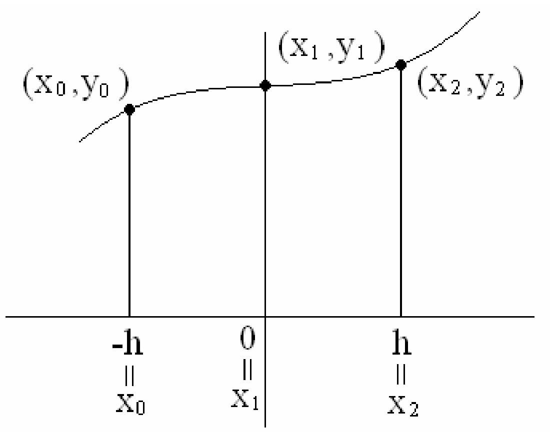
(1) Midpoint Rule：(小長方形的高=小區間中點的 f 值）

$$\int_a^b f(x) \approx \Delta x\left(f\left(\overline{x_1}\right) + \ldots\ldots + f\left(\overline{x_n}\right)\right).$$

(2)Trapezoidal：(每小區間用梯形來逼近)

$$\int_a^b f(x) \approx \frac{\Delta x}{2}\left(y_0 + 2y_1 + 2y_2 + \ldots\ldots + 2y_{n-1} + y_n\right).$$

Simpson’s Rule：二小區間的 f 圖上三點用一拋物線來逼近.



找一拋物線 $\Gamma : y = ax^2 + bx + c$ 經過 $(-h, y_0), (0, y_1), (h, y_2)$

$$\left.\begin{aligned} & y_0 = ah^2 - bh + c \\ \Rightarrow\ & y_1 = c \\ & y_2 = ah^2 + bh + c \end{aligned}\right\} \Rightarrow y_0 + y_2 \Rightarrow y_0 + 4y_1 + y_2 = 2ah^2 + 6c$$

$$\Rightarrow \int_{-h}^{h}\left(ax^2 + bx + c\right)dx = 2\int_0^h\left(ax^2 + c\right)dx$$

$$= \frac{h}{3}\left(2ah^2 + 6c\right)$$

$$= \frac{h}{3}\left(y_0 + y_2 + 4y_1\right)$$

$$= \frac{h}{3}\left(y_0 + 4y_1 + y_2\right).$$

也即

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + \ldots\ldots + 4y_{n-1} + 2y_n\right)$$

n : must be even

How accurate those approximation are？

誤差= $E_M \le \dfrac{k(b-a)^3}{24n^2}$, $\quad k = \max\limits_{a\le x\le b}\left|f''(x)\right|.$

$E_T \le \dfrac{k(b-a)^3}{12n^2}.$

$E_S \le \dfrac{\bar{k}(b-a)^5}{180n^4}$, $\quad \bar{k} = \max\limits_{a\le x\le b}\left|f^{(4)}(x)\right|$

Example 1： $\int_1^2 \frac{1}{x}\,dx$, $\quad n = 5$, midpoint rule.

Solution：

$$\ln 2 = \int_1^2 \frac{1}{x}\,dx \approx 0.2\left(\frac{1}{1.1} + \frac{1}{1.3} + \frac{1}{1.5} + \frac{1}{1.7} + \frac{1}{1.9}\right)$$

$$\approx 0.691908$$

$$\Rightarrow \ln 2 \approx 0.693147.$$

§7-8 Improper Integrals

Homework：5,7,13,21,25,27,28,33,35,49,51,53,55,59,75

What is improper integral $\int_a^b f(x)\,dx$ ？

(1) $a=\pm\infty$ or $b=\pm\infty$.

(2) $f(c)=\pm\infty$ for some $c\in[a,b]$.

以上 (1) or (2) 情況，即可稱 $\int_a^b f(x)\,dx$ 為 improper integral.

■ 幾何圖來說，improper integral 所圍出來的區域是無界的。

$$(1-a)\qquad \int_a^{\infty} f(x)\,dx=\lim_{b\to\infty}\int_a^b f(x)\,dx$$

$$(1-b)\qquad \int_{-\infty}^{\infty} f(x)\,dx=\underbrace{\int_{-\infty}^{0} f(x)\,dx}_{(1-a)}+\underbrace{\int_0^{\infty} f(x)\,dx}_{(1-a)}$$

$$(2-a)\qquad c=a\ or\ b$$

$$\lim_{d\to a^+}\int_a^b f(x)\,dx\ or\ \lim_{d\to b^-}\int_a^d f(x)\,dx$$

$$(2-b)\qquad a<c<b$$

$$\lim_{d\to c^-}\int_a^d f(x)\,dx+\lim_{d\to c^+}\int_d^b f(x)\,dx$$

註：

（1） 上述的計算過程一個 improper integral 的存在稱為收斂(convergence)，反之則稱為發散(divergence)。

（2） 有些 improper integral 是（1）和（2）的組合，如此計算時都需分開個別處理，每部分都需存在，方可說此 improper integral 存在。

（3） 若 improper integral 存在，即表示其相對應的無界區域的面積是有限的。

Example 1： $\int_1^{\infty} \frac{1}{x^P}\, dx = \lim_{b \to \infty} \int_1^{b} \frac{1}{x^P}\, dx$

Solution：

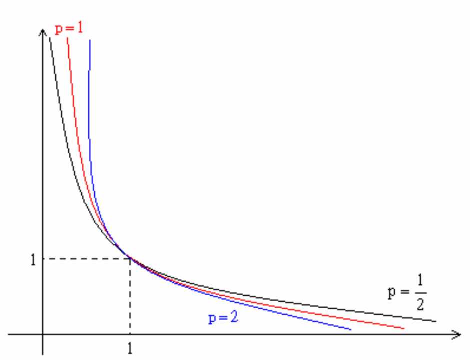
$$\int_1^b \frac{1}{x^p}\, dx = \begin{cases} \left. \dfrac{x^{-p+1}}{-p+1} \right|_1^b & p \neq 1 \\ \ln|x| \Big|_1^b & p = 1 \end{cases}$$

$$\Rightarrow \int_1^{\infty} \frac{1}{x^p}\, dx = \begin{cases} Convergence & p > 1 \\ Divergence & p \leq 1 \end{cases}$$

Example 2： $\int_0^1 \frac{1}{x^p}\, dx = \begin{cases} Convergence & p < 1 \\ Divergence & p \geq 1. \end{cases}$

Summary：

i. Example 1.中，p 要大（也即 $f(x) = \frac{1}{x^p}$ 跑到 0 的速度要夠快，當 $x \to \infty$ 時），無窮區域的面積才可能是有限。見下圖：



ii. Example 2.中，p 要小（也即 f 跑到 ∞ 的速度當 $x \to 0$ 要夠慢時），無窮區域（見上圖非斜線部份）的面積才可能是有限。

Comparison Theorem： $f \geq g \geq 0$

i. $\int_a^b g(x)\, dx : diverges \Rightarrow \int_a^b f(x)\, dx : diverges.$

ii. $\int_a^b f(x)\, dx : converges \Rightarrow \int_a^b f(x)\, dx : converges.$

Example 3： $\int_{-\infty}^{0} xe^x\, dx$.

Solution：

$$\int_{-\infty}^{0} xe^{x}dx$$
$$= xe^{x}\Big|_{-\infty}^{0} - e^{x}\Big|_{-\infty}^{0}$$
$$= 0-1$$
$$= -1 \quad （收斂）$$

Example 4： $\int_{-\infty}^{\infty}\frac{1}{1+x^{2}}\,dx$.

Solution：

$$\int_{-\infty}^{\infty}\frac{1}{1+x^{2}}\,dx$$
$$=\int_{-\infty}^{0}\frac{1}{1+x^{2}}\,dx+\int_{0}^{\infty}\frac{1}{1+x^{2}}\,dx$$
$$=\tan^{-1}x\Big|_{-\infty}^{0}+\tan^{-1}x\Big|_{0}^{\infty}$$
$$=\frac{\pi}{2}+\frac{\pi}{2}$$
$$=\pi.$$

Example 5： $\int_{0}^{3}\frac{dx}{x-1}$.

Solution：

$$\int_{0}^{3}\frac{dx}{x-1}=\underbrace{\int_{0}^{1}\frac{1}{x-1}\,dx}_{divergence}+\int_{1}^{3}\frac{1}{1-x}\,dx\,.\ (\text{發散})$$

Example 6： $\int_{0}^{1}\ln x dx$.

Solution：

$$\int_{0}^{1}\ln x dx$$
$$= x\ln x\Big|_{0}^{1} - x\Big|_{0}^{1}$$
$$=-1.$$

另解：$-\ln x$ 比 $\frac{1}{x^{\frac{1}{2}}}$ 跑到 ∞ 當 $x\to 0$ 較慢 $\Rightarrow$ 收斂 .

Example 7： $\int_{2}^{\infty}\frac{1}{\ln x}\,dx$.

Solution：

$\because \frac{1}{x} > \frac{1}{\ln x}$ 當 x 大時 $\Rightarrow$ 發散 .

Example 8： $\int_0^{\infty} e^{-x^2}\, dx$.

Solution：

$$\int_0^{\infty} e^{-x^2} dx \ \Rightarrow \ \text{收斂（和 } \frac{1}{x^2} \text{ 比）.}$$

Example 9： $\int_1^{\infty} \frac{1+e^{-x}}{x}\, dx$.

Solution：

$$\int_1^{\infty} \frac{1+e^{-x}}{x} dx \ \Rightarrow \ \text{發散（和 } \frac{1}{x} \text{ 比）.}$$

Example 10： $\int_0^{\infty} x^2 e^{-x^2}\, dx.$

Solution：

$u = x \qquad dv = xe^{-x^2} dx$

$du = dx \qquad v = -\frac{1}{2} e^{-x^2}$

$$= \int_0^{\infty} x^2 e^{-x^2} dx$$

$$= -\frac{1}{2} x e^{-x^2} \Big|_0^{\infty} + \frac{1}{2} \int_0^{\infty} e^{-x^2} dx$$

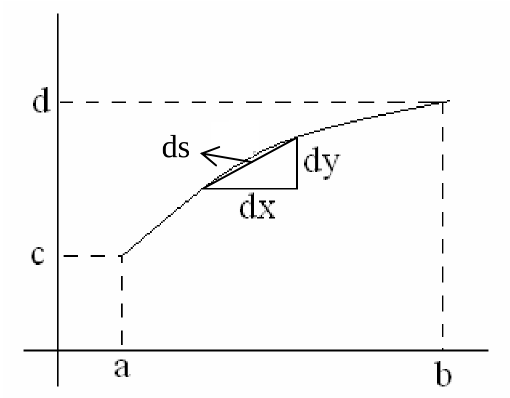
$$= \frac{1}{2} \int_0^{\infty} e^{-x^2} dx .$$

（和 $\frac{1}{x^2}$ 比可得此瑕積分為收斂）

§8-1 Arc Length

Homework：5,9,11,17,29,37

公式：弧長



$$l=\int_a^b ds$$
$$=\int_a^b\sqrt{(dx)^2+(dy)^2}$$
$$=\begin{cases}\int_a^b\sqrt{1+(\frac{dy}{dx})^2}\,dx=\int_a^b\sqrt{1+(f'(x))^2}\,dx\\ \int_c^d\sqrt{(\frac{dx}{dy})^2+1}\;dy=\int_c^d\sqrt{1+(g'(y))^2}\,dy\end{cases}$$

Example1： $y^2=x^3\qquad (1,1)\ \ to\ \ (4,8).$

Solution：

$$y=x^{\frac{3}{2}}$$
$$l=\int_1^4\sqrt{1+\frac{9}{4}x}\ dx$$
$$=\frac{3}{2}\left(1+\frac{9}{4}x\right)^{\frac{3}{2}}\Bigg|_1^4.$$

Example2： $y=x^2-\frac{1}{8}\ln x\qquad (1,1)\ \ to\ \ (x,y).$

Solution：

$$y' = 2x - \frac{1}{8x}$$

$$\Rightarrow l = \int_1^x \sqrt{1+\left(2t-\frac{1}{8t}\right)^2}\ dt$$

$$= t^2 + \frac{1}{8}\ln|t|\Big|_1^x$$

$$= x^2 + \frac{1}{8}\ln|x| - 1$$

Example3： Find the length of the curve $y = \int_1^x \sqrt{t^3-1}\ dt$, $1 \le x \le t$.

Solution：

$$y' = \sqrt{x^3-1} = f'(x)$$

$$\Rightarrow l = \int_1^4 \sqrt{1+x^3-1}\ dx$$

$$= \frac{2}{5}x^{\frac{5}{2}}\Big|_1^4$$

$$= \frac{62}{5}$$

Example4： Prove that the circumference of the circle with radius r is 2πr .

Prove：

$$y = f(x) = \sqrt{r^2-x^2} \quad \Rightarrow f'(x) = -x(r^2-x^2)^{-\frac{1}{2}}$$

$$\Rightarrow 1+(y')^2 = \frac{r^2}{r^2-x^2}$$

$$\Rightarrow 圓周長 = 4r\int_0^r \frac{1}{\sqrt{r^2-x^2}}dx$$

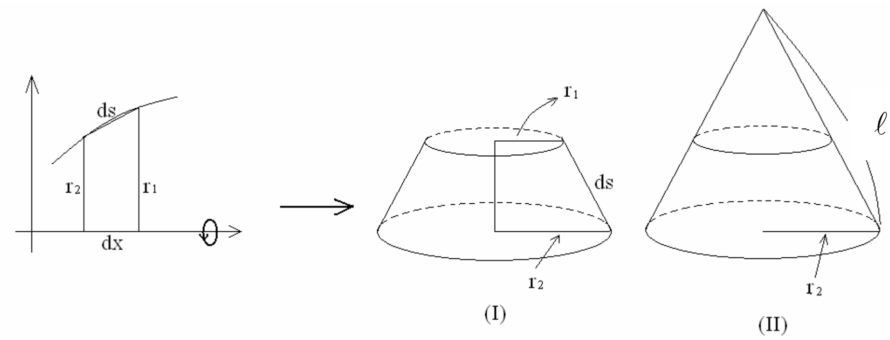
$$\overset{x=r\sin\theta}{=} \quad 4r\int_0^{\frac{\pi}{2}} \frac{1}{r\cos\theta} r\cos\theta d\theta$$

$$= \quad 2\pi r.$$

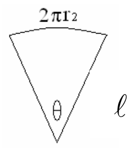
§8-2 Area of a Surface Revolution

Homework：5,11,29,33,35

公式：**Area of a Surface Revolution＝S**

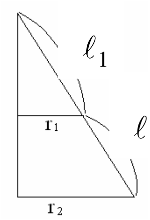


(II)圖之表面積＝



$$=\frac{1}{2}l^2\theta=\frac{1}{2}l^2\frac{2\pi r_2}{l}=\pi l r_2.$$

(I)圖之表面積 $=\pi\left(l_1+l_2\right)r_2-\pi l_1 r_1=\pi\left[\left(r_2-r_1\right)l_1+r_2 l_2\right]$



$$\Rightarrow \qquad \frac{l_1}{l_1+l_2}=\frac{r_1}{r_2}$$

$$\Rightarrow \qquad \left(r_2-r_1\right)l_1=r_1 l_2.$$

∴(I) 圓之表面積 $=\pi\left(r_1+r_2\right)l_2$

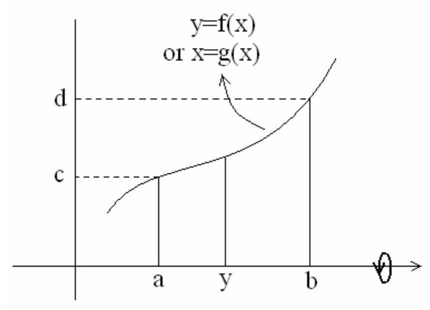
$$=\pi\left(r_1+r_2\right)ds$$

$$=2\pi r ds$$

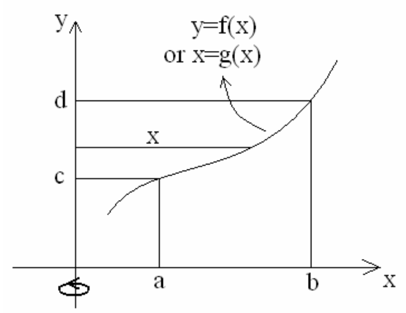
$r=\frac{r_1+r_2}{2}\left(=y=f\left(x\right)\text{當}dx\text{是無窮小時}\right).$

$$\Rightarrow \qquad S=\int 2\pi r ds.$$

$$S=\int 2\pi r ds=\begin{cases}\int_a^b 2\pi f(x)\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}\,dx\\ \int_c^d 2\pi y\sqrt{1+\left(\dfrac{dx}{dy}\right)^2}\,dy.\end{cases}$$



$$S=\int 2\pi r ds=\begin{cases}\int_a^b 2\pi x\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}\,dx\\ \int_c^d 2\pi g(y)\sqrt{1+\left(\dfrac{dx}{dy}\right)^2}\,dx.\end{cases}$$



Example 1： $y=\sqrt{r^2-x^2}$ ， $-r\le x\le r$; the *ax-ais* .

（球的表面積）

Solution：

$$y'=(-x)\left(r^2-x^2\right)^{-\frac{1}{2}} \quad\Rightarrow\quad 1+(y')^2=\frac{r^2}{r^2-x^2}$$

$$\begin{aligned}S&=\int_{-r}^{r}2\pi\sqrt{r^2-x^2}\,\frac{r}{\sqrt{r^2-x^2}}\,dx\\ &=2\pi r\int_{-r}^{r}dx\\ &=4\pi r^2.\end{aligned}$$

Example 2： $y=x^2$ ， $(1,1)$ *to* $(2,4)$; the *y-axis* .

Solution：

$$x=y^{\frac{1}{2}}$$

$$\Rightarrow\qquad x'=\frac{1}{2}y^{-\frac{1}{2}}$$

$$\Rightarrow\quad 1+(x')^2=\frac{1+4y}{4y}$$

$$S=\int_1^4 2\pi y^{\frac{1}{2}}\ \frac{\sqrt{1+4y}}{2\sqrt{y}}\ dy$$

$$=\pi\int_1^4 (1+4y)^{\frac{1}{2}}\ dy$$

$$=2\pi\frac{2}{3}\left(y+\frac{1}{4}\right)^{\frac{3}{2}}\Bigg|_1^4$$

$$=\frac{\pi}{6}\left(17\sqrt{17}-5\sqrt{5}\right).$$

註：另一解法為：

$$y'=2x\Rightarrow 1+\left(y'\right)^2=1+4x^2$$

$$S=\int_1^2 2\pi x\sqrt{1+4x^2}\ dx \qquad 1$$

$$=\frac{\pi}{6}\left(17\sqrt{17}-5\sqrt{5}\right).$$

Example 3： $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ； the *x-axis*.

（橢圓體的表面積）

Solution：

$$y=\frac{b}{a}\sqrt{a^2-x^2}$$

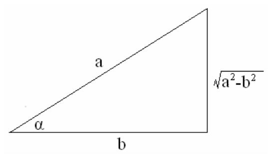
$$\Rightarrow \qquad y'=\frac{b}{a}(-x)\left(a^2-x^2\right)^{-\frac{1}{2}}$$

$$\Rightarrow 1+\left(y'\right)^2=\frac{b^2x^2+a^4-a^2x^2}{a^2\left(a^2-x^2\right)}.$$

$$S=2\int_0^a 2\pi\frac{b}{a}\sqrt{a^2-x^2}\ \frac{\sqrt{a^4-\left(a^2-b^2\right)x^2}}{a\sqrt{a^2-x^2}}\ dx$$

$$=\frac{4\pi b}{a^2}\int_0^a\sqrt{a^4-\left(a^2-b^2\right)x^2}\ dx$$

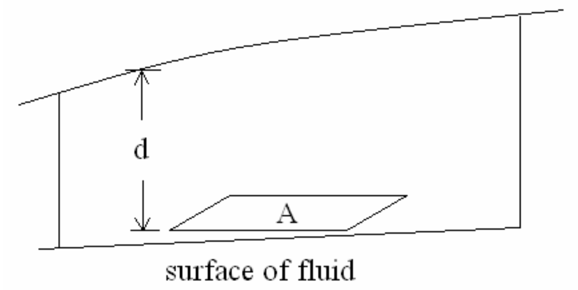
$$
\begin{aligned}
x &= \frac{a^2}{\sqrt{a^2-b^2}}\sin\theta \\
&= \frac{4\pi b}{a^2}\int_0^{\alpha} a^2\cos\theta \frac{a^2}{\sqrt{a^2-b^2}}\cos\theta d\theta \\
&= \frac{4\pi a^2 b}{\sqrt{a^2-b^2}}\int_0^{\alpha}\cos^2\theta d\theta \\
&= \frac{4\pi a^2 b}{\sqrt{a^2-b^2}}\left(\frac{\theta}{2}+\frac{\sin\theta\cos\theta}{2}\bigg|_0^{\alpha}\right). \\
&= \frac{4\pi a^2 b}{\sqrt{a^2-b^2}}\left(\frac{1}{2}\sin^{-1}\frac{\sqrt{a^2-b^2}}{a}+\frac{b\sqrt{a^2-b^2}}{2a^2}\right) \\
&= 2\pi\left[b^2+\frac{a^2 b}{\sqrt{a^2-b^2}}\sin^{-1}\frac{\sqrt{a^2-b^2}}{a}\right]
\end{aligned}
$$



§8-3 Applications to Physics and Engineering

Homework：1,11,27,34

(I) Hydrostatic Pressure and Force



$$p = \frac{F}{A} = \frac{mg}{A} = \frac{pAd}{A} = pgd.$$

F : force m : mass g : gravity acceleration.

ρ : density of a fluid in metric system.

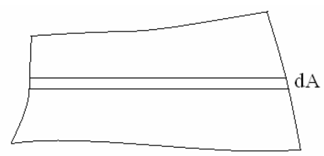
δ : density of a fluid in ft-pound system.

Water density : $1000 \, {kg}/{m^3}$ or $62.5 \, pd/ft^3$ （含重力）

(1) $P(pressure) = \begin{cases} \rho g d & (mctric\ system) \\ \delta d & (ft\text{-}pound\ system). \end{cases}$

Force againt a vertical plate or wall or dam.

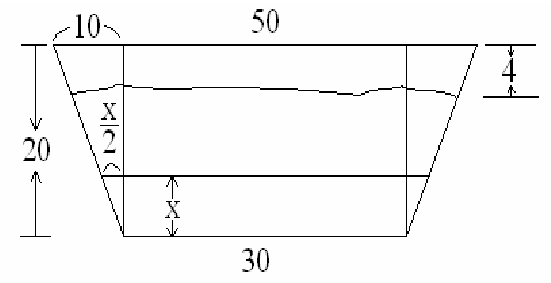
(2)



$F = \int p dA$ （ p, A 隨位置而變）

Example 1： A dam has the shape of the trapezoid. The height is 20m, and width is 50m at the top and 30m at the bottom. Find the force on the dam due to hydrostatic pressure if the water level is 4 m from the top of the dam.

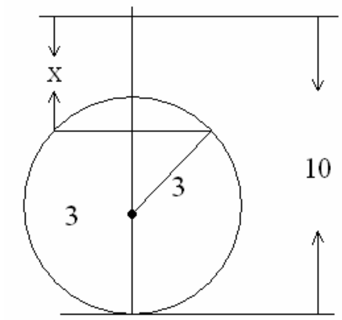
Solution：



$$F = \int_0^{16} pg(16-x)(30+x)\,dx$$

Example 2： Find the hydro static force on one end of a cylindrical drum with radius 3 ft if the drum is submerged in water 10 ft deep.

Solution：



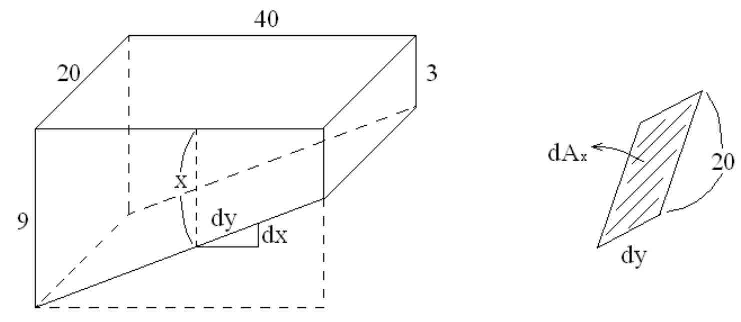
$$dA = 2\sqrt{9-(7-x)^2}\ dx$$

$$p = 2\int_4^{10} 62.5x\sqrt{9-(7-x)^2}\ dx$$

$$\underset{7-x=y}{=} \geq\int_{-3}^{3} 62.5(7-y)\sqrt{9-y^2}\ dy.$$

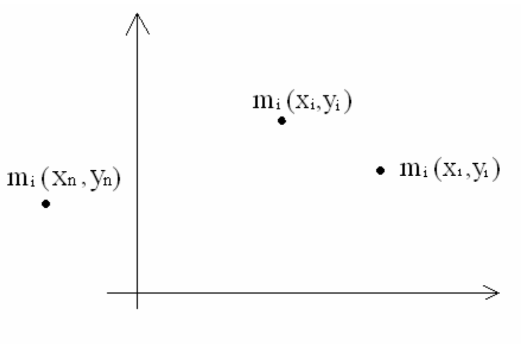
Example 3： Force againt an inclined bottom.

Solution：



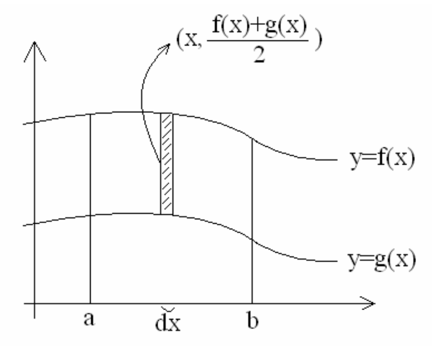
$$\int_3^9 pdA_x$$
$$= \int_3^9 \delta dA_x$$
$$= \int_3^9 \delta x 20 dy$$
$$\frac{dy}{dx} = \frac{\sqrt{1636}}{6} = \frac{\sqrt{409}}{3} \quad \Rightarrow \quad dy = \frac{\sqrt{409}}{3} dx$$
$$= \int_3^9 \frac{20\sqrt{409}}{3} \delta x dx.$$

Center $(\overline{x}, \overline{y})$ of a discrete system.



$$M_x = \sum_{i=1}^{n} m_i y_i$$
$$M_x = \sum_{i=1}^{n} m_i x_i$$
$$\Rightarrow \overline{x} = \frac{M_y}{\sum_{i=1}^{n} m_i}, \quad \overline{y} = \frac{M_x}{\sum_{i=1}^{n} m_i}$$

Center $(\overline{x}, \overline{y})$ of a continuous system with uniform density ρ.



$\left(x, \frac{f(x)+g(x)}{2}\right)$: center of infinitesimal strip at x (the shaped part on the right.).

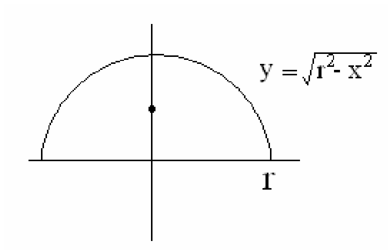
$$M_x = \int_a^b \frac{\left(f(x)+g(x)\right)}{2} \cdot \rho\left(f(x)+g(x)\right) = \rho\int_a^b \frac{\left(f^2(x)+g^2(x)\right)}{2} dx.$$
$$M_y = \int_a^b x \cdot \rho\left(f(x)+g(x)\right) = \rho\int_a^b x \cdot \left(f(x)-g(x)\right) dx.$$

$$\Rightarrow \quad \bar{y} = \frac{\frac{1}{2}\int_a^b (f^2(x) - g^2(x))dx}{\int_a^b (f(x) - g(x))dx}$$

$$\bar{x} = \frac{\int_a^b x(f(x) - g(x))dx}{\int_a^b (f(x) - g(x))dx}$$

Example 1： Find the center of a semicircular plate of radius r.

Solution：

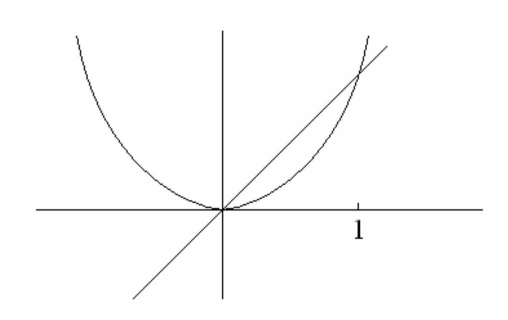


$$\Rightarrow \quad \bar{y} = \frac{\frac{1}{2}\int_{-r}^{r} (\sqrt{r^2 - x^2})^2 dx}{\frac{1}{2}\pi r^2} = \frac{\int_{-r}^{r} (r^2 - x^2)dx}{\pi r^2}$$

$$= \frac{1}{\pi r^2}\frac{1}{2}(r^2 x - \frac{x^3}{3}|_{-r}^{r}) = \frac{4r}{3\pi}$$

Example 2： Find the centroid of the region bounded by $y = x$ and $y = x^2$.

