Chapter One: Mathematical preliminaries

Electrodynamics is a synthesis of several empirical laws: Coulomb's law, Faraday's law and Ampére's law. Even though these empirical laws had been discovered for some time before Maxwell obtained his famous modification on Ampére's law, they could not form a complete self-consistent theory. Consequently, the equations of electrodynamics are called Maxwell's equations. Maxwell Equations are the absolute necessity for understanding all technological developments in electricity and magnetism. All electrical devices, radio, television, microwaves, optics and to a large extent up to X-rays and γ-rays can be fully explained in terms of Maxwell's equations. Furthermore, Maxwell's equations already satisfy the all requirement of the special relativity. In addition to the classical wave theory, electrodynamics in the form of Maxwell's equations provides a rich feast for the developing the modern field theory. Modern theories of quarks are indeed based on a theory modeled on Maxwell's equations. The same as the field theory, many diverse areas of physics can be figured out more deeply from a study of electrodynamics. Historically, Maxwell tried to interpret electrodynamics as a seismological theory and called the elastic vibrations of an all-pervading substance as the æther. The description of electrodynamics is closely associated with the knowledge of vector calculus because electrodynamics is a field theory of a vector field. Therefore, it is fundamentally important to review some of knowledge of vector calculus.

1. Gradient

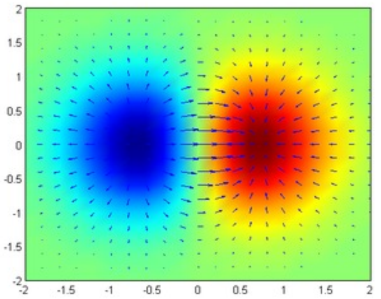
In mathematics, the gradient is a generalization of the usual concept of *derivative of a function* in one dimension to a function in several dimensions. Similarly to the usual derivative, the gradient represents the slope of the tangent of the graph of the function. More precisely, *the gradient points in the direction of the greatest rate of increase of the function and its magnitude is the slope of the graph in that direction*. The components of the gradient in coordinates are the coefficients of the variables in the equation of the tangent space to the graph. This characterizing property of the gradient allows it to be defined independently of a choice of coordinate system, as a vector field whose components in a coordinate system will transform when going from one coordinate system to another. The more general gradient, called simply the gradient in vector analysis, is a vector operator denoted ∇ and sometimes also called *del* or *nabla*. It is most often applied to a real function $f(q_1, q_2, q_3)$ of three variables , and may be denoted $\nabla f(q_1, q_2, q_3) \equiv \text{grad}(f)$. For general curvilinear coordinates, the gradient is given by

$$\nabla f = \frac{1}{h_1}\frac{\partial f}{\partial q_1}\hat{\mathbf{q}}_1 + \frac{1}{h_2}\frac{\partial f}{\partial q_2}\hat{\mathbf{q}}_2 + \frac{1}{h_3}\frac{\partial f}{\partial q_3}\hat{\mathbf{q}}_3 \quad . \tag{1.1}$$

In Cartesian coordinates, ∇f simplifies to

$$\nabla f = \frac{\partial f}{\partial x}\hat{\mathbf{x}} + \frac{\partial f}{\partial y}\hat{\mathbf{y}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}} \quad . \tag{1.2}$$

The direction of ∇f is the orientation in which the directional derivative has the largest value and $|\nabla f|$ is the value of that directional derivative.



Gradient of the 2-d function $f(x, y) = x\exp[-(x^2+y^2)]$ is plotted as blue arrows over the pseudocolor plot of the function.

The gradient of a function is called a gradient field. A (continuous) gradient field is always a conservative vector field: its line integral along any path depends only on the endpoints of the path, and can be evaluated by the gradient theorem (the fundamental theorem of calculus for line integrals). Conversely, a (continuous) conservative vector field is always the gradient of a function. The ***gradient theorem***, also known as the fundamental theorem of calculus for **line integrals**, says that a line integral through a gradient field can be evaluated by evaluating the original scalar field at the endpoints of the curve. Let $\Phi : U \subseteq \mathbf{R}^n \to \mathbf{R}$. Now suppose the domain U of Φ contains the differentiable curve γ with endpoints **p** and **q**, (oriented in the direction from **p** to **q**). Then

$$\Phi(\mathbf{q}) - \Phi(\mathbf{p}) = \int_{\gamma[\mathbf{p},\,\mathbf{q}]} \nabla\Phi(\mathbf{r}) \cdot d\mathbf{r} \, .$$

It is a generalization of the fundamental theorem of calculus to any curve in a plane or space (generally n-dimensional) rather than just the real line. The gradient theorem implies that line integrals through gradient fields are path independent. In physics this theorem is one of the ways of defining a "conservative" force. By placing Φ as potential, $\nabla\Phi$ is a conservative field. Work done by conservative forces does not depend on the path followed by the object, but only the end points, as the above equation shows. The gradient theorem also has an interesting converse: any path-independent vector field can be expressed as the