Solution：



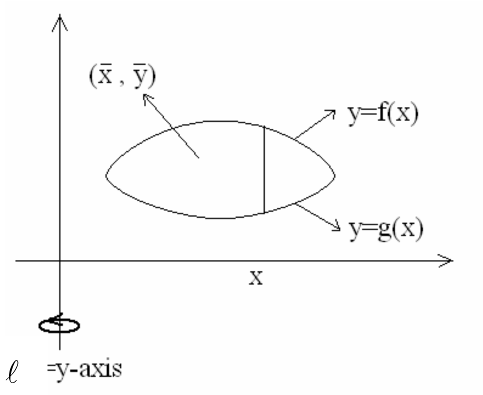
$$A = (x - x) \quad dx = \frac{1}{6}$$

$$M = \frac{1}{2}(x - x)\ dx = \frac{1}{15}$$

$$M = x(x - x) \quad dx = \frac{1}{12}$$

$$\Rightarrow \quad \bar{x} = \frac{1}{2} \quad , \quad \bar{y} = \frac{2}{5}.$$

Theorem of Pappu：



$$V=\int_a^b 2\pi x\left(f(x)-g(x)\right)dx$$

$$=2\pi\int_a^b x\left(f(x)-g(x)\right)dx$$

$$=2\pi\overline{x}\,A$$

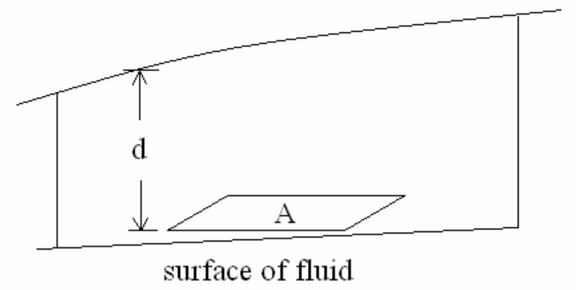
$$=\left(2\pi\overline{x}\right)A$$

=(distance traveled by of the region) × (the area of the region). 2

§8-3 Applications to Physics and Engineering

Homework：1,11,27,34

(I) Hydrostatic Pressure and Force



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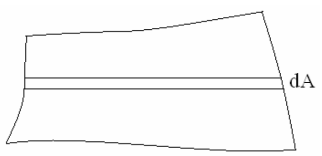
Force againt a vertical plate or wall or dam.

(2)



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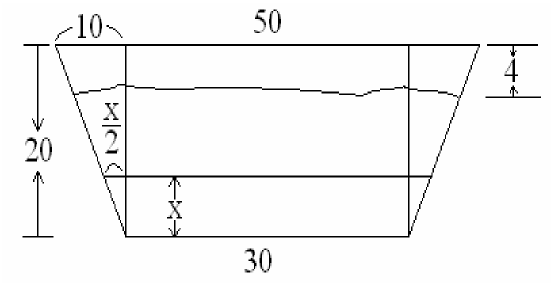

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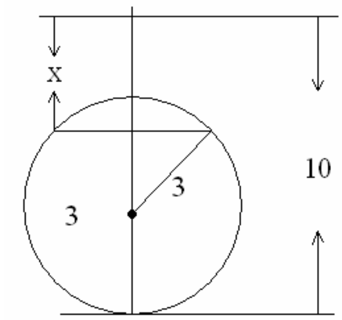
Example 2： Find the hydro static force on one end of a cylindrical drum with radius 3 ft if the drum is submerged in water 10 ft deep.

Solution：



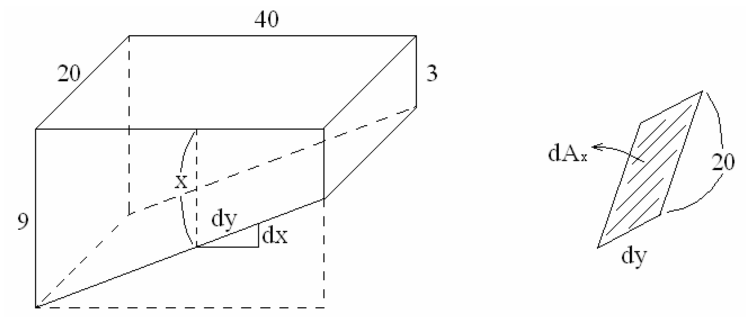
$$dA = 2\sqrt{9-(7-x)^2}\ dx$$

$$p \quad = 2\int_4^{10} 62.5x\sqrt{9-(7-x)^2}\ dx$$

$$\underline{\underline{7-x=y}}\ \geq \int_{-3}^{3} 62.5(7-y)\sqrt{9-y^2}\ dy.$$

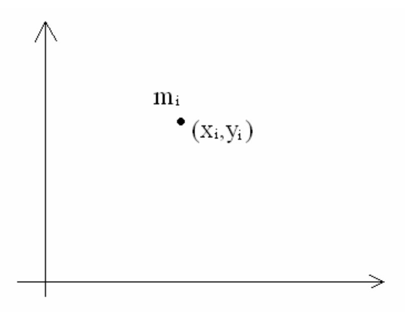
Example 3： Force againt an inclined bottom.

Solution：



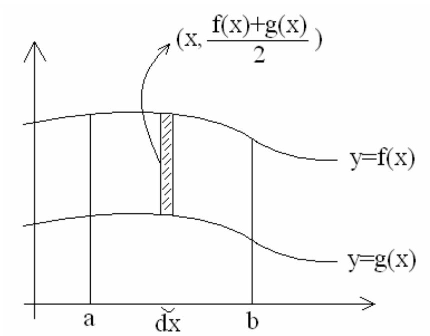
$$\int_3^9 p\,dA_x$$
$$= \int_3^9 \delta\, dA_x$$
$$= \int_3^9 \delta x 20 dy$$
$$\frac{dy}{dx} = \frac{\sqrt{1636}}{6} = \frac{\sqrt{409}}{3} \quad \Rightarrow \quad dy = \frac{\sqrt{409}}{3} dx$$
$$= \int_3^9 \frac{20\sqrt{409}}{3}\ \delta x dx.$$

Center $(\bar{x}, \bar{y})$ of a discrete system.



$$M_x = \sum_{i=1}^{n} m_i y_i$$
$$M_x = \sum_{i=1}^{n} m_i x_i$$
$$\Rightarrow \ \bar{x} = \frac{M_y}{\sum_{i=1}^{n} m_i}, \quad \bar{y} = \frac{M_x}{\sum_{i=1}^{n} m_i}$$

Center $(\bar{x}, \bar{y})$ of a continuous system with uniform density ρ.



$\left(x, \dfrac{f(x)+g(x)}{2}\right)$: center of infinitesimal strip at x (the shaped part on the right.).

$$M_x = \int_a^b \frac{\left(f(x)+g(x)\right)}{2} \cdot \rho\left(f(x)+g(x)\right) = \rho\int_a^b \frac{\left(f^2(x)+g^2(x)\right)}{2}\ dx.$$
$$M_y = \int_a^b x \cdot \rho\left(f(x)+g(x)\right) = \rho\int_a^b x \cdot \left(f(x)-g(x)\right) dx.$$

$$\Rightarrow \quad \bar{y} = \frac{\frac{1}{2}\int_a^b (f^2(x) - g^2(x))dx}{\int_a^b (f(x) - g(x))dx}$$

$$\bar{x} = \frac{\int_a^b x(f(x) - g(x))dx}{\int_a^b (f(x) - g(x))dx}$$

Example 1： Find the center of a semicircular plate of radius r .

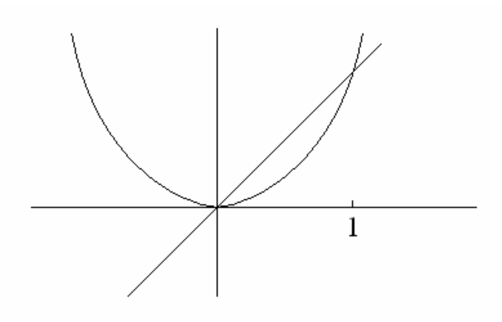
Solution：

$$\Rightarrow \quad \bar{y} = \frac{\frac{1}{2}\int_{-r}^{r} (\sqrt{r^2 - x^2})^2 dx}{\frac{1}{2}\pi r^2} = \frac{\int_{-r}^{r} (r^2 - x^2)dx}{\pi r^2}$$

$$= \frac{1}{\pi r^2}\frac{1}{2}(r^2 x - \frac{x^3}{3}|_{-r}^{r}) = \frac{4r}{3\pi}$$

Example 2： Find the centroid of the region bounded by $y = x$ and $y = x^2$.

Solution：



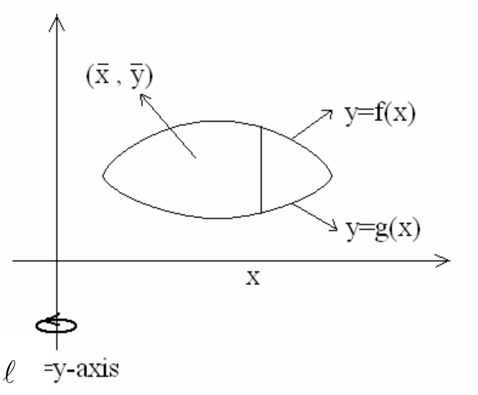
$$A = (x - x) \quad dx = \frac{1}{6}$$

$$M = \frac{1}{2}(x - x)\, dx = \frac{1}{15}$$

$$M = x(x - x) \quad dx = \frac{1}{12}$$

$$\Rightarrow \quad \bar{x} = \frac{1}{2} \quad , \quad \bar{y} = \frac{2}{5}.$$

Theorem of Pappu：



$$V = \int_a^b 2\pi x \left(f(x) - g(x)\right) dx$$

$$= 2\pi \int_a^b x \left(f(x) - g(x)\right) dx$$

$$= 2\pi \bar{x} A$$

$$= \left(2\pi \bar{x}\right) A$$

=(distance traveled by of the region) × (the area of the region).

§10.1 Curves Defined by Parametric Equations

Homework：12,24,28,39

Parameter Equations： $x = f(t)$, $y = g(t)$.

Parametric Curve：$\{ \ (x, y) : x = f(t) \ , \ y = f(t) \ \}$.

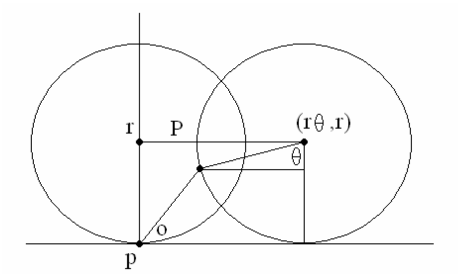
Example 1： $x = \cos t$, $y = \sin t$, $0 \le t \le 2\pi$

Solution：

Curve: the circle $x^2 + y^2 = 1$.

Example 2： The curve traced out by a point P on the circumference of a circle as the circle rolls along a straight line is called a cycloid. Find parametric equations for the cycloid.

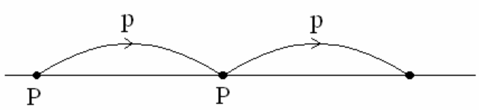
Solution：



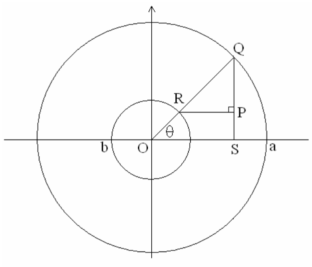
$p(x, y)$

$x = r\theta - r\sin\theta$

$y = r - r\cos\theta.$



Example 3：Find $P(x, y)$ in terms of Q. What is its parametric curve?



Solution：

$$x = \overline{SO} = a\cos\theta \text{ , } y = \overline{QS} - \overline{SP} = a\sin\theta - b\sin\theta$$

$$\Rightarrow \cos\theta = \frac{x}{a} \quad and \quad \sin\theta = \frac{y}{(a-b)}$$

$$\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{(a-b)^2} = 1.$$

§10.2 Calculus with Parametric Curves

Homework：3,15,25,31,39,51,53,59

$x = f(t)$, $y = g(t)$

1. Tangents：

   i. $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}$, $if \ \dfrac{dx}{dt} \neq 0$

   ii. $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{\frac{d}{dt}\left(\frac{dy}{dt}\right)}{\frac{dx}{dt}}$

2. Areas：

   $A = \int_a^b y\,dx = \int_\alpha^\beta g(t) f'(t)\,dt$

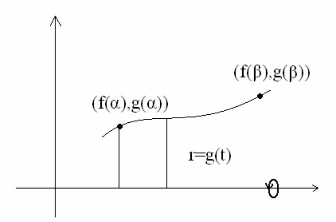
3. Arc Length：

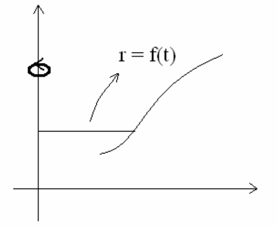
   $L = \int_a^b ds = \int_a^b \sqrt{(dx)^2 + (dy)^2} = \int_\alpha^\beta \sqrt{\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2}\,dt$

4. Surface Area：

$$S = \int_a^b 2\pi r\,ds$$

$$= \begin{cases} 2\pi \int_\alpha^\beta g(t) \sqrt{\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2}\,dt \\ \\ 2\pi \int_\alpha^\beta f(t) \sqrt{\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2}\,dt \end{cases}$$





Example 1： Given $x=t^2$, $y=t^3-3t$, $t\in R$. Find the equation(s) of the tangent(s) to the above curve at (3,0) .

Solution：

$$t^3-3t=0$$

$$\Rightarrow t=0\ (\text{不合})\ or\ \pm\sqrt{3}$$

$$\Rightarrow \frac{dy}{dx}=\frac{3t^2-3}{2t}=\frac{6}{2\left(\pm\sqrt{3}\right)}=\pm\sqrt{3}$$

$\Rightarrow$ Two tangents: $y=\sqrt{3}(x-3)$ and $y=-\sqrt{3}(x-3)$.

Example 2： Find the tangent to the cycloid $x=r(\theta-\sin\theta)$, $y=r(1-\cos\theta)$ at $\theta=\dfrac{\pi}{3}$.

Solution：

$$\frac{dy}{dx}=\frac{\sin\theta}{1-\cos\theta}\Bigg|_{\theta=\frac{\pi}{3}}=\sqrt{3}$$

$$x=r\left(\frac{\pi}{3}-\frac{\sqrt{3}}{2}\right)\ ,\ y=\frac{r}{2}$$

$\Rightarrow$ The tangent： $\sqrt{3}x-y=r\left(\dfrac{\pi}{\sqrt{3}}-2\right)$.

Example 3： Let $x=r(\theta-\sin\theta)$, $y=r(1-\cos\theta)$. Find $\dfrac{d^2y}{dx^2}$.

Solution：

$$\frac{dy}{dx}=\frac{\sin\theta}{1-\cos\theta}$$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$=\frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

$$= \frac{\cos\theta(1-\cos\theta)-\sin^2\theta}{r(1-\cos\theta)^3}$$

$$= \frac{\cos\theta - 1}{r(1-\cos\theta)^3}.$$

Example 4： Find the area and arc length under one arch of the cycloid

$$x = r(\theta - \sin\theta)\ ,\ y = r(1-\cos\theta). \quad (0 \le \theta \le 2\pi)$$

Solution：

$$A = \int_0^{2\pi} r^2 (1-\cos\theta)(1-\cos\theta)\, d\theta$$
$$= r^2 \int_0^{2\pi} 1 - 2\cos\theta + \cos^2\theta d\theta = 3\pi r^2$$
$$L = r\int_0^{2\pi} \sqrt{(1-\cos\theta)^2 + \sin^2\theta}\ d\theta$$
$$= r\int_0^{2\pi} \sqrt{2-2\cos\theta}\ d\theta$$
$$= 2r\int_0^{2\pi} \sin\frac{\theta}{2}\ d\theta$$
$$= 8r.$$

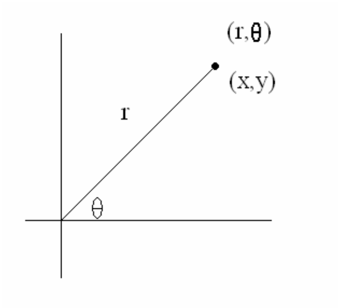
Example 5： Show that the surface area of a sphere of radius r is $4\pi r^2$.
Solution：

$$S = \int 2\pi r ds = 2\pi \int_0^{\pi} r^2 \sin\theta d\theta = 4\pi r^2.$$

§10.3 Polar Coordinates

Homework：5,13,17,25,39,55

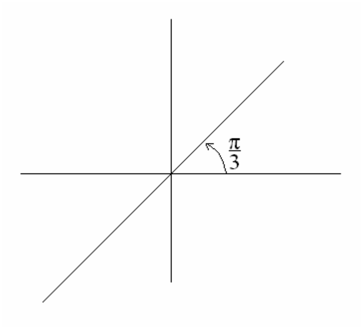
(I) 極座標



$(x,y)\leftrightarrow(r,\theta)$	
$x=r\cos\theta$ $y=r\sin\theta$	$r^2=x^2+y^2$ $\tan\theta=\dfrac{y}{x}$
i. r may be negative ii. 同一點 (r,θ) 的表法不唯一 $\left(-1,\dfrac{\pi}{4}\right)=\left(1,\dfrac{5\pi}{4}\right)=\left(1,\dfrac{13\pi}{4}\right)$	

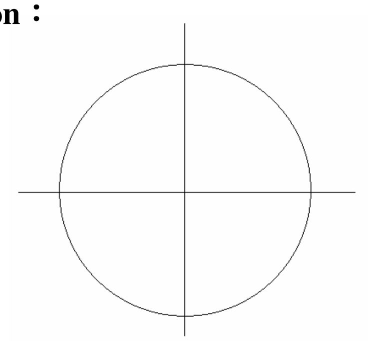
Example 1： $\theta=\dfrac{\pi}{3}$ 代表直線.

Solution：

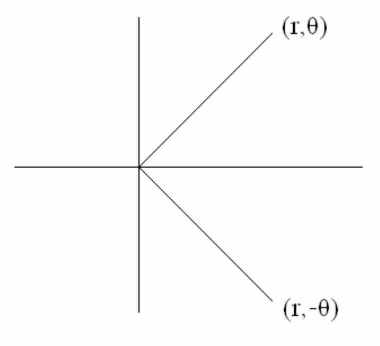
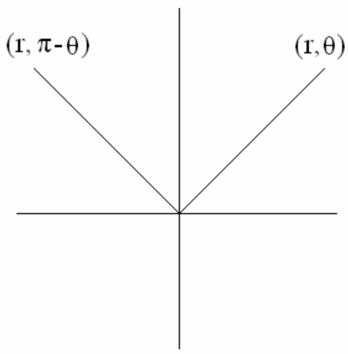
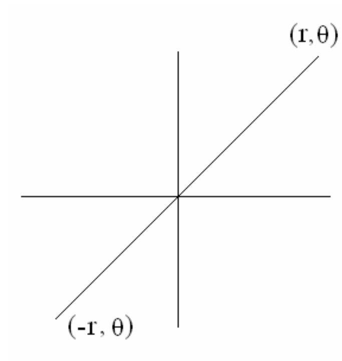


Example 2： $r=1$ 代表圓.

Solution：

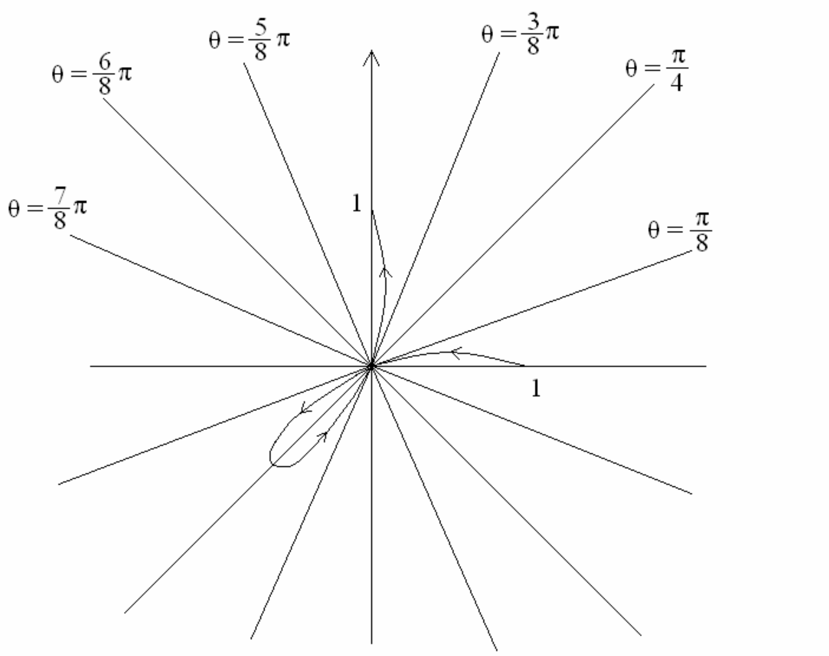


(II) Symmetry

對稱 x 軸	對稱 y 軸	對稱原點
θ 用 $-\theta$ 代 方程式不變	θ 用 $\pi-\theta$ 代 方程式不變	r 用 $-r$ 代 方程式不變
		

Example 3：Sketch $r = 2\cos 4\theta$

Solution： 對稱 x 軸和 y 軸



上題是 θ 從 $0 \to \frac{\pi}{8} \to \frac{2\pi}{8} \to \frac{3\pi}{8} \to \frac{4\pi}{8}$（也即在第一象限）的 r 的變化圖（依箭頭方向）。

將此箭頭圖 reflect on the x-axis and y-axis 我們得到一個 8-leaved rose 的圖。

(III) Tangents to Polar Curves

Polar Curve： $r = f(\theta)$

Its corresponding parametric equation： $x = f(\theta)\cos\theta$, $y = f(\theta)\sin\theta$.

Slope of the tangent： $\dfrac{dy}{dx} = \dfrac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \dfrac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$.

Remarks：

i. Horizontal tangents：points at which $\dfrac{dy}{d\theta} = \theta$.

ii. Vertical tangents: points at which $\dfrac{dx}{d\theta} = \theta$.

Example 4：

i. Sketch $r = 1 + \sin\theta$.

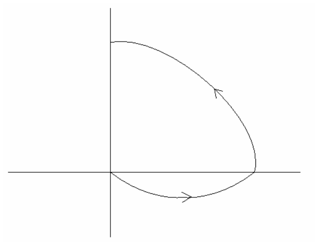
ii. Find $\left.\dfrac{dy}{dx}\right|_{\theta=\frac{\pi}{3}}$.

iii. Find the points on the curve $r = 1 + \sin\theta$ where the tangent line is horizontal or vertical.

Solution：

i. 對稱 y 軸，考慮 $-\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2}$ 的部份

θ	$-\frac{\pi}{2}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{2}$
r	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2



ii.

$$\frac{dy}{dx}=\frac{\dfrac{dy}{d\theta}\sin\theta+r\cos\theta}{\dfrac{dr}{d\theta}\cos\theta-r\sin\theta}=\frac{\cos\theta(1+2\sin\theta)}{(1+\sin\theta)(1-2\sin\theta)}$$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{3}}=-1.$$

iii.

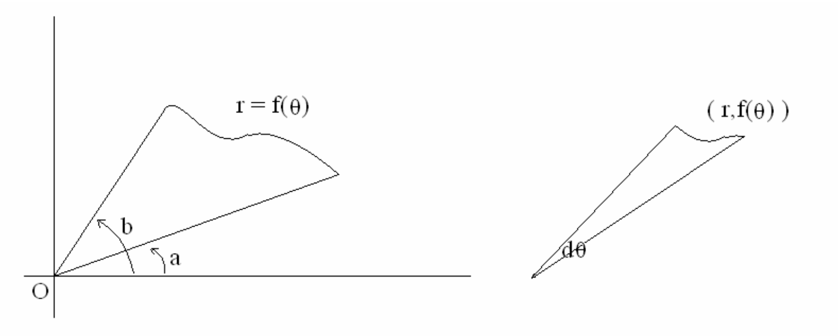
$$\frac{dy}{d\theta}=\cos\theta(1+2\sin\theta)=0 \quad \Rightarrow \quad \theta=\frac{\pi}{2}\ ,\ \frac{3\pi}{2}\ ,\ \frac{7\pi}{6}\ ,\ \frac{11\pi}{6}.$$

$$\frac{dx}{d\theta}=(1+\sin\theta)(1-2\sin\theta) \quad \Rightarrow \quad \theta=\frac{3\pi}{2}\ ,\ \frac{\pi}{6}\ ,\ \frac{5\pi}{6}.$$

§10-4 Areas and Lengths in Polar Coordinates

Homework：5,8,11,27,31,45,55

(I) Areas



$$A = \int_a^b \frac{1}{2} r^2 d\theta = \frac{1}{2}\int_a^b f^2(\theta)\, d\theta.$$

(II) Arc Length

$x = f(\theta)\cos\theta \qquad y = f(\theta)\sin\theta$

$$L = \int ds = \int \sqrt{(dx)^2 + (dy)^2} = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

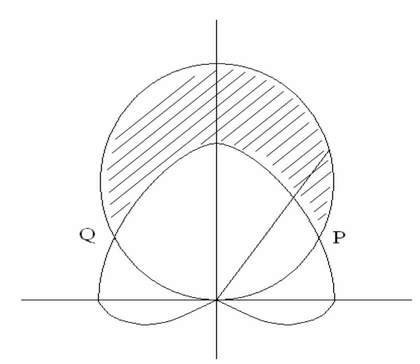
(III) Surface Area

$r = f(\theta)$, $a \le \theta \le b$, around the x-axis

$$S = \int 2\pi y\, ds = \int_a^b 2\pi \underbrace{f(\theta)\sin\theta}_{y} \underbrace{\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta}_{ds}.$$

Example 1：Find the area of the region that lies inside the circle $r = 3\sin\theta$ and outside the cardioid $r = 1 + 3\sin\theta$.

Solution：



$$\begin{cases} r = 3\sin\theta \\ r = 1 + \sin\theta \end{cases} \Rightarrow \sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6} \ or \ \frac{5\pi}{6}$$

$$A = \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left[(3\sin\theta)^2 - (1+\sin\theta)^2 \right] d\theta$$

$$= \pi.$$

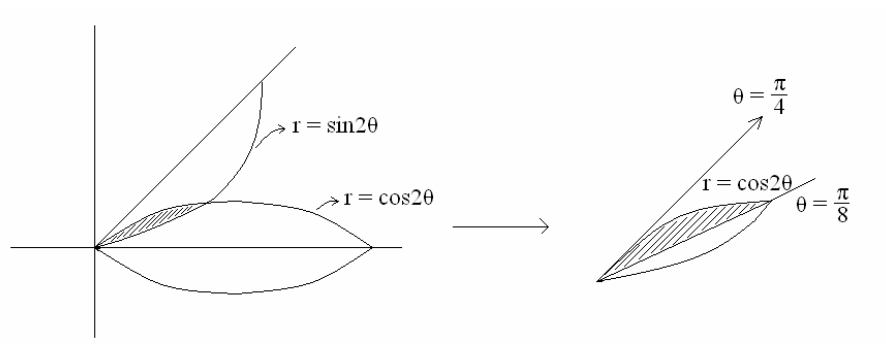
Example 2：Find the area of the region that lies inside both curves $r = \sin 2\theta$, $r = \cos 2\theta$.

Solution：

$r = \cos 2\theta$,4-leafed rose.

$$r = \sin 2\theta = \cos\left(\frac{\pi}{2} - 2\theta\right) = \cos 2\left(\frac{\pi}{4} - \theta\right) = \cos 2\left(\theta - \frac{\pi}{4}\right)$$

（將 $r = \cos 2\theta$, rotate $\frac{\pi}{4}$ 可得 $r = \sin 2\theta$ ）



$$\begin{cases} r = \sin 2\theta \\ r = \cos 2\theta \end{cases} \Rightarrow \tan 2\theta = 1 \Rightarrow \theta = \frac{\pi}{8}$$

$\Rightarrow$ 上圖斜線區域面積 $= \int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \frac{1}{2}\cos^2 2\theta d\theta$

$$= \int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \frac{1+\cos 4\theta}{4}\, d\theta$$

$$= -\left(\frac{\theta}{4} + \frac{\sin 4\theta}{16} \Bigg|_{\frac{\pi}{8}}^{\frac{\pi}{4}} \right)$$

$$= \left(\frac{\pi}{32} - \frac{1}{16} \right)$$

$$\Rightarrow \text{總面積} = 16\text{ 個斜線區域} = 16\left(\frac{\pi}{32} - \frac{1}{16} \right) = \frac{\pi}{2} - 1.$$

Example 3：Find the length of cardioid $r = 1 + \sin\theta$.

Solution：

$$L = 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{r^2 + \left(\frac{dr}{d\theta} \right)^2}\, d\theta$$

$$= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{(1+\sin\theta)^2 + \cos^2\theta}\, d\theta$$

$$= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{2+2\sin\theta}\, d\theta$$

$$= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{\frac{4-4\sin^2\theta}{2-2\sin\theta}}\, d\theta$$

$$= 4\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos\theta}{\sqrt{2-2\sin\theta}}\, d\theta \qquad u = 2-2\sin\theta \;\Rightarrow\; du = -2\cos\theta d\theta$$

$$= 2\int_0^4 \frac{du}{\sqrt{u}} = 4u^{\frac{1}{2}} \Big|_0^4 = 8.$$

§11.1 Sequences

Homework：13,19,32,33,51,61

Definition：

- A sequence is a function whose domain is N=the set of positive integers.
- In notation, we write $f(n)=a_n$.

Question：Convergence or divergence of $\{a_n\}$.

Example 1：

i. $a_n=\dfrac{1}{n}$ convergence

ii. $a_n=(-1)^n$ divergence

iii. $a_n=n$ divergence

Theorem1： $\lim\limits_{x\to\infty} f(x)=L$, $x\in R$ $\Rightarrow \lim\limits_{n\to\infty} f(n)=L$, $n\in N$.

(The converse is not true.)

Theorem2： $\lim\limits_{n\to\infty}|a_n|=0$, then $\lim\limits_{n\to\infty} a_n=0$

Theorem3： $\{r^n\}_{n=1}^{\infty}$ converges iff $-1<r\le 1$.

Example 2： $a_n=\dfrac{\ln n}{n}$

$$\lim_{n\to\infty}\frac{\ln n}{n}=\lim_{n\to\infty}\frac{\frac{1}{n}}{1}=0 \qquad \text{(X)}$$

$$\lim_{x\to\infty}\frac{\ln x}{x}=\lim_{x\to\infty}\frac{\frac{1}{x}}{1}=0+\text{Theorem1}\Rightarrow \lim_{n\to\infty}\frac{\ln n}{n}=0 \qquad \text{(O)}$$

註：

i. 在離散點上講微分是無意義的.

ii. $n^n >> n! >> e^n > n^3 > \ln n$ when n large.

Theorem4：Let $\{a_n\}$ be monotonic(單調) and bounded(有界) $\Rightarrow \{a_n\}$converges.

Example 3：Prove that $\sqrt{2+\sqrt{2+\sqrt{2+..}}}$ exists.

Proof：Let $a_1 = \sqrt{2}$, $a_{n+1} = \sqrt{2+a_n}$

(i) Clearly, a_n is monotonic increasing.

(ii) Moreover, $a_n \le 2$ for all $n \in N$.

We prove (ii) by induction.

(a) $a_1 \le 2$

(b) Suppose $a_n \le 2$, Then $a_{n+1} = \sqrt{2+a_n} \le \sqrt{2+2} = 2$

Using Theorem4, we see that $\lim_{n\to\infty} a_n$ has a limit, say x.

Then it must satisfy

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{2+a_n}$$

That is

$$x = \sqrt{2+x} \Rightarrow x = 2 \text{ or } -1(\text{不合})$$

§11.2 Series

Homework：10,17,33,39,43,45,46,49,57,62,63,65

Definition： Series $= \sum_{i=1}^{\infty} a_i = a_1 + a_2 + ... + a_n + ...$

Convergence/Divergence of a Series.

(nth) partial sum $= S_n = \sum_{i=1}^{n} a_i$

$\{S_n\}$ converges / diverges $\Rightarrow \sum_{i=1}^{\infty} a_i$ converges / diverges.

Theorem1：Geometric Series.

$\sum_{n=1}^{\infty} r^n$ converges $\Leftrightarrow$ $|r| < 1$ and $\sum_{n=1}^{\infty} r^n = \frac{r}{1-r}$.

Theorem2： $\sum_{n=1}^{\infty} a_n$ converges $\Rightarrow \lim_{n\to\infty} a_n = 0$.

($\lim_{n\to\infty} a_n \neq 0 \Rightarrow \sum_{n=1}^{\infty} a_n$ diverges.)

Remark： $\lim_{n\to\infty} \frac{1}{n} = 0$, but $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

$$\because \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \underbrace{\left(\frac{1}{3} + \frac{1}{4}\right)}_{>\frac{1}{2}} + \underbrace{\left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)}_{>\frac{1}{2}} + \underbrace{\left(\frac{1}{9} + ... + \frac{1}{16}\right)}_{>\frac{1}{2}} + ...$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

Example 1：

(i) $\sum_{n=1}^{\infty} (-1)^n \Rightarrow$ diverges.

(ii) $\sum_{n=1}^{\infty} \frac{\ln n}{n} \Rightarrow$ diverges. $\left(\because \frac{\ln n}{n} > \frac{1}{n} \text{ when } n \geq 3\right)$.

(iii) $\sum_{n=1}^{\infty} \frac{n}{n+1} \Rightarrow$ diverges $\left(\because \frac{n}{n+1} \to 1 \text{ as } n \to \infty\right)$.

Example 2： $\sum_{n=1}^{\infty} 2^{2n}3^{1-n} = \sum 4^n \cdot 3^{(-n)} \cdot 3 = \sum 3 \cdot \left(\frac{4}{3}\right)^n$

$r = \frac{4}{3} > 1 \Rightarrow$ divergence.

Example 3：

The Cantor Let , named after the German mathematician Georg Cantor (1845-1918), is constructed as follows. We start with the closed interval $[0,1]$ and remove the open interval $\left(\frac{1}{3}, \frac{2}{3}\right)$. That leaves the two intervals $\left(0, \frac{1}{3}\right)$ and $\left(\frac{2}{3}, 1\right)$, and we remove the open middle third of each. We continue this procedure indefinitely, at each step removing the open middle third of every interval that remains from the proceeding step. The Cantor set consists of the number that remain in $[0,1]$ after all those intervals have been removed.

(a) Show that the total length of all the intervals that are removed is 1.

(b) Give examples of some numbers in the Cantor set.

Proof：

(a)

$$S = \frac{1}{3} + 2\left(\frac{1}{3}\right)^2 + 2^2\left(\frac{1}{3}\right)^3 + \ldots + 2^n\left(\frac{1}{3}\right)^{n+1} + \ldots\ldots$$

$$= \frac{1}{3}\left(1 + \left(\frac{2}{3}\right) + \left(\frac{2}{3}\right)^2 + \ldots + \left(\frac{2}{3}\right)^n + \ldots\ldots\right)$$

$$= \frac{1}{3}\frac{1}{1-\frac{2}{3}}$$

$$= 1$$

(b) $\frac{1}{3}, \frac{2}{3}, \frac{1}{9}, \frac{2}{9}, \frac{7}{9}, \frac{8}{9}, \ldots\ldots$

Example 4：

The Sierpinski Carpet is a 2-dimensional counterpart of the Cantor set. We begin with a square of side 1. We remove the center one-ninth of a square. The process is then repeated just like the process done in Example3.

§11.3 The Integral Test and Estimates of Sums

Homework：1,4,5,7,12,15,21,25,29,33,38,39

- **Who：** That is what kind of the series can use the integral test to check its convergence?

a_i : (i) essentially positive
(ii)decreasing
(iii) $f(x) = a_x$ is continuous (11.3-1)

$\Downarrow$

- **What：** $\sum_{i=1}^{\infty} a_i$ and $\int_1^{\infty} a_x dx$ both converges or diverges.

Example 1： $\sum_{n=1}^{\infty} \frac{1}{n^p}$ (p-series).

Recall： $$\int_1^{\infty} \frac{1}{x^p}\, dx = \begin{cases} \text{convergence, } p > 1, \\ \text{divergence, } p \le 1. \end{cases}$$

$$\Rightarrow \text{Theorem1：} \sum_{n=1}^{\infty} \frac{1}{n^p} = \begin{cases} \text{convergence, } p > 1, \\ \text{divergence, } p \le 1. \end{cases}$$

Example 2： $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

Solution：

$$\int_1^{\infty} \frac{1}{x^2+1}\, dx = \tan^{-1} x \Big|_1^{\infty}$$
$$= \frac{\pi}{2} - \frac{\pi}{4}$$
$$= \frac{\pi}{4} \text{ (convergence)}$$

$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2+1}$ convergence.

Example 3： $\sum_{n=1}^{\infty} \frac{\ln n}{n}$

Solution：

$$\int_1^{\infty} \frac{\ln x}{x}\, dx = \int_0^{\infty} u du \text{ , } (\ln x = u).$$

$\Rightarrow$ divergence

$\Rightarrow \sum_{n=1}^{\infty} \frac{\ln n}{n}$ divergence.

Example 4： $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$

Solution：

$$\int_2^{\infty} \frac{1}{x(\ln x)^p}\, dx = \int_{\ln 2}^{\infty} \frac{du}{u^p} \qquad (u = \ln x)$$

$$\Rightarrow \begin{cases} p \le 1 \text{ , divergence} \\ p > 1 \text{ , convergence.} \end{cases}$$

Example 5： Discuss the convergence of $\sum_{n=1}^{\infty} b^{\ln n}$.

Solution：

$$\sum_{n=1}^{\infty} b^{\ln n} = \sum_{n=1}^{\infty} \left(e^{\ln b}\right)^{\ln n} = \sum_{n=1}^{\infty} \left(e^{\ln n}\right)^{\ln b} = \sum_{n=1}^{\infty} n^{\ln b} .$$

$$\Rightarrow \begin{cases} \ln b \ge -1 \ \Rightarrow\ b \ge e^{-1} \Rightarrow \text{divergence.} \\ \ln b < -1 \ \Rightarrow\ b < e^{-1} \Rightarrow \text{ convergence.} \end{cases}$$

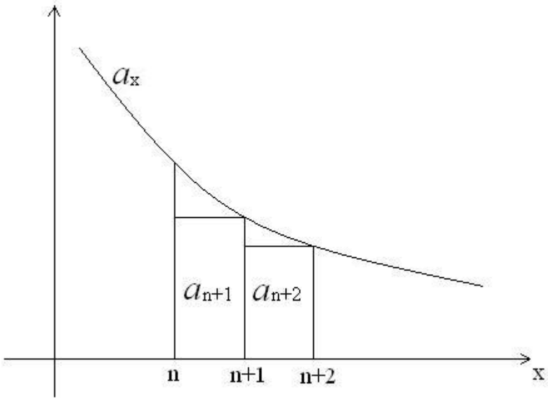
Theorem2： Let $\{a_i\}$ satisfy 11.3-1, and let $R_n = S - S_n$.

Here $S = \sum_{i=1}^{\infty} a_i$ and $S_n = \sum_{i=1}^{n} a_n$.

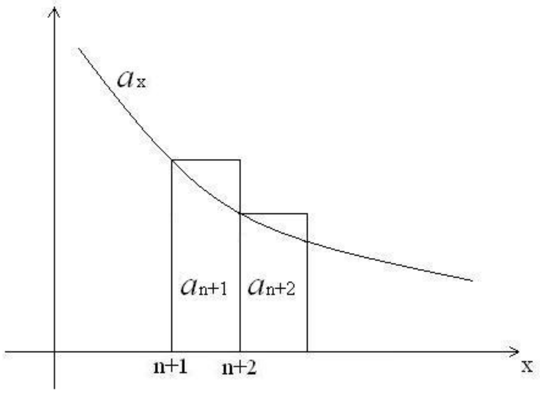
Then.

$$\int_{n+1}^{\infty} a_x dx \le R_n \le \int_n^{\infty} a_x dx$$

Proof：



（由上圖得 $R_n = a_{n+1} + a_{n+2} + \ldots \quad \leq \int_n^\infty a_x dx$）.



（由上圖得 $R_n \geq \int_{n+1}^\infty a_x dx$）.

Example 6：Use Theorem2. to estimate the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^3}$ with $n = 10$.

Solution：

$$\int_{11}^\infty \frac{1}{x^3} dx \leq S - S_{10} \leq \int_{10}^\infty \frac{1}{x^3} dx$$

$$\Rightarrow \frac{1}{2(11)^2} \leq S - S_{10} \leq \frac{1}{2(10)^2}$$

$$\Rightarrow S_{10} + \frac{1}{2(11)^2} \leq S \leq S_{10} + \frac{1}{2(10)^2}$$

$$S_{10} \approx 1.197532$$

$$\Rightarrow 1.201664 \leq \text{S} \leq 1.202532$$

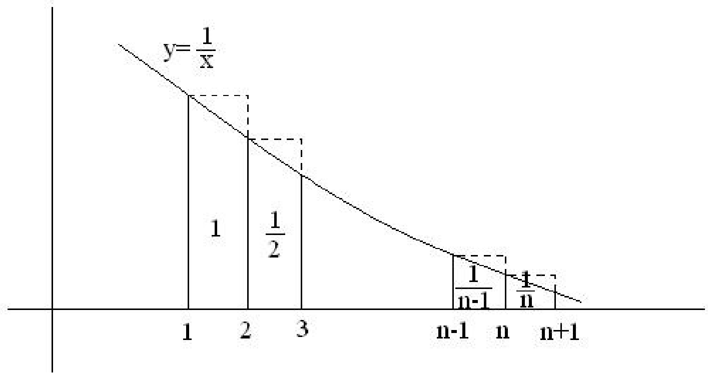
Example 7：Let $t_n = 1 + \frac{1}{2} + ... + \frac{1}{n} - \ln n.$

(a) Show that $t_n > 0$

(b) Show that $t_n - t_{n+1} > 0$

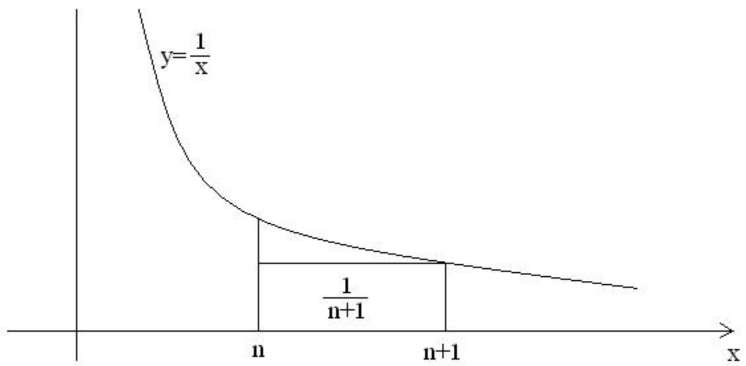
(c) Prove that $t_n \le 1$ for all n.

Proof：(a)



由上圖得知 $t_n > 0$

(b)



由上圖得知 $\ln(n+1) - \ln(n) > \frac{1}{n+1} > 0$

$\Rightarrow t_n - t_{n+1} > 0$

$\Rightarrow t_n$ is a decreasing sequence.

(c) 由(b)得，$t_1 \ge t_n$ for all n. 但$t_1 = 1$

$\Rightarrow 1 \ge t_n$ for all n.

§11.4 The Comparison Tests

Homework：3,9,15,19,23,27,31,35,39,40,41,43,45,46

Who： $\sum_{n=1}^{\infty} a_n$， $a_n > 0$ for all n (或從某項以後恆正或恆負).

即對象為 essentially positive or negative series.

What：（I）減法比較

(1) $a_n \ge b_n$ and $\sum a_n$ converges $\Rightarrow \sum b_n$ converges.

(2) $a_n \ge b_n$ and $\sum b_n$ diverges $\Rightarrow \sum a_n$ diverges.

（II）除法比較

(1) $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$（尾巴的速度相同）.

$\Rightarrow$ both $\sum a_n$ and $\sum b_n$ converge or diverge.

(2) $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ （a_n 跑到 0 的速度比較快）.

(a) $\sum b_n$ converges $\Rightarrow \sum a_n$ converges.

(b) $\sum a_n$ diverges $\Rightarrow \sum b_n$ diverges

註：若 $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$（$a_n$ 的尾巴速度比 b_n 快）.

可得類似結論.

Example 1：

	Series	Compared series	Conv./Div.
i.	$\sum \frac{1}{n^2 - n - 1}$	$\sum \frac{1}{n^2}$	Conv.
ii.	$\sum \frac{1}{\sqrt{n(n+1)(n+2)}}$	$\sum \frac{1}{n^{\frac{3}{2}}}$	Conv.

iii.	$\sum \frac{n^3}{2^n}$	$\sum \frac{1}{2^n}$	Conv.
iv.	$\sum \sin \frac{1}{n}$	$\sum \frac{1}{n}$	Div.
v.	$\sum \frac{n+5}{\sqrt[3]{n^7+n^2}}$	$\sum \frac{1}{n^{\frac{4}{3}}}$	Conv.
vi.	$\sum \frac{1}{n^{1+\frac{1}{n}}}$	$\sum \frac{1}{n}$	Div.
vii.	$\sum \frac{\ln n}{\sqrt{n}e^n}$	$\sum \frac{1}{e^n}$	Conv.

Example 2： If $\sum a_n$ converges, where $a_n > 0$, is it true that $\sum \sin a_n$ is also convergent?

Solution：

Yes.

For (i) $a_n \to 0$ as $n \to \infty$.

(ii) $\frac{\sin a_n}{a_n} \to 1$ as $n \to \infty$.

$$\Rightarrow \sum \sin a_n \text{ converges.}$$

Example 3： If $\sum a_n$, $\sum b_n$ are both convergent series with positive terms, is it true that $\sum a_n b_n$ is also convergent?

Solution：

Yes.

For $a_n \to 0$ and $b_n \to 0$.

and $a_n b_n$ approaches 0 even faster than $a_n\,(or\ b_n\ )$.

$\Rightarrow \sum a_n b_n$ converges.

§11.5 Alternating Series

Homework：3,7,13,17,23,27,35

Who：$\sum_{n=1}^{\infty}(-1)^{n+1}a_n = a_1+(-a_2)+a_3+(-a_4)+\ldots$,

$a_n > 0$ $(\text{or } a_n < 0)$ for all n.（交錯級數，正負相間）.

What：

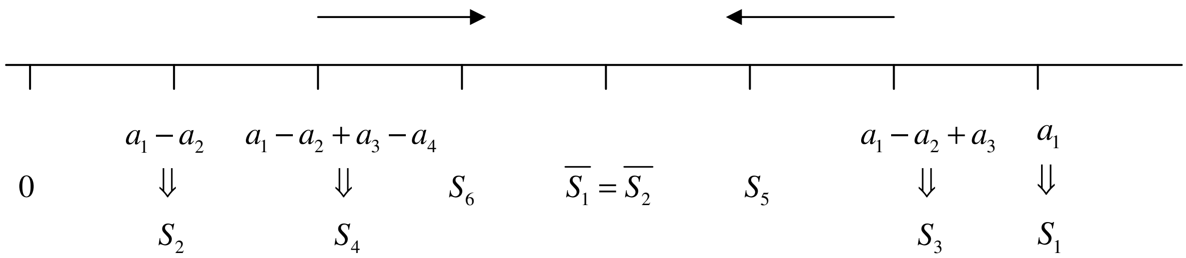
Theorem1：

(i) a_n is decreasing. (ii) $a_n \to 0$ as $n \to \infty$	(11.5-1)

$$\Rightarrow \sum_{n=1}^{\infty}(-1)^{n+1}a_n \text{ converges.}$$

Why：

(i) $a_1 > 0$



(ii) $S_{2n} \to \overline{S_2}$ and $S_{2n+1} \to \overline{S_1}$

$\Rightarrow S_{2n+1} - S_n = a_{2n+1} \to 0$

$\Rightarrow \overline{S_1} = \overline{S_2}$.

Example 1：

	Series	Conv./Div.
i.	$\sum_{n=1}^{\infty}(-1)^n \frac{1}{n}$	Conv.
ii.	$\sum_{n=1}^{\infty}\frac{(-1)^n 3n}{4n-1}$	Div. $\left(\lvert a_n\rvert \to \frac{3}{4} \neq 0\right)$.
iii.	$\sum_{n=1}^{\infty}(-1)^{n-1} \frac{\ln n}{n}$	Conv.
iv.	$\sum_{n=1}^{\infty}(-1)^n \cos\frac{\pi}{n}$	Div. $\left(\lvert a_n\rvert \to 1\right)$.
v.	$\sum_{n=1}^{\infty}(-1)^n \sin\frac{\pi}{n}$	Conv.
vi.	$\sum_{n=1}^{\infty}(-1)^{n-1} \frac{(\ln n)^p}{n}$	Conv.

$$\left(f(x)=\frac{(\ln x)^p}{x} \quad \Rightarrow \quad f'(x)=\frac{(\ln x)^{p-1}(p-\ln x)}{x^2}<0 \text{ , for large } x. \right)$$

Theorem2：(Error Estimate) $\sum_{i=1}^{\infty} a_i$ satisfies (11.5-1).

$$|R_n| = |S - S_n| \leq a_{n+1}$$

Proof： $|R_n| = |S - S_n| \leq |S_{n+1} - S_n| \leq a_{n+1}$

§11.6 Absolute Convergence and The Ratio and Root Tests

Homework：1,7,10,11,15,18,19,23,27,31,32,33,<u>40</u>

Definition：

i. $\sum_{n=1}^{\infty} a_n$ is said to be absolute convergence (AC)

if $\sum_{n=1}^{\infty} |a_n|$ is convergent.

ii. If $\sum_{n=1}^{\infty} a_n$ is convergent but not AC, then $\sum_{n=1}^{\infty} a_n$ is called conditional convergence(CC).

Example 1：

i. $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$: CC

ii. $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2}$: AC

Theorem1： AC $\Rightarrow$ convergence (反之不對).

Ratio and Root Tests (for AC)

<u>Who</u>：anyone

<u>What</u>：這 2 個方法的精神都是在檢查尾巴項的行為。且都以等比級數 $\sum_{n=1}^{\infty} a_0 r^n$ 作為比較的對象。

(i)Ratio Test : 比前後項縮小比例，即

- $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L$ (若為等比級數，L 即為公比。)
- 適用對象有 ! 項.

(ii) Root Test : • $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$

(若為等比級數，則 $\sqrt[n]{|a_0| r^n} = |a_0|^{\frac{1}{n}} r = r$,即 L 也為 r ,

即開nth 根號也為檢查尾巴項行為的方法之一).

- 適用對象有 $(\ )^n$ 項.

<u>結論</u>：

(i) $L<1 \Rightarrow$ AC

(ii) $L>1 \Rightarrow$ Divergence

(iii) $L=1 \Rightarrow$ The test fails. $\left(\sum_{n=1}^{\infty}\frac{1}{n}\ ,\ \sum_{n=1}^{\infty}\frac{1}{n^2}\right)$

(P-series 用此 2tests，2 都檢查不出來。)

註：$n^n \gg n! \gg e^n \gg n^2 \gg \ln$.

不同量級，重量級在分母，必 AC.

Example 2：Determine whether the series is AC , CC or divergent.

		Test	AC, CC or Divergent
i.	$\sum_{n=1}^{\infty}\frac{n^2}{2^n}$	$\sqrt{\ }$, $\frac{n^{\frac{2}{n}}}{2}=\frac{1}{2}$	AC
ii.	$\sum_{n=1}^{\infty}\left(\frac{n^2+1}{2n^2+1}\right)^n$	$\sqrt{\ }$, $\frac{1}{2}$	AC
iii.	$\sum_{n=1}^{\infty}\frac{(-3)^n}{n!}$	— , $\left\lvert\frac{3}{n+1}\right\rvert \to 0$	AC
iv.	$\sum_{n=1}^{\infty}(-1)^n\frac{2^n n!}{5.8......(3n+2)}$ Div.	— , $\left\lvert\frac{2(n+1)}{3(n+1)+2}\right\rvert \to \frac{2}{3}$	AC
v.	$\sum_{n=1}^{\infty}(-1)^n\frac{5^n n!}{5.8......(3n+2)}$	— , $\left\lvert\frac{5(n+1)}{3(n+1)+2}\right\rvert \to \frac{5}{3}$	Divergent
vi.	$\sum_{n=1}^{\infty}\frac{(-1)^n}{(\tan^{-1}n)^n}$	$\sqrt{\ }$, $\left\lvert\frac{1}{\tan^{-1}n}\right\rvert \to \frac{2}{\pi}$	AC
vii.	$\sum_{n=1}^{\infty}\frac{\tan^{-1}n}{n\sqrt{n}}$	$\approx \sum_{n=1}^{\infty}\frac{\frac{\pi}{2}}{n^{\frac{3}{2}}} \Rightarrow$	AC

註：P-級數 不需用（也不能用）此 2 tests.

Example 3：For which positive integers k is the following series convergent?

$$\sum_{n=1}^{\infty}\frac{(n!)^2}{(kn)!}$$

Solution：

$$\frac{\dfrac{((n+1)!)^2}{(k(n+1))!}}{\dfrac{(n!)^2}{(kn)!}}=\frac{(n+1)^2}{(kn+1)(kn+2)......(kn+k)}$$

$\Rightarrow$ 當 $k\geq 2$，$L<1$. (即收斂)

補充：

Sterling Formula.

$$\left(\frac{n}{e}\right)^n\sqrt{2\pi n}<n!<\left(\frac{n}{e}\right)^n\sqrt{2\pi n}\left(1+\frac{1}{4n}\right).$$

or

$$n!\sim\left(\frac{n}{e}\right)^n\sqrt{2\pi n}$$

Example 1：Determine the convergence / divergence of the series

$$\sum_{n=0}^{\infty}\frac{(n!)^2}{(2n)!}4^n=\sum_{n=0}^{\infty}a_n .$$

Solution：Using sterling formula, we have

$$a_n=\frac{(n!)^2}{(2n)!}4^n\sim\frac{\left[\left(\frac{n}{e}\right)^n\sqrt{2\pi n}\right]^2}{\left(\frac{2n}{e}\right)^{2n}\sqrt{2\pi(2n)}}=\frac{2\pi n}{\sqrt{4n\pi}}$$

$\not\to 0$ or $n\to\infty$

$\Rightarrow$ The series diverges

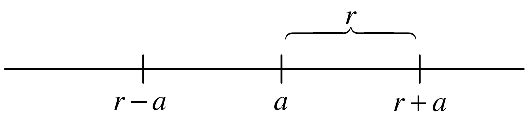
Example 2： $1.3.5\cdots(2n+1)=\dfrac{(2n+1)!}{2.4.6\cdots 2n}=\dfrac{(2n+1)!}{2^n n!}$

§11.8 Power Series

Homework：3,7,13,17,23,27,30,31,35,39,40

Definition： $\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots\ldots$ is called pourer series about a (or centered at a)(or in $(x-a)$)

Question： For what values of x do the corresponding series converge?

- 

  $\Rightarrow$ 愈靠近 a (中心)，愈可能收斂。

- 收斂半徑 (radius of convergence)
  收斂區間 (interval of convergence)

Example 1： $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$

Solution： Using ratio test, we get

$$\lim_{n\to\infty} \frac{\left|\frac{(x-3)^{n+1}}{n+1}\right|}{\left|\frac{(x-3)^n}{n}\right|} = \lim_{n\to\infty} \frac{n|x-3|}{n+1} = |x-3|.$$

If $|x-3|<1$, that is $x \in (2,4)$, the pourer series converges (absolutely).

- Check end points:

$x=2 \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \Rightarrow$ convergence.

$x=4 \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n} \Rightarrow$ divergence.

$\Rightarrow$ interval of convergence : $[2,4)$.

radius of convergence ： $r=1$.

Example 2： $\sum_{n=1}^{\infty}\frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$

Solution：

Ratio test $\Rightarrow \lim_{n\to\infty}\frac{\left|\frac{x^{2(n+1)}}{2^{2(n+1)}((n+1)!)^2}\right|}{\left|\frac{x^{2n}}{2^{2n}(n!)^2}\right|}$

$\Rightarrow \lim_{n\to\infty}\frac{x^2}{4(n+1)^2}=0$ for any $x\in R$.

$\Rightarrow$ interval of convergence: $(-\infty,\infty)$

$\Rightarrow$ radius of convergence: $r=\infty$.

註：重量級在分母 $(n!)^2 \gg x^{2n}$.

Example 3： $\sum_{n=1}^{\infty} n!x^n$

Solution：

Ratio test $\Rightarrow \lim_{n\to\infty}(n+1)x=$不存在 for any $x\in R$.

$\Rightarrow$ interval of convergence: $\{\,0\,\}$.

$\Rightarrow$ radius of convergence: $r=0$.

註：重量級 $(n!\gg x^n)$ 在分子.

Example 4： $\sum_{n=1}^{\infty}\frac{2^n(x-1)^n}{n}$

Solution：

Ratio test $\Rightarrow \lim_{n\to\infty}\left|\frac{2(x-1)n}{n+1}\right|=2|x-1|$ for any $x\in R$.

Let $2|x-1|<1 \Rightarrow |x-1|<\frac{1}{2} \Rightarrow x\in\left(\frac{1}{2},\frac{3}{2}\right)$.

- Check end points:

$x=\frac{1}{2} \Rightarrow \sum_{n=1}^{\infty}\frac{(-1)^n}{n} \Rightarrow$ convergence.

$x=\frac{3}{2} \Rightarrow \sum_{n=1}^{\infty}\frac{1}{n} \Rightarrow$ divergence.

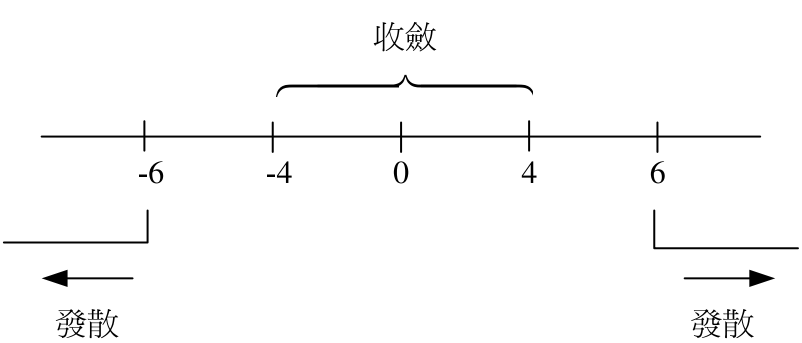
$\Rightarrow$ interval of convergence : $x\in\left[\frac{1}{2},\frac{3}{2}\right)$.

radius of convergence : $r=\frac{1}{2}$.

Example 5： If $\sum_{n=1}^{\infty}c_n x^n$ converges when $x=4$ and diverges when $x=6$, then determine the convergence of the following series.

		answers	reasons
i.	$\sum_{n=1}^{\infty}c_n$	convergence	$x=1$
ii.	$\sum_{n=1}^{\infty}c_n 8^n$	divergence	$x=8$
iii.	$\sum_{n=1}^{\infty}c_n(-3)^n$	convergence	$x=-3$
iv.	$\sum_{n=1}^{\infty}(-1)^n c_n 9^n$	divergence	$x=9$

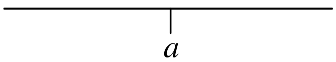
由已知可得下圖：



註：If $4 \le |x| \le 6$, then 無從判斷收斂性。

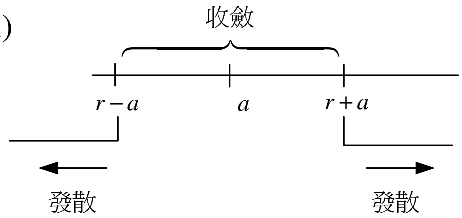
Theorem：

$\sum_{n=1}^{\infty} c_n \left(x-a\right)^n$, then either one of the following 3 possibilities holds：

(i) 

, $r = 0$(只有一點收斂)。

(ii)



, $r < \infty$(兩端點需被檢驗才可知其收斂與否)。

(iii) ____________________ , $r = \infty$(每一點皆收斂)。

§11.9 Representations of Functions as Power Series

Homework：1,7,9,11,13,15,29,37,38

- 源頭函數：$\dfrac{1}{1-x}=1+x+x^2+......=\sum_{n=0}^{\infty}x^n$, $|x|<1$.

(i) 直系函數：

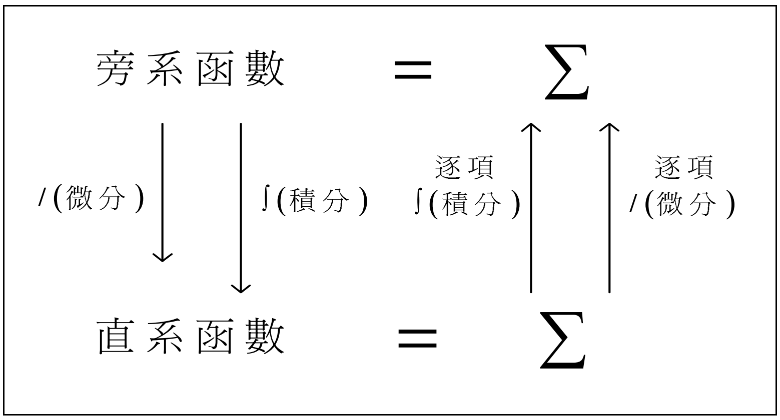
Example 1：$\dfrac{1}{1+x^2}=\dfrac{1}{1-\left(-x^2\right)}=\sum_{n=0}^{\infty}\left(-x^2\right)^n=\sum_{n=0}^{\infty}(-1)^n x^{2n}$, $\left|x^2\right|<1\Leftrightarrow|x|<1$.

Example 2：$\dfrac{1}{2-x}=\dfrac{1}{2}\dfrac{1}{1-\frac{x}{2}}=\dfrac{1}{2}\sum_{n=0}^{\infty}\left(\dfrac{x}{2}\right)^n=\sum_{n=0}^{\infty}\dfrac{x^n}{2^{n+1}}$, $|x|<2$.

Example 3：$\dfrac{x^3}{x+2}=\dfrac{x^3}{2}\dfrac{1}{1-\left(-\frac{x}{2}\right)}=\dfrac{x^3}{2}\sum_{n=0}^{\infty}\left(-\dfrac{x}{2}\right)^n=\sum_{n=0}^{\infty}(-1)^n\dfrac{x^{n+3}}{2^{n+1}}$, $|x|<2$.

(ii) 旁系函數：(related to 直系函數 by 微分和積分)

圖解：



註：

1. Power series 在絕對收斂的範圍(即收斂區間)可作逐項微分或積分的動作。
2. 有時需微分或積分不止一次才到直系。

Example 4：Find the pourer series representation of $\ln(1-x)$.

Solution：

$$\ln(1-x) = c - x - \frac{x^2}{2} - \frac{x^3}{3} \ldots\ldots = c - \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$$

$$\downarrow / \qquad\qquad \uparrow \int$$

$$\frac{-1}{1-x} \quad = -\sum_{n=0}^{\infty} x^n = -1 - x - x^2 \ldots\ldots \qquad |x| < 1$$

令 $x = 0 \quad \Rightarrow \quad \ln 1 = 0 = c$. So,

$$\ln(1-x) = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} \quad , \quad |x| < 1 \text{ .}$$

$$= -\sum_{n=1}^{\infty} \frac{x^n}{n} \quad , \quad |x| \text{ .}$$

Example 5： $f(x) = \tan^{-1} x$

Solution：

$$\tan^{-1} x = c + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \qquad |x| < 1 \text{ .}$$

$$\downarrow / \qquad \uparrow \int$$

$$\frac{1}{1+x^2} \quad = \sum_{n=0}^{\infty} (-1)^n x^{2n} \qquad |x| < 1$$

令 $x = 0 \quad \Rightarrow \quad \tan^{-1} 0 = 0 = c + 0 \quad \Rightarrow \quad c = 0$.

$$\Rightarrow \tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \quad , \quad |x| < 1 \text{ .}$$

Example 6： $f(x) = \dfrac{1}{(1-x)^2}$

Solution：

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1} \quad , \quad |x| < 1 \text{ .}$$

$$\downarrow \int \qquad \uparrow /$$

$$\frac{1}{1-x} \quad = \sum_{n=0}^{\infty} x^n \quad , \quad |x| < 1 \text{ .}$$

Example 7：Evaluate $\int \frac{1}{1+x^7}\,dx$ as a power series.

Solution：

$$\begin{aligned}
&\int \frac{1}{1+x^7}\,dx \\
&= \int \sum_{n=0}^{\infty} (-1)^n x^{7n} dx \\
&= \sum_{n=0}^{\infty} \int (-1)^n x^{7n} dx \\
&= \left(\sum_{n=0}^{\infty} (-1)^n \frac{x^{7n+1}}{7n+1} \right) + c \qquad |x| < 1 .
\end{aligned}$$

Example 8：(a) $\sum_{n=1}^{\infty} nx^{n-1} = ?\,, \quad |x| < 1.$

(b) (i) $\sum_{n=1}^{\infty} nx^{n} = ?\,, \quad |x| < 1.$

(ii) $\sum_{n=1}^{\infty} \frac{n}{2^n}.$

(c) (i) $\sum_{n=2}^{\infty} n(n-1)\,x^{n} = ?\,, \quad |x| < 1.$

(ii) $\sum_{n=2}^{\infty} \frac{n^2-n}{2^n} = ?$

(iii) $\sum_{n=1}^{\infty} \frac{n^2}{2^n}.$

Solution：

(8-a) For $|x| < 1$, we have

$$\sum_{n=1}^{\infty} nx^{n-1} = \sum_{n=1}^{\infty} \left(x^n\right)' = \left(\sum_{n=0}^{\infty} x^n \right)' = \left(\frac{1}{1-x} \right)' = \frac{1}{(1-x)^2} \quad .$$

(8-b-i) For $|x| < 1$, we have

$$\sum_{n=1}^{\infty} nx^{n} = x\sum_{n=1}^{\infty} nx^{n-1} = \frac{x}{(1-x)^2}$$ (此例子為旁系+直系).

(8-b-ii) $\sum_{n=1}^{\infty}\frac{n}{2^n}=\frac{\frac{1}{2}}{\left(1-\frac{1}{2}\right)^2}=2$.

(8-c-i) For $|x|<1$, we have $\sum_{n=2}^{\infty}n(n-1)x^n$

$$x^2\sum_{n=2}^{\infty}n(n-1)x^{n-2}=x^2\sum_{n=2}^{\infty}\left(nx^{n-1}\right)'=x^2\left(\sum_{n=1}^{\infty}nx^{n-1}\right)'=\frac{2x^2}{(1-x)^3}$$

(8-c-ii) $\sum_{n=2}^{\infty}\frac{n^2-n}{2^n}=\frac{2\left(\frac{1}{2}\right)^2}{\left(\frac{1}{2}\right)^3}=4$.

(8-c-iii) $\sum_{n=1}^{\infty}\frac{n^2}{2^n}=\sum_{n=1}^{\infty}\frac{n^2-n}{2^n}+\sum_{n=1}^{\infty}\frac{n}{2^n}=\sum_{n=2}^{\infty}\frac{n^2-n}{2^n}+\sum_{n=1}^{\infty}\frac{n}{2^n}=4+2=6$

§11.10 Taylor and Maclaurin Series

Homework：3,17,23,31,39,45,47,57,62

Questions：

(i) How to find a power series representation at a of a function f in general?
(ii)Will such power series representation equal to the function f itself?

General Method: If $f(x)$ has a pourer series representation at a, say ,

$$f(x)=\sum_{n=0}^{\infty} c_n (x-a)^n \ , \ |x-a|<R \ , \ \text{then} \quad \boxed{c_n = \frac{f^{(n)}(a)}{n!}.}$$

Definition：

(i) Taylor series of f at $x=a$ is defined to be $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$.

(ii) If $a=0$, the corresponding Taylor series is called Maclaurin series.

Remark： $f(x)$ is not necessarily equal to its Taylor series.

Example 1： $f(x)=\begin{cases} e^{-\frac{1}{x^2}} & \text{if} \quad x\neq 0. \\ 0 & \text{if} \quad x=0. \end{cases}$

Solution：

$$\Rightarrow f'(0)=f''(0)=......f^{(n)}(0)=0$$

$\Rightarrow f(x)$ 之 Maclaurin series$=0$ 但 $f(x)\neq 0$ whenever $x\neq 0$.

Example 2： Let $f(x)=e^x$. Find its Maclaurin series.

Solution：

$$\text{M.S.} = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!} . \left(\because f^{(n)}(x) = e^x\right).$$

Example 3： Let $f(x) = \sin x$. Find its Maclaurin series.

Solution：

$$\begin{aligned} f'(x) &= \cos x \\ f''(x) &= -\sin x \\ f^{(3)}(x) &= -\cos x \\ f^{(4)}(x) &= \sin x \end{aligned} \quad \Rightarrow \text{M.S.} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}.$$

Example 4： $f(x) = \cos x$

Solution：

$$\Rightarrow \text{M.S.} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2n!}.$$

誤差

↑

Taylor Inequality (誤差估計)： Let $f(x) = Tn(x) + Rn(x)$, where

$Tn = \sum_{n=0}^{n} \frac{f^{(i)}(0)}{i!}(x-a)^i$ is called the n th-degree Taylor polynomial

of f at a. If $\left|f^{(n+1)}(x)\right| \le M$ for $|x-a| \le d$, then

$$\left|R_n(x)\right| \le \frac{M}{(n+1)!}|x-a|^{n+1} \text{ for } |x-a| \le d$$

Example 5： Prove $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for any x.

Solution： $\left|R_n(x)\right| \le \frac{e^d}{(n+1)!}|x|^{n+1} \to 0$ as $n \to \infty$ for any x.

註：對任一 $x \in R$, $\sin x$ and $\cos x$ 的 Maclaurin series 分別等於 $\sin x$ and $\cos x$.

Example 6： $3+\frac{9}{2!}+\frac{27}{3!}+\frac{81}{4!}+......$

Solution：

$$3+\frac{9}{2!}+\frac{27}{3!}+\frac{81}{4!}+......$$
$$=e^3-1\ .$$

Example 7： $\frac{1}{2!}-\frac{1}{4!}+\frac{1}{6!}-\frac{1}{8!}+......$

Solution：

$$\frac{1}{2!}-\frac{1}{4!}+\frac{1}{6!}-\frac{1}{8!}+......$$
$$=1-\cos 1\ .$$

§11.11 The Binomial Series

Homework：3,7,11,15,17

牛頓推廣 binomial 的展開，即$(a+b)^k$, $k\in N$,的展開，推廣至$(a+b)^k$, $k\in R$.

Theorem1： If $k\in R$, $|x|<1$,then $(1+x)^k=\sum_{n=0}^{\infty}\binom{k}{n}x^n$,

$$\binom{k}{n}=\frac{(k)(k-1)...(k-n+1)}{n!}(n\geq 1) \text{ and } \binom{k}{0}=1.$$

Remark：目前我們學到 3 個方法得 Taylor series or power series representation.

(i) $\frac{1}{1-x}$ (ii) $c_n=\frac{f^n(a)}{n!}$ (iii) 多項式展開.

Example 1：

$$\begin{aligned}
f(x)&=\frac{1}{\sqrt{4-x}}\\
&=\frac{1}{2}\left(1-\frac{x}{4}\right)^{-\frac{1}{2}}\\
&=\frac{1}{2}\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}\left(-\frac{x}{4}\right)^n\\
&=\frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n 1.3......(2n-1)}{2^n\ n!}(-1)^n\frac{x^n}{2^{2n}}\\
&=\sum_{n=0}^{\infty}\frac{1.3......(2n-1)}{2^{3n+1}\ n!}x^n
\end{aligned}$$

$$\left(\binom{-\frac{1}{2}}{n}=\frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)......\left(-\frac{1}{2}-n+1\right)}{n!}\right)$$

Example 2： Let $f(x)=\left(1+x^2\right)^{\frac{1}{2}}$. Find $f^{(10)}(0)$.

Solution：

$$f(x)=\sum_{n=0}^{\infty}\binom{\frac{1}{2}}{n}\left(x^2\right)^n \quad \text{(多項式展開)}$$

$$=\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}x^k \quad \text{(Maclaurin Series).}$$

$$\Rightarrow k=10 \quad \Rightarrow \quad n=5.$$

$$\Rightarrow \frac{f^{(10)}(0)}{10!}=\binom{\frac{1}{2}}{5}=\frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right)}{5!}=\frac{7}{2^8}$$

$$\Rightarrow f^{(10)}(0)=\frac{7}{2^8}\ 10!$$

Example 3： Let $f(x)=\left(1+x^3\right)^{-\frac{1}{2}}$. Find $f^{(9)}(0)$.

Solution：

$$f(x)=\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}\left(x^3\right)^n$$

$$=\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}x^k$$

$$\Rightarrow k=9 \quad \Rightarrow \quad n=3.$$

$$\Rightarrow \frac{f^{(9)}(0)}{9!}=\binom{-\frac{1}{2}}{3}=\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}=-\frac{5}{2^4}$$

$$\Rightarrow f^{(9)}(0)=\frac{5}{2^4}\ 9!$$

§12 Vectors and the Geometry of Space

§12.1 Three Dimensional Coordinate Systems

Homework： 13,27,33,42

- Coordinate：
- Distance Formula：
- Equation of a Sphere：

§12.2 Vectors

Homework： 25,29,40

Vectors：

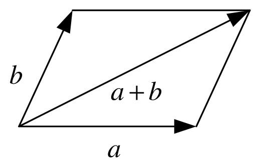
(1) 幾何(物理)觀點：有長度、方向的量

(2) 代數觀點：$\overrightarrow{op}=\langle a,b,c\rangle \quad P=(a,b,c)$.

長度$=\left|\overrightarrow{op}\right|=\sqrt{a^2+b^2+c^2}$

(3) 運算方式

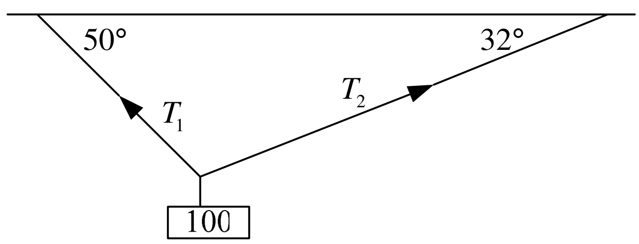
幾何：



代數：$a=<a_1,a_2,a_3>$ ， $b=<b_1,b_2,b_3>$

$\Rightarrow\ a\pm b=<a_1\pm b_1\ ,\ a_2\pm b_2\ ,\ a_3\pm b_3>$.

Example 1： A 100-*lb* weights hangs from two wires as shown below. Find the tension (forces) T_1 and T_2 in both wires and their magnitudes.



Solution：

$$|T_1|\sin 50^0 + |T_2|\sin 32^0 = 0$$
$$|T_1|\cos 50^0 = |T_2|\cos 32^0$$

$$\Rightarrow T_1 = \frac{100}{\sin 50^0 + \tan 32^0 \cos 50^0} \approx 85.64lb$$
$$T_2 = \frac{|T_1|\cos 50^0}{\cos 32^0} \approx 64.91lb$$

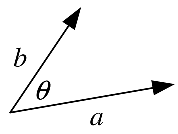
$$|T_1| = \left(|T_1|\sin 50^0\right) j - \left(|T_1|\cos 50^0\right) i \approx -55.05i + 65.60j$$
$$|T_2| = \left(|T_2|\sin 32^0\right) j + \left(|T_2|\cos 32^0\right) i \approx 55.05i + 34.40j$$

§12.3 The Dot Product

Homework： 11,17,19,23,25,27,31,37,39,51

Dot product (a scalar) Let $a=<a_1,a_2,a_3>$, $b=<b_1,b_2,b_3>$.

(1) 代數意義：$a\cdot b=a_1b_1+a_2b_2+a_3b_3$

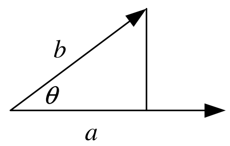


幾何意義：$a\cdot b=|a||b|\cos\theta$

(2) $a\perp b \Leftrightarrow a\cdot b=0$.

(3) $\cos\theta=\dfrac{a\cdot b}{|a||b|}$.

(4) Scalar projection of b onto $a=|b|\cos\theta=\dfrac{a\cdot b}{|a|}$.



(5) Vector project projection of b onto $a=\dfrac{a\cdot b}{|a|}\dfrac{a}{|a|}=\left(\dfrac{a\cdot b}{|a|^2}\right)a$.

(6) $a=\alpha b$ for some $\alpha\in R \Leftrightarrow a//b$

Example 1： Find the scalar and vector projection of $b=<1,1,2>$ onto $a=<-2,3,-1>$.

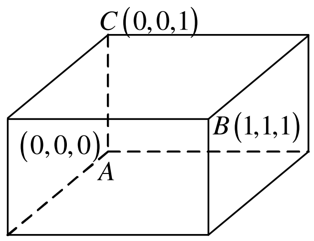
Solution：

$$\frac{a\cdot b}{|a|}=\frac{-2+3-2}{\sqrt{14}}=\frac{-1}{\sqrt{14}}$$

$$\frac{a\cdot b}{|a|}\frac{a}{|a|}=-\frac{1}{\sqrt{14}}\frac{<-2,3,-1>}{\sqrt{14}}=\left\langle\frac{1}{7},\frac{-3}{14},\frac{1}{14}\right\rangle$$.

Example 2： Find the angle between a diagonal of a cube and one of its edges.

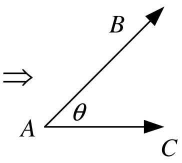
Solution：



$\overrightarrow{AC} = <0,0,1>$

$\overrightarrow{AB} = <1,1,1>$

$\Rightarrow$



$\Rightarrow \cos\theta = \dfrac{1}{\sqrt{3}} \quad \Rightarrow \quad \theta = \cos^{-1}\dfrac{1}{\sqrt{3}}$.

§12.4 The Cross Product

Homework： 5,9,15,23,27,31,34

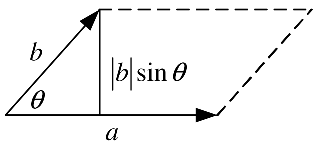
Cross Product (a vector) $a \times b$, $a = < a_1, a_2, a_3 >$, $b = < b_1, b_2, b_3 >$.

(1)

(i) 代數意義：$a \times b = \left(\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix}, \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right) = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

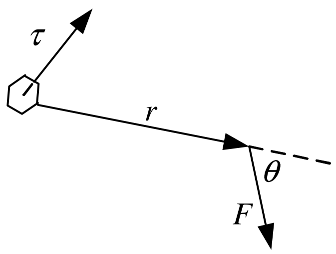
$i = < 1,0,0 >$, $j = < 0,1,0 >$, $k = < 0,0,1 >$

(ii)幾何意義：$a \times b = \begin{cases} \text{大小} : |a||b|\sin\theta . \\ \text{方向} : \text{right hand rule} . \end{cases}$



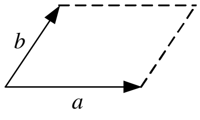
$\Rightarrow |a \times b| = |a||b|\sin\theta =$ the parallelogram determined by a and b .

(iii)物理意義：a force F acting on a rigid body at a point given by a position vector r. Then the torque τ is defined to be $\tau = r \times F$.



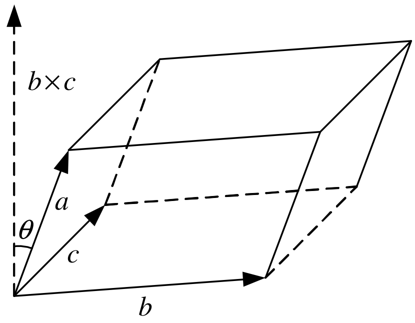
(2) $a \times b = 0 \Leftrightarrow a // b$.

(3) $a \times b \perp a$, $a \times b \perp b$.

(4) Area of  $= |a||b|\sin\theta = |a \times b|$.

(5)

Volume of



長度 ↓ 絕對值 ↓

$$=\Big|\,|a\cdot(b\times c)|\,\Big|$$

$$=\underbrace{|a||\cos\theta|}_{\text{高}}\ \underbrace{|b\times c|}_{\text{底面積}}$$

(6)

(i) $a\times b=-b\times a$.

(ii) $a\times(b+c)=a\times b+a\times c$.

(iii) $a\cdot(b\times c)=(a\times b)\cdot c$

(iv) $a\times(b\times c)=(a\cdot c)b-(a\cdot b)c\neq(a\times b)\times c$.

$i\times(i\times j)=i\times k=-j$.

$(i\times i)\times j=0\times j=0$.

Example 1： Find the area of ΔPQR,

where $P(1,4,6), Q(-2,5,-1)$ and $R(1,-1,1)$.

Solution：

$\overrightarrow{PQ}=<-3,1,-7>$, $\overrightarrow{PR}=<0,-5,-5>$

$\overrightarrow{PQ}\times\overrightarrow{PR}=<-40,-15,15>$

$$=\frac{1}{2}\left|\overrightarrow{PQ}\times\overrightarrow{PQ}\right|=\frac{5}{2}\sqrt{81}$$

Example 2： Are $a=<1,4,-7>$, $b=<2,-1,4>$ and $c=<0,-9,18>$ coplanar?

Solution：

$<1,4,-7>\cdot(<2,-1,4>\times<0,-9,18>)$

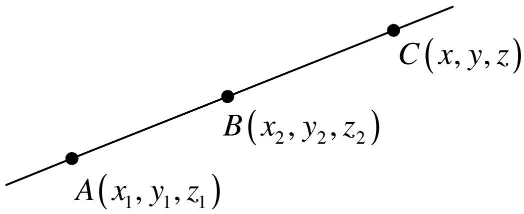
$=<1,4,-7>\cdot<18,-36,-18>=0$

$\Rightarrow$ Coplanar (共面).

§12.5 Equations of Lines and Planes

Homework： 1,5,11,15,19,37,41,49,59,63,67

(1) Equation of lines： Given A and B, find equation of the line AB.



$\overrightarrow{AB} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$, $\overrightarrow{AC} = \langle x - x_1, y - y_1, z - z_1 \rangle$

$=: \langle a, b, c \rangle$

$\overrightarrow{AB} // \overrightarrow{AC} \Rightarrow \exists\, t \in R$ s.t. $\overrightarrow{AC} = t\overrightarrow{AB}$.

$\Rightarrow$

(i) Parametric equation

$$\begin{cases} x = x_1 + t(x_2 - x_1) \\ y = y_1 + t(y_2 - y_1) \\ z = z_1 + t(z_2 - z_1) \end{cases} , \quad t \in R .$$

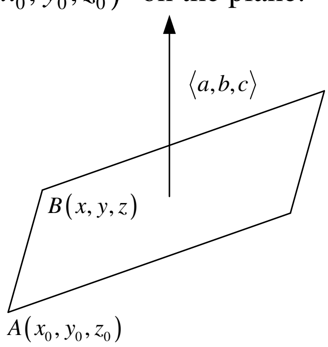
(ii) Symmetric equation

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} , \quad abc \neq 0$$

(If $a = 0$, $bc \neq 0 \Rightarrow x = x_1$, $\frac{y - y_1}{b} = \frac{z - z_1}{c}$)

Remark： $\langle a, b, c \rangle$ is called the direction of the line AB.

(2) Equation of the Plane： Given the normal direction of the plane $\langle a, b, c \rangle$ and a point (x_0, y_0, z_0) on the plane.

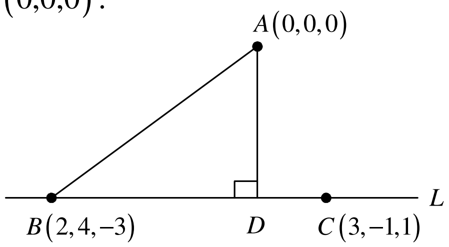


$\Rightarrow$ the equation of the plane：

(i) $\langle a,b,c\rangle \langle x-x_0, y-y_0, z-z_0\rangle = 0$

(ii) $ax+by+cz = ax_0+by_0+cz_0 \,(=d)$

Example 1： Line L through $(2,4,-3)$, $(3,-1,1)$. Find the distance between L and $(0,0,0)$.



Solution：

$\overrightarrow{BA} = \langle -2,-4,3\rangle$ and $\overrightarrow{BC} = \langle 1,-5,-4\rangle$

Vector projection of $\overrightarrow{BA}$ on $\overrightarrow{BC}$.

$$= \overrightarrow{BD} = \left(\frac{\overrightarrow{BA}\cdot\overrightarrow{BC}}{\left|\overrightarrow{BC}\right|}\right)\frac{\overrightarrow{BC}}{\left|\overrightarrow{BC}\right|} = \left\langle \frac{1}{7}, \frac{-5}{7}, \frac{-4}{7}\right\rangle$$

$$\overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{BD} = \langle 2,4,-3\rangle + \left\langle \frac{1}{7}, \frac{-5}{7}, \frac{-4}{7}\right\rangle$$

$$= \left\langle \frac{15}{7}, \frac{23}{7}, \frac{-25}{7}\right\rangle$$

The distance $= \left|\overrightarrow{AD}\right|$.

Example 2： Find the equation of the plane through $(1,3,2),(3,-1,6)$ and $(5,2,0)$.

Solution：

$$法向量 = <2,-4,4> \times <4,-1,-2>$$
$$= <12,20,14>$$
$$= 2<6,10,7>$$
$$<6,10,7> \cdot <x-5\ ,\ y-2\ ,\ z-0> = 0$$
$$\Rightarrow 6x+10y+7z = 50.$$

Example 3：

(i) Find the angles between two planes $P_1 : x + y + z = 1$ and $P_2 : x - 2y + 3z = 1$.

(ii) Let $l = P_1 \cap P_2$. Find the equation of l.

Solution：

(i) 平面夾角=法向量夾角

$$\Rightarrow \cos\theta = \frac{<1,1,1>\cdot<1,-2,3>}{\sqrt{3}\ \sqrt{14}} = \frac{2}{\sqrt{42}} = \frac{\sqrt{42}}{21}$$

$$\Rightarrow \theta = \cos^{-1}\frac{\sqrt{42}}{21}$$

(ii) l 的方向和 P_1 ， P_2 平面的法向量垂直

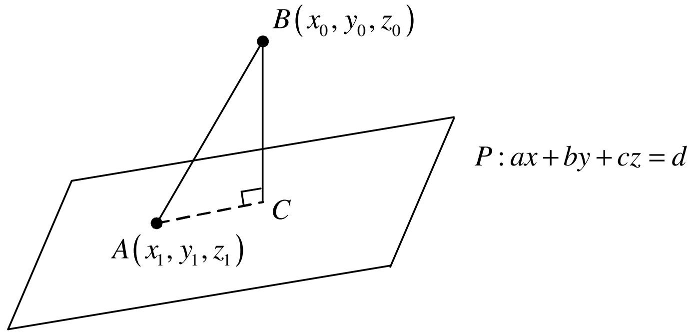
$\Rightarrow <1,1,1>\times<1,-2,3> = <5,-2,-3> =$ 直線 L 的方向.

$\begin{cases} x + y + z = 1 \\ x - 2y + 3z = 0 \end{cases}$ $\Rightarrow$ 令 $z = 0$ $\Rightarrow$ $y = \frac{1}{3}$ and $x = \frac{2}{3}$.

$$\Rightarrow \frac{x - \frac{2}{3}}{5} = \frac{y - \frac{1}{3}}{-2} = \frac{z}{-3} \quad .$$

Example 4： Find the distance between two parallel planes $10x + 2y - 2z = 5$ and $5x + y - z = 1$.

Solution：



$$\overrightarrow{BC} = \left| \text{ Scalar projection of } \overrightarrow{BA} \text{ on } \overrightarrow{BC} \right|.$$

$$= \left| \frac{\overrightarrow{BA}\cdot\overrightarrow{BC}}{\overrightarrow{BC}} \right|$$

$$= \left| \frac{ax_0 + by_0 + cz_0 - d}{\sqrt{a^2 + b^2 + c^2}} \right|$$

＝點到面之距離公式 .

$\left(\overrightarrow{BA} = \langle\ x_1 - x_0\ ,\ y_1 - y_0\ ,\ z_1 - z_0\ \rangle\ ,\ \overrightarrow{BC} = \langle a, b, c\rangle \right)$

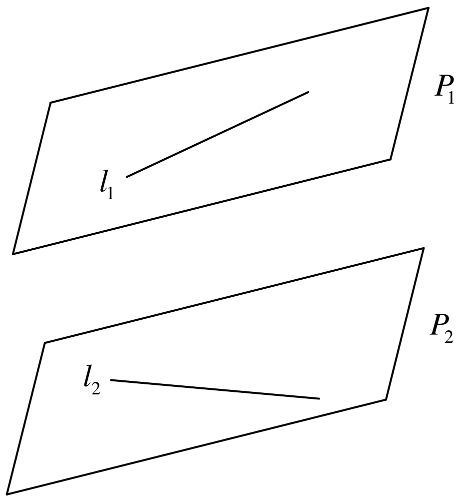
Now, 令 $x=0$, $y=\frac{5}{2}$, $z=0$,

則＝平行平面之距離＝ $\dfrac{\frac{5}{2}}{\sqrt{27}}=\dfrac{5\sqrt{3}}{18}$.

Example 5： Find the distance between two skew lines

$$l_1:\begin{cases} x=1+t \\ y=-2+3t \\ z=4-t \end{cases},\ t\in R\ ,\quad l_2:\begin{cases} x=2s \\ y=3+s \\ z=-3+4s \end{cases},\ s\in R\ .$$

Solution：



找 $P_1 // P_2$ 且 $l_1 \in P_1$, $l_2 \in P_2$.

$\Rightarrow P_1$ and P_2 的法向量 $n \perp l_1$, l_2 .

$\Rightarrow n=\langle 1,3,1\rangle \times \langle 1,1,4\rangle = \langle 11,-3,-2\rangle$

將 P_1 , P_2 方程式找出

$\Rightarrow d(l_1,l_2)=d(P_1,P_2)$.

§12.6 Cylinders and Quadrate Surfaces

Homework： 9,21-28,29,31,35,43,46,48

(1) Cylinders： A surface consisting of all lines (called rulings) that are // to a given line and pass through a given plane curve.

Example 1： (i) $z = x^2$ (Parabolic cylinder)
(ii) $x^2 + y^2 = 1$

Remark： If one of the variables x , y or z is missing from the equation of a surface , then the surface is a cylinder.

(2) Quadratic Surfaces：

$ax^2 + by^2 + cz^2 + dxy + eyz + fxz + gx + hy + iz + j = 0$

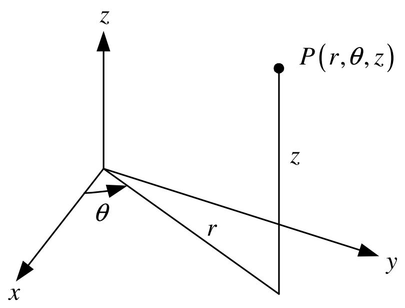
See P.836 of the test book for the classification of quadratic surface.

(3) Cross-sections or traces

§12.7 Cylindrical and Spherical Coordinates

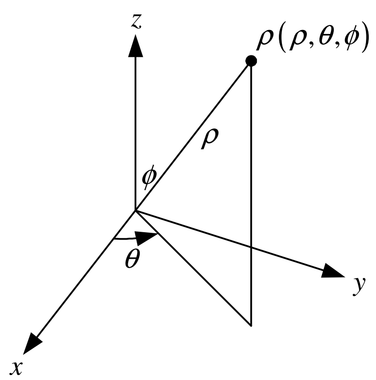
Homework： 21,25,29,35,39

(i) Cylindrical Coordinate



$$(r,\theta,z) \leftrightarrow (x,y,z)$$
$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$

(ii) Spherical Coordinates



$$(x,y,z) \leftrightarrow (\rho,\theta,\phi)$$
$$x = \rho\sin\phi\cos\theta$$
$$y = \rho\sin\phi\sin\theta$$
$$z = \rho\cos\phi$$

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Homework： 21,25,29,35,39

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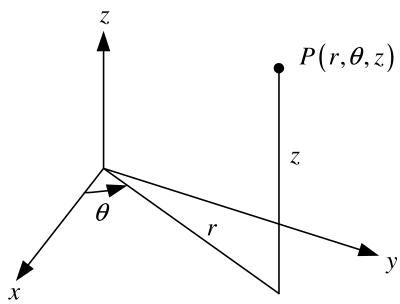


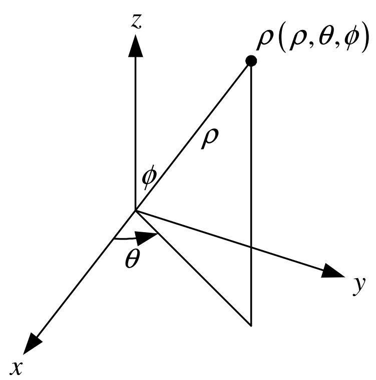
$(r,\theta,z) \leftrightarrow (x,y,z)$

$x = r\cos\theta$

$y = r\sin\theta$

$z = z$

(ii) Spherical Coordinates



$(x,y,z) \leftrightarrow (\rho,\theta,\phi)$

$x = \rho\sin\phi\cos\theta$

$y = \rho\sin\phi\sin\theta$

$z = \rho\cos\phi$

§13 Vector Functions

§13.1 Vector Functions

Homework：3,14,19-24,35

Functions：

(i) $f: R \to R$

(ii) $f: R \to R^m$

(iii) $f: R^n \to R$

(iv) $f: R^n \to R^m$

(i)型的函數之微分和積分是我們之前微積分課程學的東西。(ii)和(iii)型函數的微分和積分是這學期剩下的時間要學習的。至於(iv)型函數的微分在高微的課程會學到。

(1) Vector Functions or Vector-valued Functions

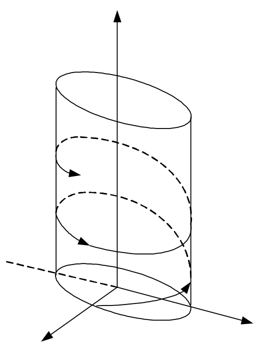
(ii)型函數即積 Vector Functions.

如 $r(t) = \langle f(t), g(t), h(t) \rangle = f(t)i + g(t)j + h(t)k$,其中 f, g, h 是實函數.

(2) Space curve：If f, g, h are continuous real-valued functions, then the orbit of $r(t) = \langle f(t), g(t), h(t) \rangle$ is called a space curve.

Example 1： $r(t) = \langle\ 1+t\ ,\ 2+5t\ ,\ -1+6t\ \rangle$ 此函數圖形為直線.

Example 2： $r(t) = \langle\ \cos t\ ,\ \sin t\ ,\ t\ \rangle$



$x^2 + y^2 = 1$
$z = t$

此函數圖形稱helix(螺旋).

Example 3： Find a vector function that represents the curve of intersection of the two surfaces：the cone $z=\sqrt{x^2+y^2}$ and the plane $z=1+y$.

Solution：

$$(1+y)^2 = x^2+y^2$$

$$\Rightarrow \quad x^2 = 1+2y$$

$$r(t) = \left(t \ , \ \frac{t^2-1}{2} \ , \ \frac{t^2+1}{2} \right)$$

Example 4： 19-24 of P.855.

Match the parametric equations with the graphs (labeled I-VI).
Give reasons for your choices.

19. $x=\cos 4t$, $y=t$, $z=\sin 4t$
沿著 y 軸而上的 helix. $\Rightarrow$ VI

20. $x=t$, $y=t^2$, $z=e^{-t}$
y 、z 皆為正，x 愈大，z 愈小. $\Rightarrow$ II

21. $x=t$, $y=\dfrac{1}{1+t^2}$, $z=t^2$
y 、z 皆為正，x 愈大，z 愈大. $\Rightarrow$ IV

22. $z=e^{-t}\cos 10\text{t}$, $y=e^{-t}\sin 10\text{t}$, $z=e^{-t}$
$x^2+y^2=z^2$ (cone，軸為 z -axis)，且 z 皆為正. $\Rightarrow$ I

23. $x=\cos t$, $y=\sin t$, $z=\sin 5t$
$x^2+y^2=1$ $\quad x, y, z$ 皆為週期. $\Rightarrow$ V

24. $x=\cos t$, $y=\sin t$, $z=\ln t$
$x^2+y^2=1$ 且 $z\to\infty$ as $t\to 0^+$. $\Rightarrow$ III

§13.2 Derivatives and Integrals of Vector Functions

Homework：10,19,21,23,33

Definition： Let $r(t)=\langle f(t),g(t),h(t)\rangle$. Then

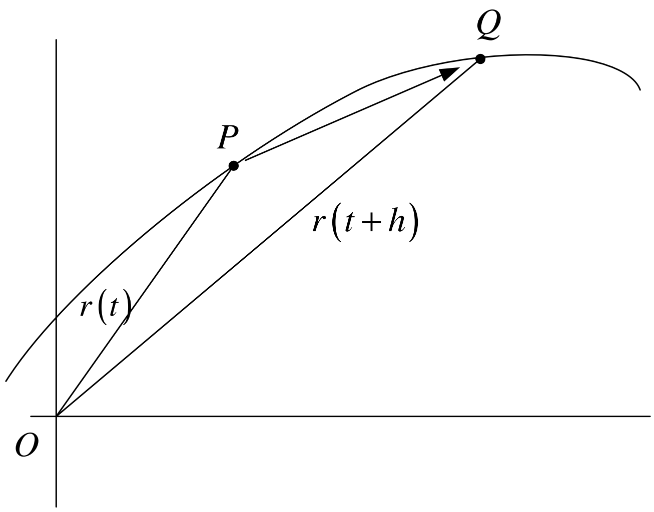
$\Delta r(t)=\langle \Delta f(t),\Delta g(t),\Delta h(t)\rangle$. Here

$\Delta = \lim\limits_{t\to a}$, $\dfrac{d}{dt}$ or $\int$

(極限) (微分) (積分)

(1) $r'(t)=$ the tangent vector to the curve defined by r at the point P.

(過 P 點的 r 曲線的切向量)＝在 t 時間的(瞬間)速度向量



$\overrightarrow{PQ}=\overrightarrow{OP}-\overrightarrow{OQ}=r(t)-r(t+h)=$ 割向量

$\Rightarrow \lim\limits_{h\to 0}\dfrac{r(t)-r(t+h)}{h}=r'(t)=$ 過 P 點的切向量.

Example 1： Let $x = t^5$, $y = t^4$, $z = t^3$, $t \in R$. Find the equation of tangent to the curve at $(1,1,1)$.

Solution：

$$r'(t) = \langle\ 5t^4, 4t^3, 3t^2\ \rangle_{t=1} = \langle 5,4,3\rangle.$$

Example 2： Let $r(t) = \langle\ \cos t\ ,\ \sin t\ ,\ t\ \rangle$. Find

(i) velocity when $t = 2\pi$ (ii)distance traveled over $[0, 2\pi]$.

Solution：

(i) $v(t) = r'(t) = \langle -\sin t, \cos t, 1\rangle_{t=2\pi} = \langle 0,1,1\rangle$

$|v(t)| =$ 速度(speed) $= \sqrt{2}$

(ii) $d = \int_0^{2\pi} |v(t)|\, dt = 2\pi\sqrt{2} = 2\sqrt{2}\pi.$

(2) Differentiation Rules

(i) $\frac{d}{dt}\big(u(t)\cdot v(t)\big) = u'(t)\cdot v(t) + u(t)\cdot v'(t)$

(ii) $\frac{d}{dt}\big(u(t)\times v(t)\big) = u'(t)\times v(t) + u(t)\times v'(t)$

(3) $\frac{d}{dt}|r(t)| = \frac{r'(t)\cdot r(t)}{|r(t)|}$

Proof： $|r(t)|^2 = r(t)\cdot r(t)$

$$\Rightarrow 2|r(t)|\frac{d}{dt}|r(t)| = 2r'(t)\cdot r(t)$$

$$\Rightarrow \frac{d}{dt}|r(t)| = \frac{r'(t)\cdot r(t)}{|r(t)|}$$

§13.3 Arc Length and Curvature

Homework： 13,19,21,39

(1) Arc Length

$L=\int_a^b |r'(t)|\ dt$

Why?

(i) 上學期學的 $x=f(t), y=g(t)$, 參數式的距離公式.

$$
\begin{aligned}
L &= \int_a^b \sqrt{(dx)^2+(dy)^2} \\
&= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\ dt \\
&= \int_a^b \sqrt{f'(t)^2+g'(t)^2}\ dt \\
&= \int_a^b\ |r'(t)|\ dt
\end{aligned}
$$

Here $r(t)=\langle f(t)\ ,\ g(t)\ \rangle$.

三維的參數式同理可得類似的弧長公式.

(ii) 從物理角度看， $|r'(t)|$ 代表物體在 t 時間的瞬間速度， $|r'(t)|\Delta t=$ 小範圍的距離 $\Rightarrow \int_a^b |r'(t)|\ dt=$ 代表此物體從 $t=a$ 到 $t=b$ 所走過的距離.

(2) Arc Length Function (or Distance Function)

$s(t)=\int_a^t\ |r'(u)|\ du.$

$\Rightarrow \dfrac{ds}{dt}=|r'(t)|=$ 距離的變化率 $=$ 速度.

(3) 將一個 Curve 用 arc length (s) 來作參數是一個非常有用的想法和技巧.

(如此的表達方式不隨著不同的座標系統而改變.)

(4) Unit Tangent vector： $T(t)=\dfrac{r'(t)}{\left|r'(t)\right|}$.

(5) Curvature(曲率)：a measure of how quickly the curve changes direction at a given point.

Definition： κ=曲率 $=\left|\dfrac{dT}{ds}\right|$

定理 (i) $\kappa=\left|\dfrac{dT}{ds}\right|=\left|\dfrac{dT/dt}{ds/dt}\right|=\left|\dfrac{T'(t)}{r'(t)}\right|=\dfrac{\left|r'(t)\times r''(t)\right|}{\left|r'(t)\right|^{3}}$

(ii) Given a plane curve $y=f(x)$, then its curvature K at a given point x is

$$\kappa(x)=\frac{\left|f''(x)\right|}{\left[1+\left(f'(x)^{2}\right)\right]^{3/2}}$$

Proof of (i)：

$$r'=\left|r'\right|T=\frac{ds}{dt}T \qquad (1)$$

$$\Rightarrow r''=\frac{d^{2}s}{dt^{2}}T+\frac{ds}{dt}T' \qquad (2)$$

(1)×(2)

$$\Rightarrow r'\times r''=\left(\frac{ds}{dt}\right)^{2}\left(T\times T'\right).$$

$$\Rightarrow \left|r'\times r''\right|=\left(\frac{ds}{dt}\right)^{2}\left|T\right|\left|T'\right|=\left(\frac{ds}{dt}\right)^{2}\left|T'\right|$$

$$\Rightarrow \left|T'\right|=\frac{\left|r'\times r''\right|}{\left(ds/dt\right)^{2}}=\frac{\left|r'\times r''\right|}{\left|r'\right|^{2}}$$

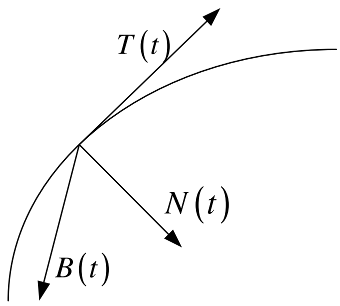
$$\Rightarrow \kappa=\frac{\left|T'\right|}{\left|r'\right|}=\frac{\left|r'\times r''\right|}{\left|r'\right|^{3}}$$

(6)Principal unit normal vector $N(t)$.

$$N(t)=\frac{T'(t)}{\left|T'(t)\right|}$$

(7)Binormal Vector $B(t)$.

$$B(t)=T(t)\times N(t)$$



§14 Partial Derivatives

§14.1 Functions of Several Variables

Homework：9,13,19,27,30,37,41,53,59,63

函數型： $f: R^2 \to R$ or $f: R^3 \to R$

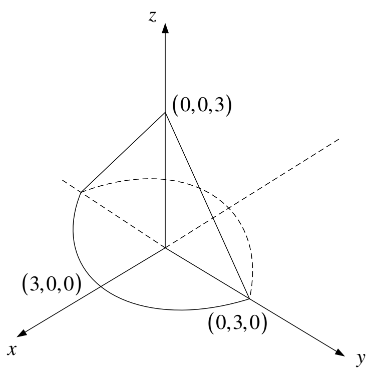
(1) Domain (2) Range (3) Graph (4) Level curves, surfaces or Cant our Curves

Example 1： $f(x, y) = \sqrt{9 - x^2 - y^2} = z$

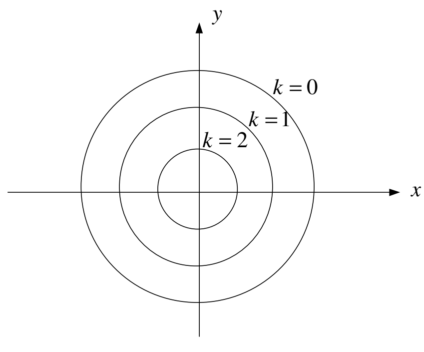
(1) Domain： $9 - x^2 - y^2 \geq 0$; $\{(x, y) : x^2 + y^2 \leq 9\}$

(2) Range：$\{ z : 0 \leq z \leq 3 \}$

(3) Graph：



(4) Level curve： (等高線) ： $\sqrt{9 - x^2 - y^2} = k$

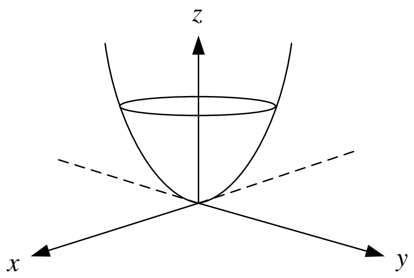


Example 2： $f(x,y)=4x^2+y^2=z$

(1) Domain： R^2

(2) Range：$\{\, z : z \ge 0 \,\}$

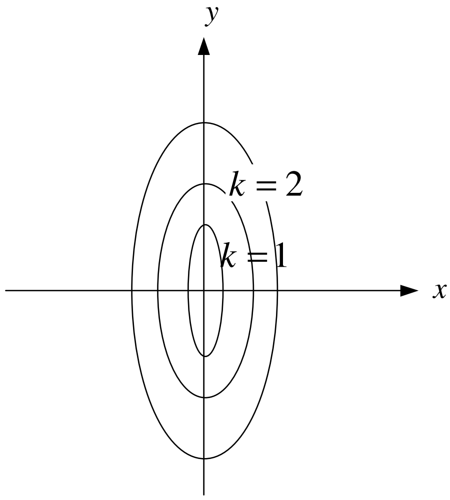
(3)



抛物橢圓體

(4) $4x^2+y^2=k \;\Rightarrow\; \dfrac{x^2}{\left(\dfrac{\sqrt{k}}{2}\right)^2}+\dfrac{y^2}{\left(\sqrt{k}\right)^2}=1$

Level curve： 橢圓



Example 3： $f(x,y,z)=x^2+y^2+z^2=u$

(1) Domain： R^3

(2) Range： $\{\, u : u \ge 0 \,\}$

(3) 四度空間

(4) Level surfaces： $x^2+y^2+z^2=k$ (球)

Example 4： Match the function with its graph (labeled I-VI) as given in #30. of P.899 of the text.

(a) $f(x,y)=|x|+|y|$ ：$z=0$ 為唯一最低點 $\Rightarrow$ VI.

(b) $f(x,y)=|xy|$ ：z 的高度在 x 軸和 y 軸皆為 0 $\Rightarrow$ V.

(c) $f(x,y)=\dfrac{1}{1+x^2+y^2}$ ：z 的高度隨著 $r=x^2+y^2$ 愈大愈小至於 0 $\Rightarrow$ I.

(d) $f(x,y)=\left(x^2-y^2\right)^2$ ： 在 $x=y$ 和 $x=-y$ 上 z 的高度為 0 $\Rightarrow$ IV.

(e) $f(x,y)=(x-y)^2$ 唯一沒對稱 $y-z$ 平面 $\Rightarrow$ II.

(f) $f(x,y)=\sin\left(|x|+|y|\right)$ ： 高度有界且呈震盪週期型 $\Rightarrow$ III.

§14.2 Limits and Continuity

Homework： 7,12,13,17,19,22,23,29,32,35,37,39

(1) 極限： $f: R^2 \to R$. Its domain $= D$

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

Definition：Given (any) $\varepsilon > 0$, $\exists$ a $\delta > 0$ s.t.

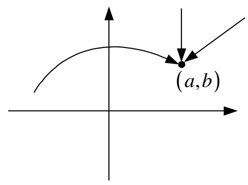
$$|f(x,y) - L| < \varepsilon \quad \text{whenever} \quad 0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$$

and $(x,y) \in D$.

(i) 極限存在必唯一

(ii) $(x,y) \to (a,b)$ (二維逼近)和 $x \to a$ (一維逼近)的差別

- 在 2 維有無窮多的方向逼近到 (a,b)
- 沿著曲線也可以



(iii)驗證不存在：只需找 2 個方向，其極限值不同。

(iv)驗證存在：定義或夾擊或 polar coordinate.

Example 1： $f(x,y) = \dfrac{x^2 - y^2}{x^2 + y^2}$. Show that $\lim\limits_{(x,y)\to(0,0)}$ does not exist.

Proof：

Key：分子和分母為齊次(各項次數相同)，則可沿直線証明極限不存在。

Let： $y = mx \Rightarrow f(x, mx) = \dfrac{1-m^2}{1+m^2}$.

$$\lim_{(x,mx)\to(0,0)} f(x,mx) = \lim_{(x,mx)\to(0,0)} \frac{1-m^2}{1+m^2} = \lim_{x\to 0} \frac{1-m^2}{1+m^2} = \frac{1-m^2}{1+m^2}$$

隨 m 而變的極限 $\Rightarrow$ 不存在

Example 2： Show that $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy^2}{x^2 + y^4}$ does not exist.

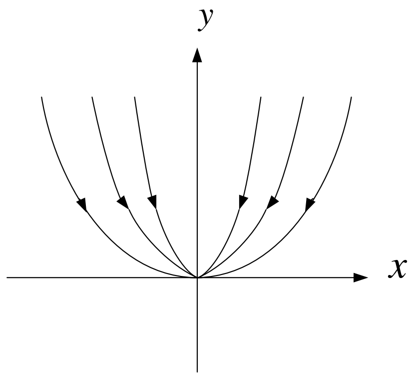
Proof：

Key：分子和分母各項如何變齊次。

Let： $x = my^2 \Rightarrow \dfrac{xy^2}{x^2+y^4} = \dfrac{my^4}{m^2y^4+y^4} = \dfrac{m}{1+m^2}$

$\Rightarrow$ 沿著 $x = my^2$ 的拋物線趨近 $(0,0)$ 則其極限為 $\dfrac{m}{1+m^2}$ （隨 m 而變）

$\Rightarrow$ 不存在



Example 3： Prove that $\lim\limits_{(x,y)\to(0,0)} \dfrac{3x^2y}{x^2+y^2} = 0$

Proof：

$$0 \le \left|\frac{3x^2y}{x^2+y^2}\right| \le 3|y|$$

當 $(x, y) \to (0,0)$，上、下界的極限皆趨近於 0

$$\Rightarrow \lim_{(x,y)\to(0,0)} \left|\frac{3x^2y}{x^2+y^2}\right| = 0$$

$$\Rightarrow \lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0$$

Example 4： Prove that $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y^3}{x^2+y^2} = 0$

(1)By definition：Skip.

(2)Sequence Theorem：

$$0 \le \left|\frac{x^2y^3}{x^2+y^2}\right| \le \left|y^3\right|$$

$\swarrow$ 0 $\Downarrow$ $\searrow$ 0

$\longrightarrow 0$

(3)Polar coordinate $x = r\cos\theta$, $y = r\sin\theta$.

$$\frac{x^2y^3}{x^2+y^2} = r^3\cos^3\theta\sin^3\theta \to 0 \text{ as } (x,y)\to 0 \text{ (and so } x^2+y^2=r\to 0\text{)}$$

Example 5： Decide whether $\lim_{(x,y)\to(0,0)} \frac{x^3+y^2}{x^2+y^2}$ exist or not by polar coordinate.

Solution：

$$\frac{x^3+y^2}{x^2+y^2} = \frac{r^3\cos^3\theta + r^2\sin^2\theta}{r^2} = r\cos^3\theta + \sin^2\theta$$

$\to \sin^2\theta$ as $r\to 0$ $\Rightarrow$ 不存在

Example 6： Decide whether $\lim_{(x,y)\to(0,0)} \frac{x^2+\sin^2 y}{2x^2+y^2}$ exist or not.

Solution：

Let $y = mx$

$$\Rightarrow \frac{x^2+\sin^2 y}{2x^2+y^2} = \frac{x^2 + \frac{\sin^2 mx}{m^2x^2}m^2x^2}{2x^2+m^2x^2} = \frac{1+\left(\frac{\sin mx}{mx}\right)^2 m^2}{2+m^2}$$

$\to \frac{1+m^2}{2+m^2}$ as $x\to 0$ $\Rightarrow$ 不存在

(此例子可當成齊次式)

Summary：

$\lim_{(x,y)\to(0,0)} \frac{f}{g}$, where f and g are polynomials in x and y.

(1) g 的某一項的次數 $\geq f$ 的某一項的次數. $\Rightarrow$ 不存在.

(2) $g(x,y)\neq 0$ for all $(x,y)\neq(0,0)$ 和 g 的每一項的次數 $< f$ 的每一項次數. $\Rightarrow$ 存在.

Example 7： Decide whether $\lim_{(x,y)\to(0,0)} \frac{x+y^3}{x^2+y^2}$ exist or not by polar coordinate.

Solution：

$$\lim_{(x,y)\to(0,0)} \frac{x+y^3}{x^2+y^2}$$

$$= \lim_{r\to 0} \frac{r\cos\theta + r^3\sin^3\theta}{r^2}$$

$$= \lim_{r\to 0}\left(\frac{\cos\theta}{r} + r\sin^3\theta\right) \Rightarrow \text{不存在}.$$

Example 8：Decide whether $\lim_{(x,y)\to(0,0)} \dfrac{x^2+y^2}{x+y^3}$ exist or not by polar coordinate.

Solution：

$$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{x+y^3}$$

$$= \lim_{r\to 0} \frac{r}{\cos\theta + r^2 \sin^3\theta}$$

$$= \lim_{r\to 0} \left(\frac{1}{\frac{\cos\theta}{r} + r\sin^3\theta} \right) = \begin{cases} \text{不存在 .} & \theta = \frac{\pi}{2} \\ 0 & \theta \neq \frac{\pi}{2} \end{cases} \Rightarrow \text{不存在 .}$$

Example 9：Prove that $\lim_{(x,y,z)\to(0,0,0)} \dfrac{xyz}{x^2+y^2+z^2} = 0$

Proof：

$$\frac{x^2+y^2+z^2}{3} \geq \sqrt[3]{x^2y^2z^2} = (xyz)^{\frac{2}{3}}$$

$$\Rightarrow 0 \leq \left| \frac{xyz}{x^2+y^2+z^2} \right| \leq \left| \frac{xyz}{3(xyz)^{\frac{2}{3}}} \right| = \frac{1}{3} \bullet |xyz|^{\frac{1}{3}} \to 0$$

as $(x,y,z) \to (0,0,0)$.

(2) f is continuous at (a,b) provided that $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$.

Example 10： Let $f(x,y) = x^2 + y^2$. f is continuous at $(0,0)$.

Example 11： $f(x,y) = \ln \dfrac{y}{x}$

Domain：$\{ (x,y) : xy > 0 \}$

f is continuous on its domain.

Example 12：

$$f(x,y)=\begin{cases} \dfrac{x^2+y^2}{\sqrt{x^2+y^2+1}\ -1} & (x,y)\neq(0,0) \\ 0 & (x,y)=(0,0) \end{cases}$$

Find the set of points of continuity of f .

Solution：

$$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}\ -1}$$

$$=\lim_{r\to 0}\frac{r}{\sqrt{r+1}-1}$$

$$=\lim_{r\to 0}\frac{r\left(\sqrt{r+1}+1\right)}{r+1-1}$$

$$=\lim_{r\to 0}\left(\sqrt{r+1}+1\right)=2$$

$\Rightarrow$ f is continuous on $R^2-\{(0,0)\}$.

Example 13：

$$f(x,y)=\begin{cases} \left(x^2+y^2\right)\ln\left(x^2+y^2\right) , & (x,y)\neq(0,0). \\ a , & (x,y)=(0,0). \end{cases}$$

Choose an a so that f is continuous at $(0,0)$.

Solution：

$$\lim_{(x,y)\to 0}\left(x^2+y^2\right)\ln\left(x^2+y^2\right)=\lim_{r\to 0^+} r\ln r=0$$

$\Rightarrow a=0$

§14.3 Partial Derivatives

Homework：5,15,17,29,37,43,51,55,61,83,88,89,90,91

(1) Partial Derivatives (偏導數) $f_x(a,b)$ of f with respect to x at (a,b).

Definition：

(i) $f_x(a,b)=\lim\limits_{h\to 0}\dfrac{f(a+h,b)-f(a,b)}{h}$

(ii)同理可定義對 y 的偏導數作用在(a,b)為

$$f_y(a,b)=\lim_{h\to 0}\frac{f(a,b+h)-f(a,b)}{h}.$$

Notation： $\dfrac{\partial f}{\partial x}=f_x=f_1$; $\dfrac{\partial f}{\partial y}=f_y=f_2$

Example 1： $f(x,y)=x^2+xy+y^2$

$\Rightarrow f_x=2x+y$ (把 y 當常數對 x 微分)

$f_y=2y+x$ (把 x 當常數對 y 微分)

$$f_{xx}=\frac{\partial}{\partial x}\left(\frac{f}{\partial x}\right)=\frac{\partial^2 f}{\partial x^2}=f_{11}=2$$

$$(f_x)_y=f_{xy}=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)=f_{12}=1$$

$$(f_y)_x=f_{yx}=f_{21}=1$$

$$f_{yy}=\frac{\partial^2 f}{\partial y^2}=f_{22}=2$$

(2)

(i) Clairau's Theorem： If f_{xy} and f_{yx} are continuous on a region D, then

$f_{xy}(a,b)=f_{yx}(a,b)$ for any $(a,b)\in D$.

(ii) $f_{xyy} = \left(\left(f_x\right)_y\right)_y$, f_{yxy} , f_{yyx} are continuous on D, then

$f_{xyy}(a,b) = f_{yxy}(a,b) = f_{yyx}(a,b)$ for $(a,b) \in D$.

Example 2： Let $u(x,y) = e^x \sin y$. Prove that $\Delta u =: \dfrac{\partial^2 u}{2x^2} + \dfrac{\partial^2 u}{2y^2} = 0$.

Proof：

$$u_x = e^x \sin y \qquad u_y = e^x \cos y$$
$$u_{xx} = e^x \sin y \qquad u_{yy} = -e^x \sin y$$

$$\Rightarrow u_{xx} + u_{yy} = 0 \quad \text{(Laplace's Equation)}$$

Example 3： Let $u(x,t) = f(x - at)$. Prove that $\dfrac{\partial^2 u}{2t^2} = a^2 \dfrac{\partial^2 u}{2x^2}$.

Proof：

$$u_t = -af'(x-at) \qquad u_{tt} = a^2 f''(x-at)$$
$$u_x = f'(x-at) \qquad u_{xx} = f''(x-at)$$

$$\Rightarrow u_{tt} = a^2 u_{xx} \quad \text{(Wave's Equation)}$$

Example 4： Let $xyz = \cos(x+y+z)$. Find $\dfrac{\partial z}{\partial x}$, $\dfrac{\partial z}{\partial y}$.

Solution：

將 z 想成以 x 和 y 為獨立變數的函數

$$yz + xy\frac{\partial z}{\partial x} = \left(-\sin(x+y+z)\right)\left(\frac{\partial z}{\partial x} + 1\right)$$

$$\Rightarrow \frac{\partial z}{\partial x} = \frac{-\sin(x+y+z) - yz}{xy + \sin(x+y+z)}$$

$$\overset{\text{同理}}{\Rightarrow} \frac{\partial z}{\partial y} = \frac{-\sin(x+y+z) - xz}{xy + \sin(x+y+z)}$$

Example 5： $f(x,y)=\left(x^3+y^3\right)^{\frac{1}{3}}e^{-\sin y}$. Find $f_x(1,0)$.

Solution：

$$f(x,0)=\sqrt[3]{x^3}=x \quad \Rightarrow \quad f'(x)=1$$

$\Rightarrow f_x(1,0)=1$.($f_x(1,0)$將 y 固定為 0 對 x 的微分.)

Example 6：

$$f(x,y)=\begin{cases} \dfrac{x^3y-xy^3}{x^2+y^2} & , \quad (x,y)\neq(0,0) \ . \\ 0 & , \quad (x,y)=(0,0) \ . \end{cases}$$

(1) Find $f_x(x,y)$, $f_y(x,y)$ for $(x,y)\neq(0,0)$.

(2) Find $f_x(0,0)$ and $f_y(0,0)$.

(3) Find $f_{xy}(0,0)$ and $f_{yx}(0,0)$.

Solution：

(1) For $(x,y)\neq(0,0)$, we have

$$f_x(x,y)=\frac{x^4y+4x^2y^3-y^5}{\left(x^2+y^2\right)^2} \ , \ f_y(x,y)=\frac{x^5-4x^3y^2-xy^4}{\left(x^2+y^2\right)^2} \ .$$

(2) By definition, we have

$$f_x(0,0)=\lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}=0$$

$$f_y(0,0)=\lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h}=\lim_{h\to 0}\frac{-h^5}{h^4}=0$$

(3) Let

$$g(x,y)=f_x(x,y)=\begin{cases} \dfrac{x^4y+4x^2y^3-y^5}{\left(x^2+y^2\right)^2} & , \quad (x,y)\neq(0,0) \ . \\ 0 & , \quad (x,y)=(0,0) \ . \end{cases}$$

$$\Rightarrow g_y(0,0)=f_{xy}(0,0)=\lim_{h\to 0}\frac{g(0,h)-g(0,0)}{h}=\frac{\frac{-h^5}{h^4}}{h}=-1\ .$$

Let

$$h(x,y)=f_y(x,y)=\begin{cases}\dfrac{x^5-4x^3y^2-xy^4}{\left(x^2+y^2\right)^2} &, \ (x,y)\neq(0,0)\ .\\ 0 &, \ (x,y)=(0,0)\ .\end{cases}$$

$$\Rightarrow h_x(0,0)=f_{yx}(0,0)=\lim_{l\to 0}\frac{h(l,0)-h(0,0)}{l}=\frac{\frac{l^5}{l^4}}{l}=1\ .$$

Since $f_{xy}(0,0)\neq f_{xy}(0,0)$，則由 Clairaut's Theorem 得知

f_{xy} and f_{yx} 在$(0,0)$附近必不連續。

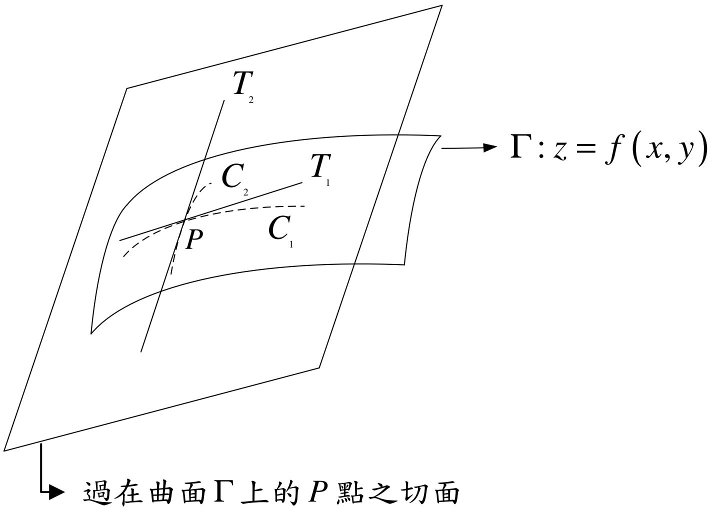
<u>檢查</u>：If $(x,y)\neq(0,0)$，則$f_{xy}(x,y)$ and $f_{yx}(x,y)$ 皆為分母、分子為次數 8 的<u>齊次項</u>。

$\Rightarrow \lim_{(x,y)\to(0,0)} f_{xy}(x,y)$ and $\lim_{(x,y)\to(0,0)} f_{yx}(x,y)$ 不存在。

§14.4 Tangent Planes and Linear Approximation

Homework： 3,5,11,15,19,23,27,31,33,39,42

(1)切面方程式



設此切面方程式為

$$A(x-x_0)+B(y-y_0)+(z-z_0)=0 \qquad (1)$$

令 $x=x_0$ ，則

$B(y-y_0)+(z-z_0)=0$ 為過 P 點和曲線 $C_1 : z=f(x_0,y)$ 相切的直線方程式 T_1 .

$\Rightarrow f_y(x_0,y_0)=T_1$ 的斜率 $=-B$.

同理 $f_x(x_0,y_0)=T_2$ 的斜率 $=-A$

公式：(i) 切面的法向量為

$$\langle A,B,-1\rangle=\langle f_x(x_0,y_0), f_y(x_0,y_0), -1\rangle$$

(ii) 切面方程式

$$z-z_0=f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)$$

Example 1： $z = 2x^2 + y^2$. 過 $P(1,1,3)$，找過 P 點的切面方程式 .

Solution：

$$f_x(x,y) = 4x \Rightarrow f_x(1,1) = 4 .$$
$$f_y(x,y) = 2y \Rightarrow f_y(1,1) = 2 .$$

Tangent plane： $4(x-1) + 2(y-1) = z - 3$.

(2) Linear approximation of $z = f(x,y)$ at (a,b) .

(以切平面來逼近曲面) .

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) .$$

Example 2： $z = \sqrt{x^2 + y^2}$ at $(3,4)$. Find $z(3.01, 3.99)$.

Solution：

$$f_x(x,y) = \frac{1}{2}\left(x^2+y^2\right)^{-\frac{1}{2}}(2x) \Rightarrow f_x(3,4) = \frac{3}{5}$$
$$f_y(x,y) = \frac{1}{2}\left(x^2+y^2\right)^{-\frac{1}{2}}(2y) \Rightarrow f_y(3,4) = \frac{4}{5}$$
$$\Rightarrow z = 5 + \frac{3}{5}(0.01) + \frac{4}{5}(-0.01) = 4.998 \approx \sqrt{(3.01)^2 + (3.99)^2} .$$

(3)Differentiable of f at (a,b)：

Definition： If $\Delta z = f(a+\Delta x\ ,\ b+\Delta y) - f(a,b)$

$$= f_x(a,b)\Delta x + f_y(a,b)\Delta y + \underbrace{\varepsilon_1 \Delta x + \varepsilon_2 \Delta y}_{\text{誤差}} ,$$

then ε_1 , $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$ (切面 $\approx$ 曲面)

Example 3： $f(x,y)=\begin{cases} \dfrac{xy}{x^2+y^2} & ,\ (x,y)\neq(0,0). \\ 0 & ,\ (x,y)=(0,0). \end{cases}$

・ $f_x(0,0)=f_y(0,0)=0$.

・Tangent plane through $(0,0)$: $z=0$.

・But $f(x,y)=\dfrac{1}{2}$ whenever $x=y\neq 0$.

$\Rightarrow f$ is not differentiable at (a,b).

Theorem： f is differentiable at (a,b) provided that

(i) f_x , f_y exist near (a,b).

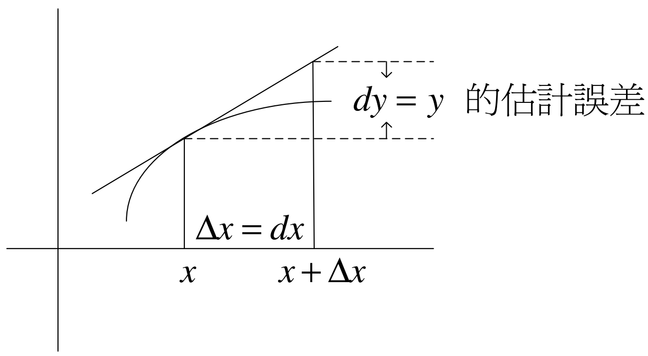
(ii) f_x , f_y are continuous at (a,b).

Remark： Differentiable $\Rightarrow$ Continuous.

(4) Differentiables

Recall：

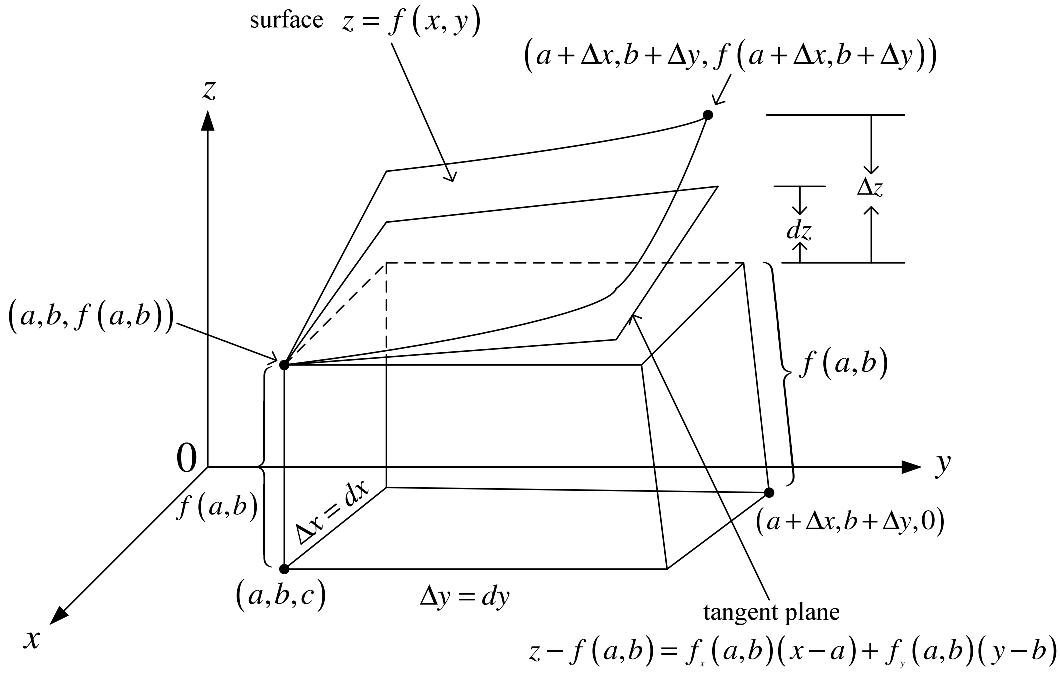
・1-D：



$\Delta y = f(x+\Delta x)-f(x)=$ 實際誤差.

$\Delta y \approx dy = f'(x)dx$ (切線代曲線得到的誤差).

・2-D：



$\Delta z=f(\ a+\Delta x\ ,\ b+\Delta y\ )-f(a,b)=$實際誤差

$dz=$估計誤差(利用切平面代曲面得到的誤差)

$\quad =f_x(a,b)dx+f_y(a,b)dy$.

公式 $\boxed{\Delta z\approx dz=f_x(a,b)dx+f_y(a,b)dy}$

Example 4： $z=f(x,y)=x^2+3xy-y^2$

(i) Find dz (ii) Use differentials to estimate $f(2.05,2.96)$.

Solution：

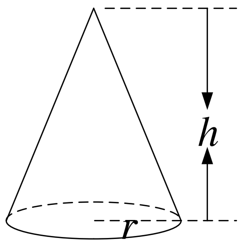
(i) $dz=(2x+3y)dx+(3x-2y)dy$

(ii) $dz=(2.2+3.3)(0.05)+(3.2-2.3)(-0.04)=0.65$

(令 $a=2$, $b=3$ $\Rightarrow \Delta x=0.05$, $\Delta y=-0.04$).

註 $\Delta z=f(2.05,2.96)-f(2,3)=0.6449$.

Example 5：



$r = 10 \pm 0.1$

$h = 25 \pm 0.1$

Use differentials to estimate the maximum error in the calculated volume of the cone.

Solution：

$$V = \frac{1}{3}\pi r^2 h$$

$$dV = \frac{2}{3}\pi hrdr + \frac{1}{3}\pi r^2 dh$$

$$= 20\pi$$

§14.5 The Chain Rule

Homework：5,7,11,13,19,25,26,29,33,39,43,45,47,49,51

(1) The Chain Rule

$$\boxed{z(t)=f\left(x(t),y(t)\right)\Rightarrow \frac{dz}{dt}=f_x\frac{dx}{dt}+f_y\frac{dy}{dt}\ .}$$

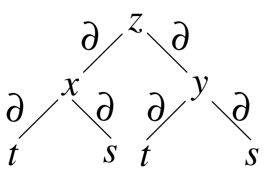
(由上節的 Differentials 或 Differentiable 的定義來感受上式.)

Example 1： $z=f\left(x,y\right)=x^2+xy+y^2$, $x=\cos t$, $y=\sin t$.

$$\begin{aligned}
&z=f\left(x,y\right)=x^2+xy+y^2\\
\Rightarrow\ &z=\cos^2 t+\cos t\sin t+\sin^2 t\\
&\frac{dz}{dt}=2\cos t\left(-\sin t\right)+\left(-\sin t\right)\sin t+\cos t\cos t+2\sin t\cos t\\
&\quad=\left(2\cos t+\sin t\right)\left(-\sin t\right)+\left(\cos t+2\sin t\right)\cos t\\
&\quad=\left(2x+y\right)\frac{dx}{dt}+\left(x+2y\right)\frac{dy}{dt}\ .
\end{aligned}$$

(2) Tree Diagram

$z=\left(t,s\right)=f\left(x\left(t,s\right),y\left(t,s\right)\right)$



$$\Rightarrow\begin{cases}\dfrac{\partial z}{\partial t}=f_x\dfrac{\partial x}{\partial t}+f_y\dfrac{\partial y}{\partial t}\\[2ex]\dfrac{\partial z}{\partial s}=f_x\dfrac{\partial x}{\partial s}+f_y\dfrac{\partial y}{\partial s}\end{cases}$$

(3) Implicit Differentiation

If $f\left(x,y,z\right)=0$ and $z=g\left(x,y\right)$

$\Rightarrow f\left(x,y,g\left(x,y\right)\right)=0$

$\Rightarrow f_x+f_y\dfrac{dy}{dx}+f_z\dfrac{\partial z}{\partial x}=0$

Since $\dfrac{dy}{dx}=0$ $\Rightarrow$ $\dfrac{\partial z}{\partial x}=-\dfrac{f_x}{f_z}$.　　　　同理 $\dfrac{\partial z}{\partial y}=-\dfrac{f_y}{f_z}$.

Example2： $u = x^4 y + y^2 z^2$, $x = rse^t$, $y = rs^2 e^{-t}$, $z = r^2 s \sin t$.

Find $\left.\dfrac{\partial u}{\partial s}\right|_{\substack{r=2\\ s=1\\ t=0}}$.

Solution：

$$\left.\frac{\partial u}{\partial s}\right|_{\substack{r=2\\ s=1\\ t=0}} = 4x^3 y\left(re^t\right) + \left(x^4 + 2yz^2\right)\left(2sre^{-t}\right) + 2y^2 z\left(r^2 \sin t\right)$$
$$= 128 + 64$$
$$= 192 \ .$$

Example 3： $x^3 + y^3 + z^3 + 6xyz = 1$. Find $\dfrac{\partial z}{\partial x}$.

Solution：

$$3x^2 + 3z^2 \frac{\partial z}{\partial x} + 6yz + 6xy\frac{\partial z}{\partial x} = 0 \ .$$
$$\Rightarrow \frac{\partial z}{\partial x} = \frac{-x^2 - 2yz}{z^2 + 2xy} \ .$$

Example 4： Let f has continuous $2nd - order$ partial derivatives.

$z = f(x, y)$, $\begin{cases} x = r^2 + s^2 \\ y = 2rs \end{cases}$, find $\dfrac{\partial^2 z}{2r^2}$.

Solution：

$$\frac{\partial z}{\partial r} = f_x 2r + f_y 2s$$

$$\frac{\partial^2 z}{2r^2} = \left(f_{xx} 2r + f_{xy} 2s\right)2r + 2f_x + \left(f_{yx} 2r + f_{yy} 2s\right)2s$$
$$= 4r^2 f_{xx} + 8rsf_{xy} + 4s^2 f_{yy} + 2f_x$$
$$\left(f_{xy} = f_{yx}\right)$$

Example 5： $PV = 8.31T$. Find $\dfrac{dP}{dt}$ when $T = 300K$ and is increasing at a rate of $0.1\ K/s$, and $V = 100L$ and is increasing at a rate of $0.2\ L/s$.

Solution：

$$P=\frac{8.31T}{V}=\frac{8.31\times 300}{100}=24.93$$

$$\frac{dP}{dt}V+P\frac{dV}{dt}=8.31\frac{dT}{dt} \qquad (1)$$

$$\frac{dT}{dt}=0.1\ ,\ \frac{dV}{dt}=0.2 \quad T=300\ ,\ V=100\ ,\ P=24.93$$

代入(1)式求$\frac{dP}{dt}$.

Example 6： $z=f\left(x,y\right)$, $\begin{cases} x=r\cos\theta \\ y=r\sin\theta \end{cases}$

(i) Find $\frac{\partial^2 z}{\partial r^2}$, $\frac{\partial^2 z}{\partial \theta^2}$

(ii) Prove that $\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=\frac{\partial^2 z}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2}+\frac{1}{r}\frac{\partial z}{\partial r}$.

Solution：

(i)

$$\frac{\partial z}{\partial r}=f_x\cos\theta+f_y\sin\theta$$

$$\frac{\partial^2 z}{\partial r^2}=\left(f_{xx}\cos\theta+f_{xy}\sin\theta\right)\cos\theta+\left(f_{yx}\cos\theta+f_{yy}\sin\theta\right)\sin\theta$$

$$=f_{xx}\cos^2\theta+2f_{xy}\sin\theta\cos\theta+f_{yy}\sin^2\theta$$

$$\frac{\partial z}{\partial \theta}=f_x\left(-r\sin\theta\right)+f_y r\cos\theta$$

$$\frac{\partial^2 z}{\partial \theta^2}=\left[f_{xx}\left(-r\sin\theta\right)+f_{xy}r\cos\theta\right]\left(-r\sin\theta\right)$$

$$+f_x\left(-r\cos\theta\right)+f_y\left(-r\sin\theta\right)+\left[f_{yx}\left(-r\sin\theta\right)+f_{yy}r\cos\theta\right]r\cos\theta\ .$$

$$=r^2\sin^2\theta f_{xx}-2r\cos\theta\sin\theta f_{xy}+r^2\cos^2\theta f_{yy}-r\cos\theta f_x-r\sin\theta f_y\ .$$

(ii) 由 (i)可得.

Example 7： If $g(s,t) = f(s^2 - t^2, t^2 - s^2)$ and f is differentiable.

Show that g satisfies $t\frac{2g}{\partial s} + s\frac{\partial g}{\partial t} = 0$.

Proof：

$$\frac{\partial g}{\partial s} = 2sf_x - 2sf_y$$

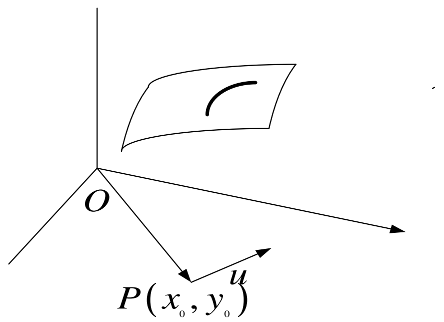
$$\frac{\partial g}{\partial t} = -2tf_x + 2tf_y$$

$$\Rightarrow t\frac{\partial g}{\partial s} + s\frac{\partial g}{\partial t} = 0$$

§14.6 Directional Derivatives and Gradient Vector

Homework：9,17,19,23,31,35,43,47,53,59

(1) 方向導數(沿著 $u=\langle a,b\rangle$, $a^2+b^2=1$，方向的微分.)



定義：$u=\langle a,b\rangle$, $a^2+b^2=1$.

$$D_u f(x_0,y_0)=\lim_{h\to 0}\frac{f\left(\overline{op}+h\vec{u}\right)-f\left(\overline{op}\right)}{h}$$
$$=\lim_{h\to 0}\frac{f(\ x_0+ha\ ,\ y_0+hb\ )-f(x_0,y_0)}{h}$$

Remarks：(i) $f_x=Df_i$ $i=\langle 1,0)$

(ii) $f_y=Df_j$ $j=\langle 0,1\rangle$

Theorem：How to compute $D_u f(x_0,y_0)$

$$D_u f(x_0,y_0)=\left\langle f_x(x_0,y_0)\ ,f_y(x_0,y_0)\ \right\rangle\bullet\langle a,b\rangle$$
$$=\ a f_x(x_0,y_0)+b f_y(x_0,y_0)$$

Why：? Let $g(h)=f(\ x_0+ah\ ,\ y_0+bh\ )$

Then $D_u f(x_0,y_0)=g'(0)$ Now,

$$g'(h)=f_x a+f_y b$$
$$\Rightarrow g'(0)=f_x(x_0,y_0)a+f_y(x_0,y_0)b$$

(2) ・The gradient vector (梯度向量) of f at (x_0,y_0)

定義 $\langle\ f_x(x_0,y_0)\ ,f_y(x_0,y_0)\ \rangle\ =:\ \nabla f$

- $D_u f(x_0,y_0)=$ gradient 和 u 的內積 .

Example 1： $f(x,y)=x^3-3xy+6y^2$. Find the directional derivative of f at $(1,2)$ in the direction of the vector $\langle\sqrt{3},1\rangle$.

Solution：

$$D_u f(1,2)=\langle 3x^2-3y,-3x+12y\rangle\Big|_{\substack{x=1\\y=2}}\bullet\left\langle\frac{\sqrt{3}}{2},\frac{1}{2}\right\rangle$$

$$=\frac{-3\sqrt{3}+21}{2}$$

(3) Facts

(i) $D_u f=\nabla f\bullet u=f$ 在 u 方向的變化率

(ii) $\max\limits_{u}\ D_u f=|\nabla f|$ when $u\,/\!/\,\nabla f$ (same direction)

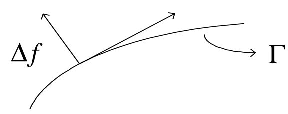
$$\because \nabla f\bullet u=|\nabla f||u|\cos\theta\ ,\ |u|=1$$

$$=|\nabla f| \qquad \text{if } \theta=0$$

(iii) $\min\limits_{u}\ D_u f=-|\nabla f|$. When $u\,/\!/\,\nabla f$ (opposite direction).

(iv) (a) $\nabla f\perp$ level curve

(b) $\nabla f\perp$ 任何在 level surface 的曲線.



(a) Consider $\Gamma: f(x(t),y(t))=C$

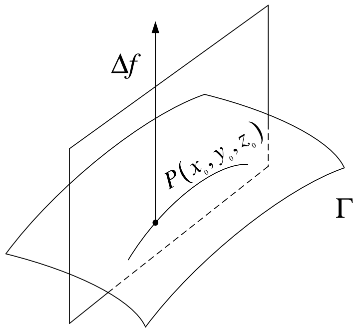
$$\Rightarrow f_x\frac{dx}{dt}+f_y\frac{dy}{dt}=0$$

$$\Rightarrow \nabla f\ \bullet\left(\frac{dx}{dt},\frac{dy}{dt}\right)=0$$

(b) Consider $\Gamma: f(x,y,z)=C$. Let

$f(x(t),y(t),z(t))=C$ be any curve on the level surface $\Gamma: f(x,y,z)=C$.

Then $\nabla f \bullet \left(\frac{dx}{dt},\frac{dy}{dt},\frac{dz}{dt}\right)=0$.



(v) Consider surface $\Gamma: z=f(x,y)$. Let $g(x,y,z)=f(x,y)-z=0$ 則 Γ 可

當成 g 函數的一個 level surface. 此 g 函數的 gradient 為 $\nabla g=\langle f_x, f_y, -1\rangle$.

(即將二維函數看成三維函數的等高曲面).

由(iv-b) ∇g 為過點 $P(x_0,y_0,z_0)$ 和 Γ 相切的平面的法向量.

Example 3： Given a level surface：$\frac{x^2}{4}+\frac{y^2}{1}+\frac{z^2}{9}=3$. 求過 $(-2,1,-3)$ 且此等高曲面相切的切面方程式.

Solution：

$$\bar{n}=\left\langle \frac{x}{2},2y,\frac{2z}{9}\right\rangle\bigg|_{\substack{x=-2\\ y=1\\ z=-3}}=\left\langle -1,2,-\frac{2}{3}\right\rangle$$

切面方程式 $\left\langle -1,2,-\frac{2}{3}\right\rangle \bullet \langle x+2, y-1, z+3\rangle=0$

$$\Rightarrow -(x+2)+2(y-1)-\frac{2}{3}(z+3)=0$$

Example 4： $\begin{cases} f: z=x^2+y^2 \\ g: 4x^2+y^2+z^2=9 \end{cases}$

求過 $(-1,1,2)$ 和其交集的曲線相切的切線方程式.

Solution：

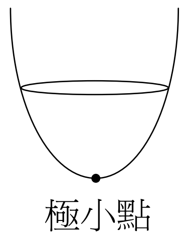
此切線的方向和 $\nabla f, \nabla g$ 垂直

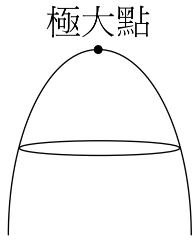
$\Rightarrow$ 切線的方向為 $\nabla f\times\nabla g$.

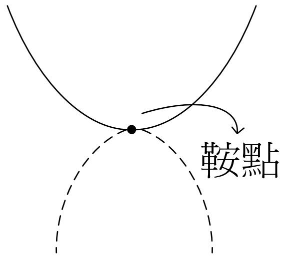
$$\nabla f = \langle 2x, 2y, -1 \rangle = \langle -2, 2, -1 \rangle$$
$$\nabla g = \langle 8x, 2y, 2z \rangle = 2\langle -4, 1, 2 \rangle$$
$$\langle -2, 2, -1 \rangle \times \langle -4, 1, 2 \rangle = \langle 5, 8, 6 \rangle$$
$$\Rightarrow \frac{x+1}{5} = \frac{y-1}{8} = \frac{z-2}{6}$$

§14.7 Maximum and Minimum Values

Homework：5,11,17,29,33,37,41,45,51,53,54







(saddle point)

(Minimum in all directions)

(One direction is a maximum and another direction is a minimum.)

(I) How to find such points?

(1) Find critical points： (a,b)

(a) $f_x(a,b) = f_y(a,b) = 0$. or

(b) f_x or f_y does not exist at (a,b).

(2) $2nd$ Derivative Tests.

Let $\alpha = f_{xx}(a,b)$, $\beta = f_{xy}(a,b) = f_{yx}(a,b)$, $\gamma = f_{yy}(a,b)$

Set $D = \begin{vmatrix} \alpha & \beta \\ \beta & \gamma \end{vmatrix} = \alpha\gamma - \beta^2$

(i) $D > 0$, $d > 0$ (or $\gamma > 0$) ($\because D > 0 \Rightarrow \alpha$, γ 同號).

$\Rightarrow f(a,b)$ local minimum and (a,b) is a minimum point.

(ii) $D > 0$, $\alpha < 0$ (or $\gamma < 0$) $\Rightarrow f(a,b)$ local max.

(iii) $D < 0$: (a,b) is a saddle point.

(iv) For any other cases, the test fails.

Why：?

(2-i) 的條件中

$\alpha > 0 \Rightarrow$ 在 x 軸方向 $(a,b,f(a,b))$ 是凹向上.

$D > 0 \Rightarrow$ 其它所有方向也凹向上.

Proof ：

$u = \langle h,k \rangle$.

$D_u f(a,b) = f_x\ h + f_y\ k$.

$$D_u^{\ 2} f = D_u(D_u f) = (f_{xx}\ h + f_{xy}\ k)h + (f_{yx}\ h + f_{yy}\ k)k$$
$$= f_{xx}\ h^2 + 2f_{xy}\ hk + f_{yy}\ k^2$$
$$= \alpha h^2 + 2\beta hk + rk^2$$
$$= \alpha\left(h + \frac{\beta k}{\alpha}\right)^2 + \frac{k^2}{\alpha}(\alpha\gamma) - \beta^2 .$$

If $\alpha > 0 + D > 0 \Rightarrow D_u^{\ 2} f > 0$ for any direction u.

$\Rightarrow$ All directions at $(a,b,f(a,b))$ are concave up.

Example 1： $f(x,y) = x^4 + y^4 - 4xy + 1$. Find local max, min and saddle point.

Solution：

$$\begin{cases} f_x = 4x^3 - 4y = 0 \\ f_y = 4y^3 - 4x = 0 \end{cases}$$

$\Rightarrow$ $(0,0)$, $(1,1)$ $(-1,-1)$ are critical points.

$f_{xx} = 12x^2 \quad f_{xy} = -4 \quad f_{yy} = 12y^2$

$D(x,y) = 144x^2y^2 - 16$

$\Rightarrow D(0,0) = -16 < 0.$

$\Rightarrow (0,0)$ is a saddle point.

$D(1,1) = 128 > 0$, $f_{xx}(1,1) = 12 > 0$

$\Rightarrow f(1,1) = -1$ is a local min.

$D(-1,-1)=128>0$, $f_{xx}(-1,-1)=12>0$

$\Rightarrow f(-1,-1)=-1$ is also a local min.

(II) Theorem： f is continuous on a closed and bounded set D.

$\Rightarrow$(i) There exist absolute minimum and absolute maximum.

(ii) Those extreme points occur at the critical points of f or the boundary of D.

Example 2： $f(x,y)=x^2-2xy+2y$ on $0\le x\le 3$, $0\le y\le 2$.

Find absolute max and min.

Solution：

$$\begin{aligned} f_x &= 2x-2y \\ f_y &= -2x+2 \end{aligned} \quad \Rightarrow \quad x=1 \ , \ y=1 .$$

(x,y)	$(1,1)$	$x=0$ $0\le y\le 2$	$x=3$ $0\le y\le 2$	$y=0$ $0\le x\le 3$	$y=2$ $0\le x\le 3$
$f(x,y)$		$2y$	$9-4y$	x^2	$(x-2)^2$
values	1	0,4	1,9	0,9	0,4

$\Rightarrow 0$： absolute minimum.

9： absolute maximum.

Example 3： $f(x,y)=2x^2+x+y^2-2$, $x^2+y^2=4$. Find absolute extreme.

Solution：

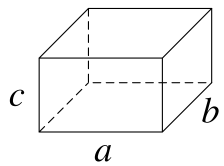
$$\begin{aligned} f_x &= 4x+1 \\ f_y &= 2y \end{aligned} \quad \Rightarrow \quad x=-\frac{1}{4} \ , \ y=0 .$$

(x,y)	$\left(-\frac{1}{4},0\right)$	$x^2+y^2=4$	
$f(x,y)$		x^2+x-2 $-2\le x\le 2$	
values	$-\frac{17}{8}$	$0,4,-\frac{9}{4}$	

$\Rightarrow$ max = 4 occurs at $x=2$, $y=0$.

min = $-\frac{9}{4}$ occurs at $x=-\frac{1}{2}$, $y=\pm\frac{\sqrt{15}}{2}$.

Example 4： A box with open top. with constraint：



$$ab + 2bc + 2ac = 12$$

Find max volume.

Solution：

$$\frac{ab + 2bc + 2ac}{3} \geq \sqrt[3]{4a^2b^2c^2}$$

$$\Rightarrow \quad 64 \geq 4a^2b^2c^2$$

$$\Rightarrow \quad 4 \geq abc$$

Maximum volume = 4 when $ab = 2bc = 2ac$

$\Rightarrow$ $a = 2$, $b = 2$, $c = 1$.

Example 5： (x, y, z) satisfies $x + 2y + 3z = 6$.

Find max $V = xyz$.

Solution：

$$\frac{6}{3} \geq \sqrt[3]{6xyz} \Rightarrow xyz \leq \frac{4}{3}$$

Max $V = \frac{4}{3}$ when $x = 2y = 3z$ $\Rightarrow$ $x = 2$, $y = 1$, $z = \frac{2}{3}$.

§14.8 Lagrange Multipliers

Homework：1,3,7,11,15,19,23,39

Problem：Maximize or Minimize functions $f(x,y)$ or $f(x,y,z)$ under certain constraints.

cases	1	2	3
objective function	$f(x,y)$	$f(x,y,z)$	$f(x,y,z)$
constraints	$g(x,y)=k$	$g(x,y,z)=k$	$g(x,y,z)=k$ and $h(x,y,z)=k$

Lagrange Multipliers： 解上述問題的一個方法

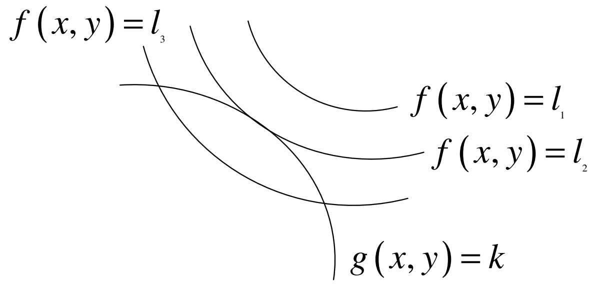
想法：

(i) 極值產生的地方：

f (變動) 的 level curve or level surface 和

g (固定) 的 level curve or level surface 相切.

(ii)Level curve of $f \perp \nabla f$.



由 (i) + (ii) $\Rightarrow \nabla f \,//\, \nabla g$

<u>結論</u>：Extreme occur when

(i) $\nabla f=\lambda\nabla g$ for some λ (cases 1 and 2).

(ii) $\nabla f=\lambda\nabla g+\mu\nabla h$ for some λ,μ (case 3).

($\because \nabla f$ 躺在由 ∇g 和 ∇h 所張出來的平面上.)

Example 1： A rectangular box without a lid is to be made from $12\,m^2$ of cardboard. Find the maximum volume of such box.

Solution：

(Method I)

$$f(x,y,z)=xyz$$

constraint g : $g(x,y,z)=2xy+2yz+xz$ (1)

$$\nabla f=\langle\ yz\ ,\ xz\ ,\ xy\ \rangle$$
$$\nabla g=\langle\ 2y+z\ ,\ 2x+2z\ ,\ 2y+x\ \rangle$$

$$\overset{\nabla f\,//\,\nabla g}{\Rightarrow}\begin{cases} yz=\lambda(2y+z)\\ xz=\lambda(2x+2z)\\ xy=\lambda(2y+x)\end{cases}$$

$$\Rightarrow\begin{cases}\dfrac{y}{x}=\dfrac{2y+z}{2x+2z}\\ \dfrac{z}{y}=\dfrac{2x+2z}{2y+x}\\ \dfrac{x}{z}=\dfrac{2y+x}{2y+z}\end{cases}\Rightarrow\begin{cases}2y=x & (2)\\ z=2y & (3)\\ x=z\end{cases}$$

將(2),(3)代入(1)

$\Rightarrow 4y^2+4y^2+4y^2=12$

$\Rightarrow y=1$

$\Rightarrow x=2 \quad z=2$

$\Rightarrow V=4$

(Method II)

$$\frac{12}{3}=\frac{2xy+2yz+xz}{3}\geq\left(4x^2y^2z^2\right)^{\frac{1}{3}}$$

$\Rightarrow 64\geq 4x^2y^2z^2$

$\Rightarrow 4\geq xyz$

$\Rightarrow$ Maximum volume = 4 .

Example2： $f(x,y)=x^2+2y^2$. $g(x,y)=x^2+y^2=1$.

(Method I)

$$\text{高中解法}\begin{cases} 1.\text{算幾不等式} \\ 2.\text{柯西不等式} \\ 3.\text{配方法} \end{cases}$$

$\Rightarrow f(x,y)=x^2+2y^2=1+y^2$ ， $(-1\le y\le 1)$.

$\Rightarrow f$ 的最大值$=2$ ； f 的最小值$=1$.

(Method II)

$\nabla f=\langle\ 2x\ ,\ 4y\ \rangle$, $\nabla g=\langle\ 2x\ ,\ 2y\ \rangle$

$\Rightarrow \begin{cases} x=\lambda x \\ 2y=\lambda y \end{cases} \Rightarrow\ x=0$ or $y=0$.

$\Rightarrow (\ 0\ ,\pm 1\ )$ and $(\pm 1\ ,\ 0\ )$

$\Rightarrow$ max $f=2$ min $f=1$.

Example 3： $f(x,y,z)=x+2y+3z$. Constraints： $\begin{cases} x-y+z=1 & (1) \\ x^2+y^2=1 & (2) \end{cases}$

(Method I)

$f(x,y,z) \overset{(\text{由}1)}{=\!=} x+2y+3(1-x+y)=-2x+5y+3$

柯西不等式　得：

$(x^2+y^2)\left((-2)^2+5^2\right)\ge(-2x+5y)^2$

$\Rightarrow \sqrt{29}\ge -2x+5y\ge -\sqrt{29}$

$\Rightarrow \sqrt{29}+3\ge -2x+5y+3\ge -\sqrt{29}-3$

↓ ↓

Maximum minimum

(Method II)

$$\begin{aligned} \nabla f&=\langle\ 1,2,3\ \rangle \\ \nabla g&=\langle\ 1,-1,1\ \rangle \\ \nabla h&=\langle\ 2x,2y,o\ \rangle \end{aligned} \Rightarrow \begin{cases} 1=\lambda+2\mu x \\ 2=-\lambda+2\mu y \\ 3=\lambda \end{cases}$$

$$\Rightarrow \begin{cases} 2\mu x=-2 \\ 2\mu y=5 \end{cases} \Rightarrow \frac{x}{y}=\frac{-2}{5}$$

Since $x^2+y^2=1 \Rightarrow x=\frac{\mp 2}{\sqrt{29}}$, $y=\frac{\pm 5}{\sqrt{29}}$

$\Rightarrow z=1\pm\frac{7}{\sqrt{29}}$

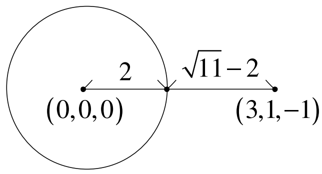
$\Rightarrow$ max $=3+\sqrt{29}$, min $=-3-\sqrt{29}$

Example 4： $x^2+y^2+z^2=4$. Find max or min of

$(x-3)^2+(y-1)^2+(z+1)^2$. Where those extreme occur?

Solution：

(Method I)



$\Rightarrow$ min $=\sqrt{11}-2$ occurs at $\left(\frac{6}{\sqrt{11}},\frac{2}{\sqrt{11}},\frac{-2}{\sqrt{11}}\right)$ and

max $=\sqrt{11}+2$ occurs at $\left(\frac{-6}{\sqrt{11}},\frac{-2}{\sqrt{11}},\frac{2}{\sqrt{11}}\right)$.

Example 5： $\begin{cases} f(x,y,z)=x+y+z \\ \frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1 \end{cases}$

Solution：

$$\left(\left(\sqrt{x}\right)^2+\left(\sqrt{y}\right)^2+\left(\sqrt{z}\right)^2\right)\left(\left(\frac{1}{\sqrt{x}}\right)^2+\left(\frac{1}{\sqrt{y}}\right)^2+\left(\frac{1}{\sqrt{z}}\right)^2\right)\geq(1+1+1)^2=9$$

$$\Rightarrow(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)=(x+y+z)\geq 9$$

$\Rightarrow$ minimum $=9$.

Example 6： $\begin{cases} f(x,y,z)=x^2+2y^2+2z^2 \\ x+y+z=1 & (1) \\ x-y+2z=2 & (2) \end{cases}$

Solution：

(Method I)

$\langle 1,1,1\rangle \times \langle 1,-1,2\rangle = \langle 3,-1,-2\rangle$

Let $z=0 \Rightarrow \begin{cases} x+y=1 \\ x-y=2 \end{cases} \Rightarrow x=\dfrac{3}{2}$, $y=-\dfrac{1}{2}$.

$\Rightarrow$ $(1)\cap(2)$ 之直線方程式為

$$\begin{cases} x=\dfrac{3}{2}+3t \\ y=-\dfrac{1}{2}-t \\ z=0-2t \end{cases} \quad , \quad t\in \mathrm{R} . \qquad (3)$$

將 (3) 代入 f 利用配方法可得極值.

(Method II) Langrage Multipliers

Example 7： The plane $x+y+2z=2$ intersects the paraboloid $z=x^2+y^2$ in an ellipse. Find the points on this ellipse that are nearest to and farthest from the origin.

Solution：

$$\text{Max or Min} \begin{cases} f(x,y,z)=x^2+y^2+z^2 & \\ x+y+2z=2 & h \\ z=x^2+y^2 & g \end{cases}$$

$\nabla h=\langle 1,1,2 \rangle$, $\nabla g=\langle 2x,2y,-1\rangle$, $\nabla f=\langle 2x,2y,2z\rangle$

$\Rightarrow \nabla f=\lambda\langle 1,1,2 \rangle+\mu\langle 2x,2y,-1 \rangle$

$\Rightarrow x=y$

$\Rightarrow x=\dfrac{1}{2}$ or -1

$\Rightarrow$ Two points to check are $\left(\dfrac{1}{2},\dfrac{1}{2},\dfrac{1}{2}\right)$ and $(-1,-1,2)$.

$\Rightarrow f\left(\dfrac{1}{2},\dfrac{1}{2},\dfrac{1}{2}\right)=\dfrac{3}{4}$ & $f(-1,-1,2)=6$

↓ 最近值　　↓ 最遠值

§15 Multiple Integrals

§15.1 Double Integrals over Rectangles

Homework：3,11,13,17,18

Problem： $z=f(x,y)$ $\leftrightarrow$ 房子的屋頂 $(f\geq 0)$

$D=[a,b]\times[c,d]$ $\leftrightarrow$ 房子的地基 $(f\geq 0)$

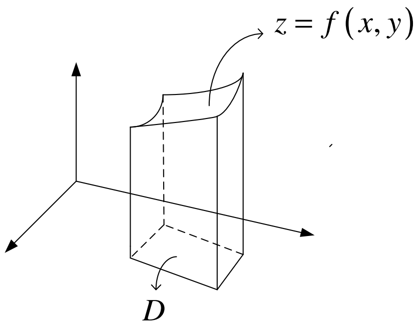
＝a rectangle

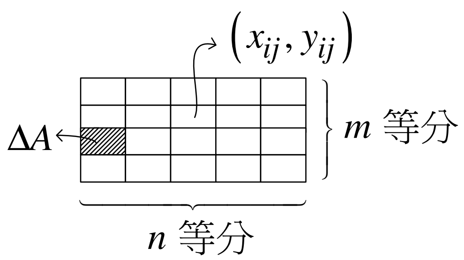
則此房子的體積(Volume)為何?

Definition：The double integral of f over a rectangle D is

$$\lim_{n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n}\underbrace{f\left(x_{ij},y_{ij}\right)}_{\text{高}}\underbrace{\Delta A}_{\text{底}}$$ = volum under the roof $z=f(x,y)$ on

$D(f\geq 0)$＝若 $f\geq 0$,此式代表屋子的體積.





Remarks：

(i) $\sum_{i=1}^{m}\sum_{j=1}^{n}f\left(x_{ij},y_{ij}\right)\Delta A$ is called double Riemann sum

＝一堆小長方形之體積和

＝這是二維的算法

(ii) Midpoint Rule：Double Riemann sum with $\left(x_{ij},y_{ij}\right)$ being center of the corresponding rectangle.

(iii) average value of f on $D = \dfrac{\iint_D f(x,y)\,dA}{area\ D}$ = 房子的平切高度.

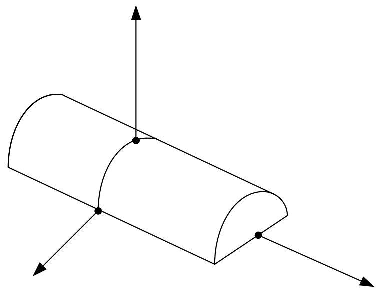
(iv) Properties of double integral.

(v) linear： $\iint_D (af+bg)\,dA = a\iint_D f dA + b\iint_D g dA$

(vi) $\iint_D f \ge \iint_D g$ provided that $f \ge g$ on D.

Example 1：屋頂： $f(x,y)=\sqrt{1-x^2}$

地基 $= D = [-1,1]\times[-2,2]$



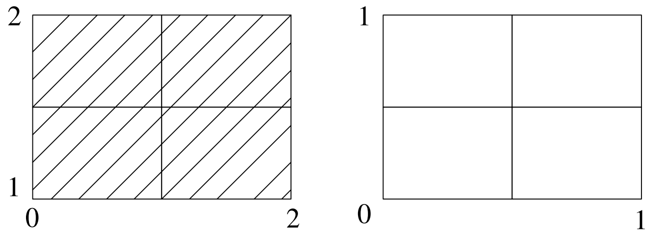
Solution：

Evaluate $\iint_D \sqrt{1-x^2}\ dA$ = 圓柱體的一半

$$=\frac{1}{2}\pi(1)^2 \ \bullet\ 4 = 2\pi$$

Example 2： $z=x^2+y^2$. $D=[0,1]\times[0,1]$. Using midpoint rule to compute the double Riemann sum with $n=m=2$.

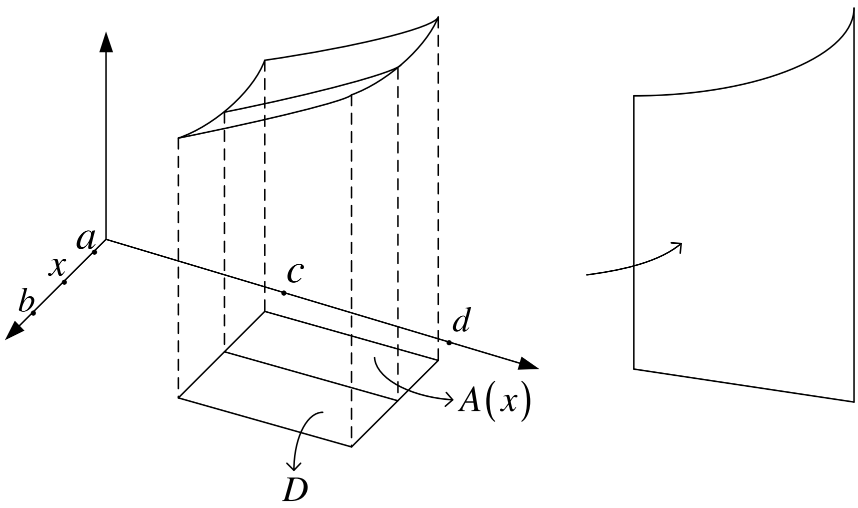
$$\iint_D (x^2+y^2)\,dA \approx \frac{1}{4}\left[f\left(\frac{1}{4},\frac{1}{4}\right)+f\left(\frac{3}{4},\frac{1}{4}\right)+f\left(\frac{1}{4},\frac{3}{4}\right)+f\left(\frac{3}{4},\frac{3}{4}\right)\right]=?$$



§15.2 Iterated Integrals

Homework： 1,5,11,15,17,21,23,25,29,36

Double Integral：二維計算屋子體積的方法
Iterated Integrals：一維計算屋子體積的方法(要算 2 次).



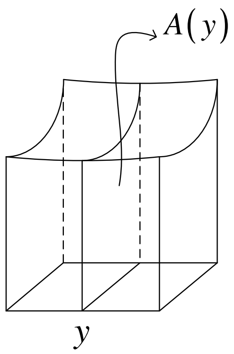
(i) $\int_c^d f(x,y)\,dy = A(x)$

$=$固定 x ，屋子的縱裁面面積

$\int_a^b A(x)\,dx=$將所有縱裁面面積連續相加

$=\int_a^b\left(\int_c^d f(x,y)dy\right)\,dx=\int_a^b\int_c^d f(x,y)\,dy\,dx$

$=$一維算法算 2 次



同理可得，先固定 y 的縱裁面面積算法而得 $\int_c^d\int_a^b f(x,y)\,dx\,dy$

Problem：什麼樣的屋頂$\left(f(x,y)=z\right)$此三種屋子體積算法相等。

Theorem：(Fubini：) If f is continuous on D (屋頂沒破)，then

$$\iint_D f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\ dy\ dx = \int_c^d \int_a^b f(x,y)\ dx\ dy.$$

Remark：

(i) In fact, f 只有些許小裂縫，三種體積算法也是一樣。

(ii) Clairaut's Theorem： When the order of partial differentiation does not matter?

Fubini Theorem：When the order of iterated integrals does not matter?

Example 1：Find $\iint_R y\sin(xy)\,dA : R = [1,2]\times[0,\pi]$

Solution：

$$\iint_R y\sin(xy)\,dA = \int_0^\pi \int_1^2 y\sin(xy)\ dx\ dy$$

$$= \int_0^\pi \left[-\cos xy \Big|_1^2 \right] dy$$

$$= \int_0^\pi -\cos 2y + \cos y dy$$

$$= -\frac{1}{2}\sin 2y \Big|_0^\pi + \sin y \Big|_0^\pi$$

$= 0$ (有地下室, 體積相消 .)

Example 2：$\iint_R (x-3y^2)\,dA \qquad R = [0,2]\times[1,2]$

Solution：

- $\int_1^2 \int_0^2 (x-3y^2)\ dx\ dy = \int_1^2 \left(\frac{1}{2}x^2 - 3y^2 x \Big|_0^2 \right) dy = \int_1^2 (2-6y^2)\,dy = -124$
- $\int_0^2 \int_1^2 (x-3y^2)\ dy\ dx = \int_0^2 \left(xy - 9y^3 \Big|_1^2 \right) dx = -124$

Example 3：$\begin{cases} x^2+2y^2+z=16 \\ x=2 \\ y=2 \\ \text{Three coordinate planes} \end{cases}$ 所圍的物體體積.

Solution：

屋頂：$z = 16 - x^2 - y^2$

地基：$[0,2]\times[0,2]$

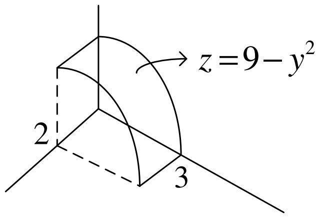
$$V=\int_0^2\int_0^2\left(16-x^2-y^2\right)\,dx\,dy\;=48$$

Example 4： Find the volume of the solid in the first octant bounded by the cylinder $z=9-y^2$ and the plane $x=2$.

Solution：

屋頂： $z=9-y^2$

地基： $[0,2]\times[0,3]$



$$V=\int_0^3\int_0^2\left(9-y^2\right)\,dx\,dy\;=36.$$

Example 5： If f is continuous on $[a,b]\times[c,d]$ and $g(x,y)=\int_a^x\int_c^y f(s,t)\,dt\,ds$ for $a<x<b$, $c<y<d$. Prove that $g_{xy}=g_{yx}=f(x,y)$.

Proof：

$$g_x \overset{FTC}{=} \int_c^y f(x,t)\,dt \Rightarrow g_{xy} \overset{FTC}{=} f(x,y).$$

$$g(x,y) \underset{Fubini}{=} \int_c^y\left(\int_a^x f(s,t)\,ds\right)dt$$

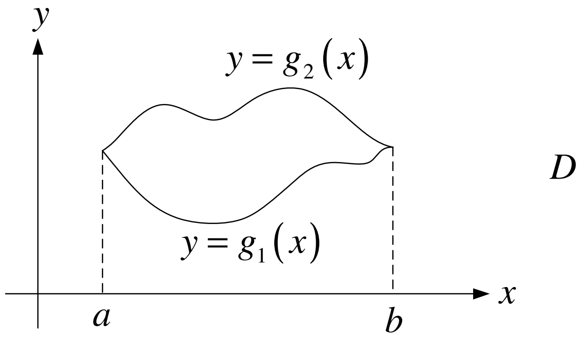
$$\Rightarrow g_y \overset{FTC}{=} \int_a^x f(s,y)\,ds \Rightarrow g_{yx} \overset{FTC}{=} f(x,y).$$

§15.3 Double Integrals over General Regions

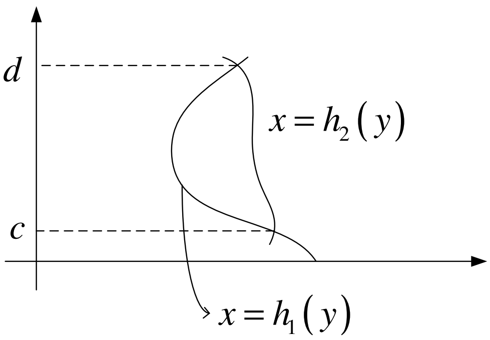
Homework：3,9,13,17,21,27,28,31,37,41,47,49,51,54,55

General Regions

(1) Tape I



(2) Tape II



- 只要屋頂“不太壞”（如：沒破洞，只有些許裂縫），則 general region 的邊界“不太壞”（如：沒斷點），則一維和二維計算面積方式仍相同 ie。

(Tape I) $\iint_D f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx$

(Tape II) $\iint_D f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy$

Summary：

(1) D 可能是 (i) Tape I (ii) Tape II (iii) 是 Tape I 也是 Tape II (iv)二者皆不是但可表成兩種聯集.

(2)若 $D_1 \cup D_2 = D$, $D_1 \cap D_2 =$ 邊界 $\Rightarrow \iint_D f dA = \iint_{D_1} f dA + \iint_{D_2} f dA$.

(3) linear： $\iint_D (\alpha f + g) dA = \alpha \iint_D f dA + \iint_D g dA$.

(4) If $m \le f(x, y) \le M$, $\Rightarrow mA(D) \le \iint_D f(x, y) dA \le MA(D)$.

Here $A(D) =$ 地基的面積.

(5) $\iint_D f(x) dA = 0$ Provided that

(i) D 對稱 y 軸 (利用對稱性) (ii) $f(x)$:odd functions.

Example 1： $\iint_D (x^2 \tan x + y^3 + 4) \, dA \quad D: x^2 + y^2 = 2$

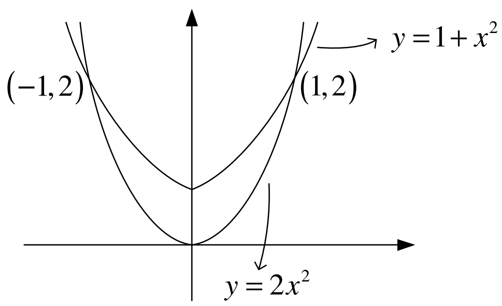
Solution：

$$\begin{aligned}
&\iint_D (x^2 \tan x + y^3 + 4) \, dA \\
&= \iint_D x^2 \tan x dA + \iint_D y^3 dA + \iint_D 4 dA \\
&= 0 + 0 + 8\pi \\
&= 8\pi
\end{aligned}$$

Example 2： $\iint_D (x + 2y) \, dA$, $D: \begin{cases} y = 2x^2 \\ y = 1 + x^2 \end{cases}$

Solution：

D:



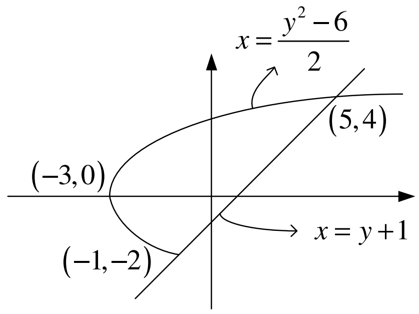
Tape I

$$= \int_{-1}^{1} \int_{2x^2}^{1+x^2} (x + 2y) \, dx = \frac{32}{15}$$

Example 3： $\iint_D xydA$, $D:\begin{cases} y = x-1 \\ y^2 = 2x+6 \end{cases}$

Solution：

$D:$

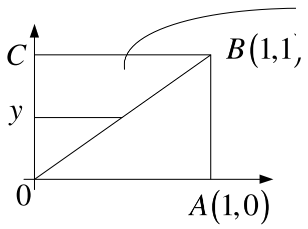


Tape II

$$=\int_{-2}^{4}\int_{\frac{y^2}{2}-3}^{y+1} xydxdy = 36$$

Example 4： $\int_0^1\int_x^1 \sin^2 ydydx$

Solution：



$D = \Delta OBC =$ 是 Tape I 也是 Tape II

Tape I 寫法不容易積出 $\int_x^1 \sin^2 ydy$ 改成 Tape II 寫法

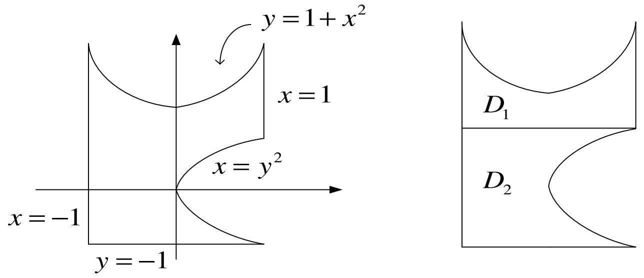
$$=\int_0^1\int_0^y \sin^2 dxdy = \int_0^1 y\sin^2 ydy$$

$$= -\frac{\cos y^2}{2}\Big|_0^1 = -\frac{\cos 1}{2}+\frac{1}{2}$$

Example 5： $\iint_D xydA$

Solution：

$D:$



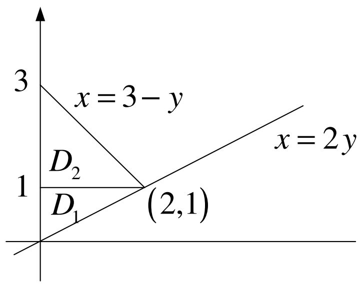
$$= \iint_{D_1} xydA + \iint_{D_2} xydA$$

$$= \int_{-1}^{1}\int_{1}^{1+x^2} xydydx + \int_{-1}^{1}\int_{-1}^{y^2} xydxdy$$

$= 0$ (利用對稱來算)

Example 6： $\iint_D f(x,y)\,dA = \int_0^1\int_0^{2y} f(x,y)\,dxdy + \int_1^3\int_0^{3\text{-}y} f(x,y)\,dxdy$ .Find D.

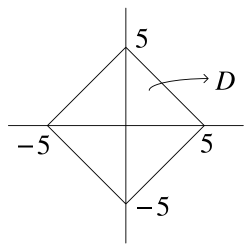
Solution：



$\Rightarrow D = D_1 \cup D_2$

Example 7： $\iint_D (2 - 3x + 4y)\ dA$

Solution：



(利用對稱性)

$$\iint_D (2 - 3x + 4y)\ dA$$

$$= \iint_D 2dA - \iint_D 3xdA + \iint_D 4ydA$$

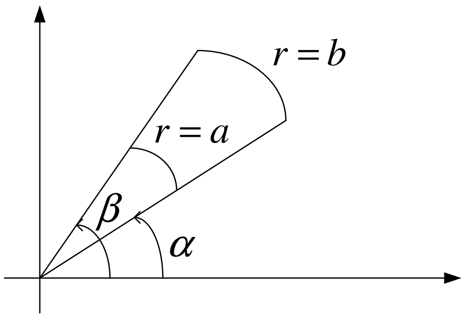
$$= 2.50 - 0 + 0$$

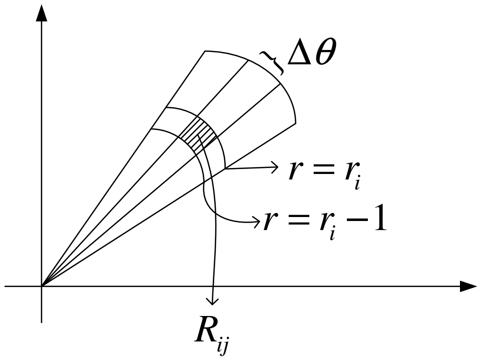
$$= 50$$

§15.4 Double Integrals in Polar Coordinates

Homework： 5,7,11,15,19,21,25,29,31,35

直角座標之積分變成極座標積分





$R=\{\ (r,\theta): a\le r\le b\ ,\ \alpha\le\theta\le\beta\ \}.$

$A_{ij}=\text{Area of }\ R_{ij}=\frac{1}{2}\left(r_i^2-r_{i-1}^2\right)\Delta\theta=\frac{1}{2}\left(r_i+r_{i-1}\right)\Delta r\Delta\theta$

當等分成“無窮”細時

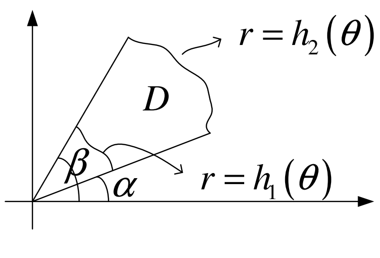
$A_{ij}\to dA=rdrd\theta$

Theorem：

(i) $\iint\limits_R f(x,y)\,dA=\int_\alpha^\beta\int_a^b f(r\cos\theta,r\sin\theta)\ rdrd\theta$

(ii) $D=\{\ (r,\theta):\alpha\le\theta\le\beta\ ,\ h_1(\theta)\le r\le h_2(\theta)\ \}$

$$\Rightarrow\iint\limits_D f(x,y)\,dA=\int_\alpha^\beta\int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta,r\sin\theta)\ rdrd\theta$$



Example 1： $\iint_D \left(3x+4y^2\right)\,dA.\quad D:1\le x^2+y^2\le 4\ ,\ y\ge 0.$

Solution：

$$\iint_D \left(3x+4y^2\right)\,dA$$

$$=\int_0^{\pi}\int_1^2\left(3r\cos\theta+4r^2\sin^2\theta\right)\,rdrd\theta$$

$$=\frac{15\pi}{2}$$

$$\boxed{=\int_{-2}^{2}\int_{\sqrt{1-x^2}}^{\sqrt{4-x^2}}\left(3x+4y^2\right)\,dydx}$$

Example 2： Volume of $\begin{cases} z=1-x^2-y^2 \\ z=0 \end{cases}$

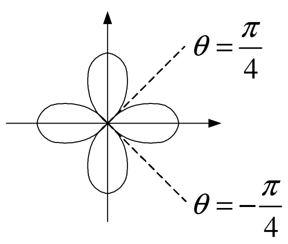
Solution：

$$\iint_D \left(1-x^2-y^2\right)\,dA\qquad D:x^2+y^2\le 1.$$

$$=\int_0^{2\pi}\int_0^1\left(1-r^2\right)rdrd\theta=\frac{\pi}{2}$$

Example 3： Area of $r=\cos 2\theta$ (Area=Volume with height 1)

Solution：



$$\text{Area}=4\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\int_0^{\cos2\theta}(1)\,rdrd\theta$$

$$=4\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\int_0^{\cos2\theta}rdrd\theta$$

$$=\frac{\pi}{2}$$

Example 4： Volume of the solid that lies $\begin{cases} \text{under}\quad z=x^2+y^2 \\ \text{inside } x^2+y^2=2x \\ \text{above}\quad z=0 \end{cases}$

Solution：

$$V=\iint_D \left(x^2+y^2\right)\,dA$$
$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\cos\theta} r^2 r\,dr\,d\theta$$
$$=\frac{3\pi}{2}$$

地基

$D:(x-1)^2+y^2=1$

$D:r=2\cos\theta$

屋頂

$z=x^2+y^2$

$z=r^2$

Example 5： Volume of $\begin{cases} x^2+y^2=4 \\ \dfrac{x^2}{16}+\dfrac{y^2}{16}+\dfrac{z^2}{64}=1 \end{cases}$

Solution：

$$V=2\int_0^{2\pi}\int_0^2 2\sqrt{16-r^2}\;r\,dr\,d\theta$$
$$=\frac{8\pi}{3}\left(64-24\sqrt{3}\right)$$

地基

$D:x^2+y^2=4$

$D:r=2$

屋頂

$z=\sqrt{64-4x^2-4y^2}$

$=\sqrt{64-4r^2}$

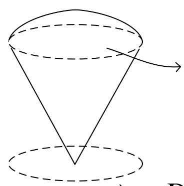
$=2\sqrt{16-r^2}$

Example 6： Volume of the solid that lies $\begin{cases} \text{below } x^2+y^2+z^2=1 \\ \text{above } z=\sqrt{x^2+y^2} \end{cases}$

$$\begin{cases} x^2+y^2+z^2=1 \\ z=\sqrt{x^2+y^2} \end{cases}$$

$$\Rightarrow x^2+y^2+\left(\sqrt{x^2+y^2}\right)^2=1$$

$$\Rightarrow x^2+y^2=\frac{1}{2}$$



$x^2+y^2=\dfrac{1}{2}$

$\left(z=\sqrt{\dfrac{1}{2}}\right)$

$D\;\; x^2+y^2=\dfrac{1}{2}$

$(z=0)$

Solution：

$$V=\int_0^{2\pi}\int_0^{\frac{1}{\sqrt{2}}}\left(\sqrt{1-r^2}-r\right)\ rdrd\theta$$
$$=\frac{\pi}{3}\left(2-\sqrt{2}\right)$$

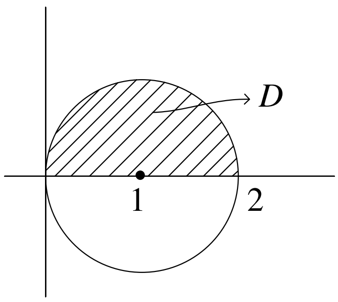
地基 $D: x^2+y^2=\frac{1}{2}$ $r=\frac{\sqrt{2}}{2}$
屋頂 $z=\sqrt{1-x^2-y^2}=\sqrt{1-r^2}$ $z=\sqrt{x^2+y^2}=r$

Example 7：Evaluate the iterated integral by converting to polar coordinates.

$$\int_0^2\int_0^{\sqrt{2x-x^2}}\sqrt{x^2+y^2}\ dydx$$

Solution：

$$y=\sqrt{2x-x^2}\Rightarrow x^2+y^2=2x\Rightarrow r=2\cos\theta$$



$$=\int_0^{\frac{\pi}{2}}\int_0^{2\cos\theta} r^2drd\theta$$
$$=\frac{16}{9}$$

§15.6 Surface Area

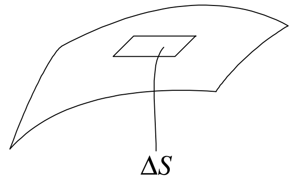
Homework：1,5,11,13,21,23

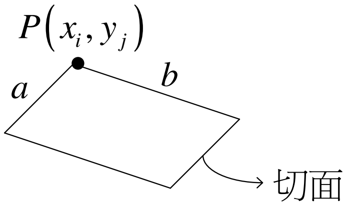
單變數函數

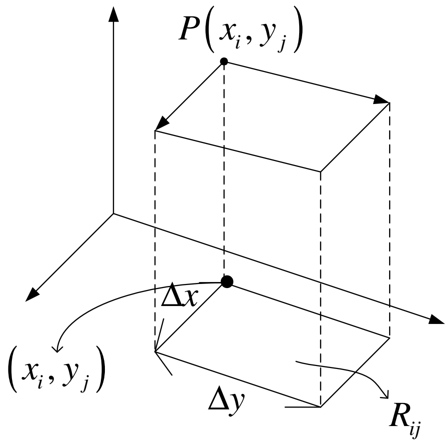
弧長 $\rightarrow$ 用線段逼近

多變數函數

Surface Area $\rightarrow$ 切面面積逼近







過 P_{ij} 的切面來逼近.

過 P_{ij} 點的曲面.

$$a = \Delta x i + o j + f_x \Delta x k$$
$$b = o i + \Delta y j + f_y \Delta y k$$

曲面面積 $= \Delta S \approx$ 切面面積

$\Rightarrow$ 切面面積 $= |a \times b|$

$$= \sqrt{1 + f_x^2 + f_y^2}\,(\Delta x \Delta y)$$
$$= \sqrt{1 + f_x^2 + f_y^2}\,(\Delta A)$$

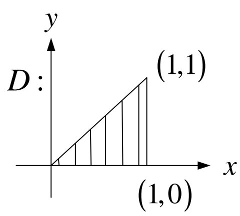
$\Rightarrow$ 作無窮切割後

$$dS = \sqrt{1 + f_x^2 + f_y^2}\ dA$$

公式：地基為 D，屋頂為 $z = f(x, y)$

$\Rightarrow$ 其屋頂面積 $= S = \iint_D dS = \iint_D \sqrt{1 + f_x^2 + f_y^2}\ dA$

Example 1： $z = x^2 + 2y$.



Solution：

$$\Rightarrow S = \int_0^1 \int_x^1 \sqrt{1 + (2x)^2 + 2^2}\, dy dx = \frac{1}{12}\left(27 - 5\sqrt{5}\right)$$

Example 2： Find surface area of $z = x^2 + y^2$ under $z = 9$.

Solution：

$$S = \iint_{x^2+y^2\le 9} \sqrt{1+4x^2+4y^2}\, dA$$
$$= \int_0^{2\pi}\int_0^3 \sqrt{1+4r^2}\, r dr d\theta$$
$$= \frac{\pi}{6}\left(37\sqrt{37}-1\right)$$

Example 3： Find surface area of $z = y^2 - x^2$ between $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

Solution：

$$S = \iint_{1\le x^2+y^2\le 4} \sqrt{1+4y^2+4x^2}\, dA = \text{極座標轉換}$$

Example 4： Find the area of the finite part of the paraboloid $y = x^2 + z^2$ cut off by the plane $y = 25$.

Solution：

地基 $\begin{cases} x^2 + z^2 = y \\ y = 25 \end{cases} \Rightarrow x^2 + z^2 = 5^2 \ (y = 0)$

屋頂 $y = x^2 + z^2$

$$S = \iint_{x^2+z^2\le 5^2} \sqrt{1+4x^2+4z^2}\, dA$$
$$= \int_0^{2\pi}\int_0^5 r\sqrt{1+4r^2}\, dr d\theta$$
$$= \frac{\pi}{6}\left(101\sqrt{101}-1\right)$$

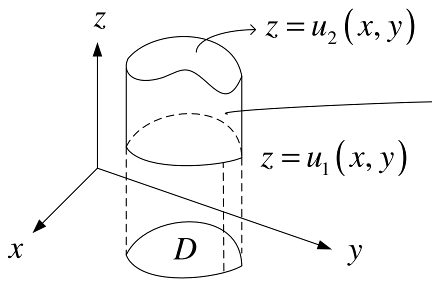
§15.7 Triple Integrals

Homework：3,7,9,11,15,19,25,29,31,33

屋頂： $u = f(x, y, z)$

地基： 在三維空間的物體E.

例如



$$E = D \times [\, u_1(x, y) \,,\, u_2(x, y) \,]$$
$$= \{\, (x, y, z) : x, y \in D \,,\, u_1(x, y) \le z \le u_2(x, y) \,\}$$

Triple Integral：

$$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\, dz \right] dA \qquad (1)$$

註：屋頂夠好，上述等式成立，事實上，

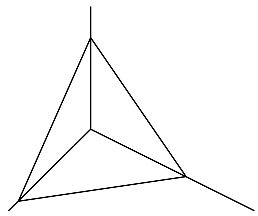
Fubini Theorem： 屋頂夠好（沒三維破洞，或只有有限的二維或一維的裂縫），則 Triple Integral 可寫成遞迴式(iterated)的三個一維積分。

Example 1： $\iiint_E z\,dV$, where E is the solid bounded by the four planes

$x = 0$, $y = 0$, $z = 0$ and $x + y + z = 1$

Solution：

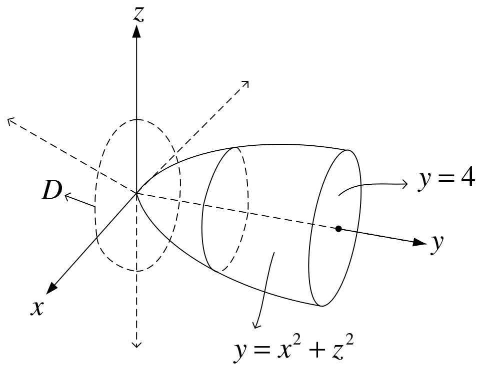
$$\iiint_E z\,dV$$
$$= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} z\,dz\,dy\,dx$$
$$= \frac{1}{24}$$
$$= \int_0^1 \int_0^{1-z} \int_0^{1-y-z} z\,dx\,dy\,dz$$



Example 2： $\iiint_E \sqrt{x^2+z^2}\ dV$. E is bounded by $\begin{cases} y = x^2+z^2 \\ y = 4 \end{cases}$.

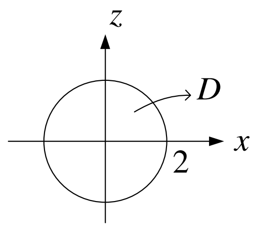
Solution：

地基 E



$E = D \times \left[\ x^2+z^2, 4\ \right]$

$D = x^2+z^2 \le 4$



$$\Rightarrow \iiint_E \sqrt{x^2+z^2}\ dV$$

$$= \iint_D \left(\int_{x^2+z^2}^{4} \sqrt{x^2+z^2}\ dy \right) dA$$

$$= \iint_{x^2+z^2 \le 4} \left(\sqrt{x^2+z^2} \right) \left(4 - x^2 - z^2 \right)\ dA \boxed{= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left(4 - x^2 - z^2 \right) \sqrt{x^2+z^2}\ dzdx}$$

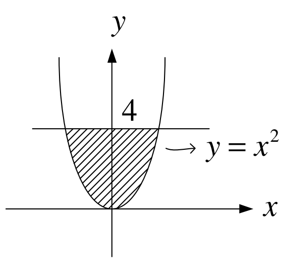
$$= \int_0^{2\pi} \int_0^2 r^2 \left(4 - r^2 \right)\ drd\theta$$

$$= \frac{128\pi}{15}$$

E 的另一看法：

$E = D' \times \left[\ -\sqrt{y-x^2}\ ,\ \sqrt{y-x^2}\ \right]$

$D' =$

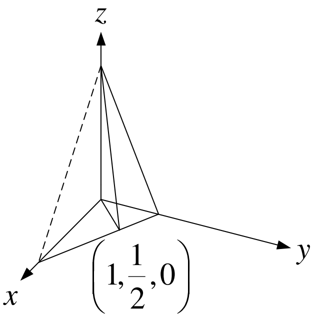


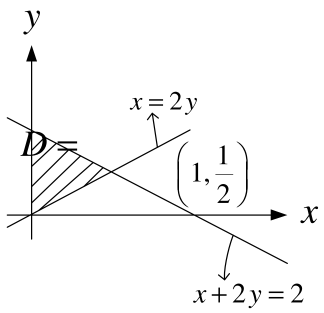
$$\text{Triple Integral} = \iint_{D'} \int_{-\sqrt{y-x^2}}^{\sqrt{y-x^2}} \sqrt{x^2+z^2}\ dzdA$$

$$= \int_{-2}^{2} \int_{x^2}^{4} \int_{-\sqrt{y-x^2}}^{\sqrt{y-x^2}} \sqrt{x^2+z^2}\ dzdydx \text{ (不好算)}$$

Example 3： Use a triple integral to find the volume of the tetrahedron T bounded by the planes $x+2y+z=2$, $x=2y$, $x=0$ and $z=0$.

Solution：





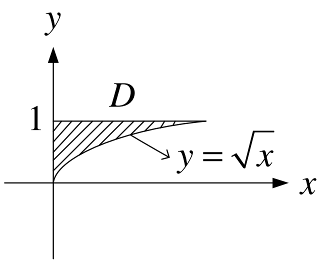
$$V = \iint_{D} \int_{0}^{2-x-2y} 1\ dz\ dA$$

$$= \int_{0}^{1} \int_{\frac{x}{2}}^{\frac{2-x}{2}} \int_{0}^{2-x-2y} 1\ dz\ dy\ dx$$

$$= \frac{1}{3}$$

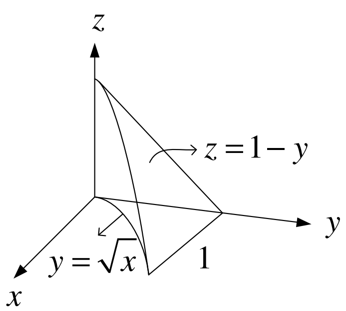
Example 4： Rewrite $\int_{0}^{1} \int_{\sqrt{x}}^{1} \int_{0}^{1-y} f(x,y,z)\ dzdydx$ as an equivalent iterated integral in the five other orders.

Solution：



$$E = D \times [\ 0\ ,\ 1-y\ ]$$

$$= \{\ (x,y,z) : (x,y) \in D \text{ and } 0 \le z \le 1-y\ \}$$

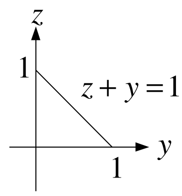


$$\text{Triple Integral} = \iint_D \int_0^{1-y} f \; dz \; dA$$

$$= \int_0^1 \int_0^{y^2} \int_0^{1-y} f \; dz \; dx \; dy$$

$$= \int_0^1 \int_0^{1-z} \int_0^{y^2} f \; dx \; dy \; dz$$

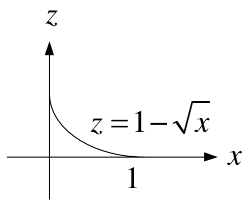
$$= \int_0^1 \int_0^{1-y} \int_0^{y^2} f \; dx \; dz \; dy$$



$$= \int_0^1 \int_0^{1-\sqrt{x}} \int_{\sqrt{x}}^{1-z} f \; dy \; dz \; dx$$

$$\begin{cases} y = \sqrt{x} \\ z = 1 - y \end{cases} \Rightarrow z = 1 - \sqrt{x}$$

$$= \int_0^1 \int_0^{(1-z)^2} \int_{\sqrt{x}}^{1-z} f \; dy \; dx \; dz$$



§15.8 Triple Integrals in Cylindrical and Spherical Coordinates

(1) Cylindrical Coordinates (C.C.)

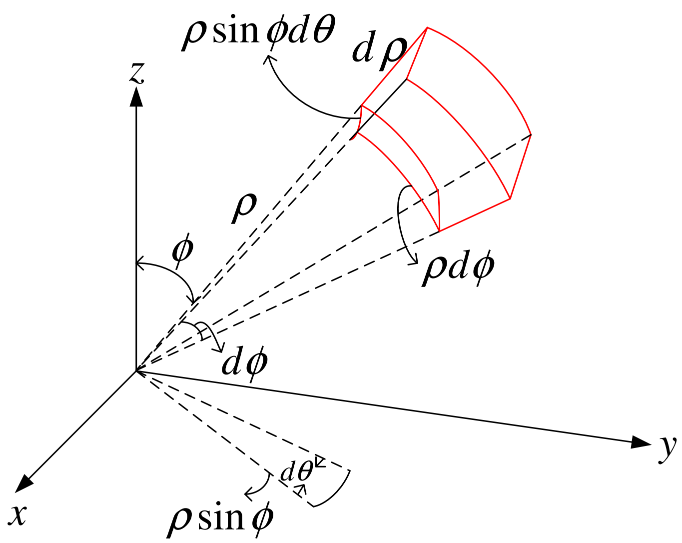
$E = D\times\left[u_1(x,y),u_2(x,y)\right]$ (圓柱狀適用 C.C.)

D: 用 polar coordinate 來表達.

$$\Rightarrow \iiint_E f(x,y,z)\,dz = \int_\alpha^\beta \int_{h_2(\theta)}^{h_2(\theta)} \int_{u_1}^{u_2} rf(r\cos\theta, r\sin\theta, z)\,dz\,dr\,d\theta$$

(2) Spherical Coordinates (S.C.)

$E=$ 球狀物體



Volume element in SC
$dV = \rho^2 \sin\phi\, d\theta\, d\phi$

Let $E = \{ (\rho,\theta,\phi) \mid a\le\rho\le b,\ \alpha\le\theta\le\beta,\ c\le\phi\le d \}$

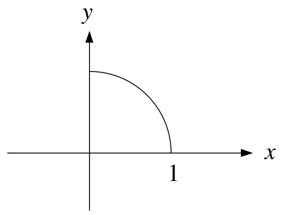
$$\iiint_E f(x,y,z)\,dV = \int_c^d \int_\alpha^\beta \int_a^b f\rho^2 \sin\phi\, d\rho\, d\theta\, d\phi$$

Example 1： $\displaystyle\iiint_{B(\text{單位球})} e^{(x^2+y^2+z^2)^{\frac{3}{2}}}\, dV$

$$= \int_0^\pi \int_0^{2\pi} \int_0^1 e^{\rho^3} \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi = \frac{4\pi}{3}(e-1)$$

Example 2： $\int_0^1 \int_0^{\sqrt{1-y^2}} \int_{x^2+y^2}^{\sqrt{x^2+y^2}} xyz \; dzdxdy$

$D =$



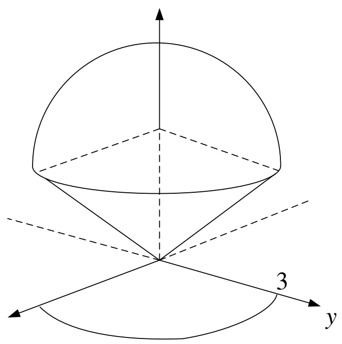
$$= \iint_D \frac{xy}{2}\left[x^2+y^2-\left(x^2+y^2\right)^2\right] dA$$

$$= \int_0^{\frac{\pi}{2}} \int_0^1 \frac{\sin 2\theta}{4}\left(r^2-r^4\right) r^3 dr d\theta = \frac{1}{96}$$

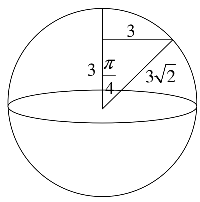
Example 3： $\int_0^3 \int_0^{\sqrt{9-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{18-x^2-y^2}} (x^2+y^2+z^2) \; dzdxdy$

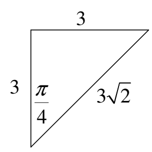
Solution：

E



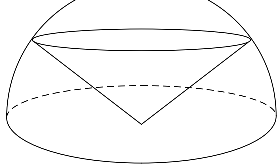
$$\sqrt{x^2+y^2} = \sqrt{18-x^2-y^2}$$
$$\Rightarrow x^2+y^2=9$$





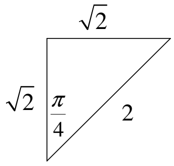
$$= \int_0^{\frac{\pi}{4}} \int_0^{\frac{\pi}{2}} \int_0^3 \rho^2 \rho^2 \sin\phi d\rho d\theta d\phi$$
$$= 486\pi\left(\frac{\sqrt{2}-1}{5}\right)$$

Example 4： Find the volume of the solid that lies within the sphere $x^2+y^2+z^2=4$, above the xy-plane and below the cone $z=\sqrt{x^2+y^2}$.



Solution：

$$\begin{cases} x^2+y^2+z^2=4 \\ \sqrt{x^2+y^2}=z \end{cases} \Rightarrow \ x^2+y^2=2$$

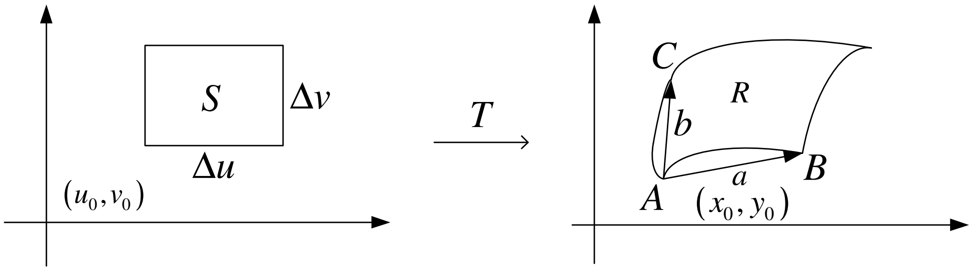


$$V=\int_0^{2\pi}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^2 1\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$=\frac{8\sqrt{2}}{3}\pi$$

§15.9 Change of Variables in Multiple Integrals

Homework： 3,5,9,11,13,15,17,19,21,23



$$T(u,v)=(x,y)=\big(x(u,v),\,y(u,v)\big)$$

$$\vec{a}=\overrightarrow{AB}=\big(\ x(u_0+\Delta u\ ,\ v_0)\ ,y(u_0+\Delta u\ ,\ v_0)\ \big)-\big(\ x(u_0,v_0)\ ,\ y(u_0,v_0)\ \big)$$

$$=\big(\ x(u_0+\Delta u\ ,\ v_0)-x(u_0,v_0)\ ,\ y(u_0+\Delta u\ ,\ v_0)-y(u_0,v_0)\ \big)$$

$$=\left(\frac{\partial x}{\partial u}\Delta u,\frac{\partial y}{\partial u}\Delta u\right)\Bigg|_{(u^*,v^0)}\qquad u^*\text{介於}u_0+\Delta u\ and\ u_0\text{之間}.$$

$$=\Delta u\left(\frac{\partial x}{\partial u},\frac{\partial y}{\partial u}\right)\Bigg|_{(u^*,v^0)}$$

同理

$$\vec{b}=\overrightarrow{AC}=\left(\frac{\partial x}{\partial v}\Delta v,\frac{\partial y}{\partial v}\Delta v\right)\Bigg|_{(u_0,v^*)}$$

$$=\Delta v\left(\frac{\partial x}{\partial v},\frac{\partial y}{\partial v}\right)\Bigg|_{(u_0,v^*)}$$

當 Δu 和 Δv 很小時

絕對值

$$R\text{ 的面積}\approx\left|\ \vec{a}\times\vec{b}\ \right|=\left|\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}\right|\Delta u\Delta v$$

$$=\left|\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}\right|\text{S 的面積}.$$

Definition：The Jacobian of the transf 倍率 T given by $x=g(u,v)$ and

$$y = h(u,v) \text{ is } \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = (\text{行列式值})$$

註(i) 當Δu ,Δv 夠小時轉換後的面積倍率為 Jacobian 的絕對值

(ii) $\Rightarrow dA = dxdy = |\text{Jacobian}|\ dudv$

(iii) 若 T 為 Linear Transformation, 則任一區域 S(不須小)經 T 轉換後之區域 R 面積為原面積的$|\text{Jacobian}|$倍.

Theorem：

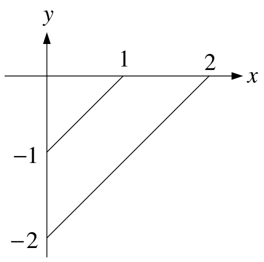
(i) $$\iint_R f(x,y)\,dA = \iint_S f\left(x(u,v),\, y(u,v)\right)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv$$

(ii) $$\iiint_R f(x,y,z)\,dV = \iiint_v f\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|\ dudvdw$$

註：(1) $\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} = -\rho^2 \sin\phi$.　(2) $\frac{\partial(x,y)}{\partial(r,\theta)} = r$.

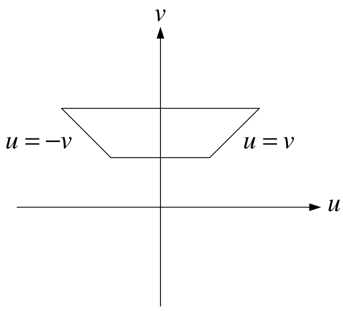
Example 1：$\iint_R e^{\frac{x+y}{x-y}}\ dA$.

R:



令 $u = x + y$
$v = x - y$

$\Rightarrow x = \frac{1}{2}(u+v)$
$y = \frac{1}{2}(u-v)$



$$= \iint_S e^{\frac{u}{v}} \left|\frac{\partial(x,y)}{\partial(u,v)}\right|\ dA = \frac{}{2}\iint_S e^{v}\ dA$$

$$= \frac{1}{2}\int_1^2\int_{-v}^{v} e^{\frac{u}{v}} dudv = \frac{3}{4}\left(e - e^{-1}\right)$$

Example 2： $\iiint\limits_E dV \qquad E: \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$

Set $x=au$, $y=bv$, $z=cw$

$$= \iiint\limits_{u^2+v^2+w^2\le 1} abc\ dV = \frac{4}{3}\pi abc$$

Example 3： $\iint\limits_R e^{x+y}\ dA. \qquad R: |x|+|y|\le 1$

Solution：

$$u = x + y \ , \ v = x - y$$

$$\Rightarrow x = \frac{1}{2}(u-v) \ , \ y = \frac{1}{2}(u+v)$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{2}$$

$$\Rightarrow \iint\limits_R e^{x+y}\ dA = \iint\limits_S \frac{1}{2}e^u dA$$

$$= \frac{1}{2}\int_{-1}^{1}\int_{-1}^{1} e^u du dv = e - e^{-1}$$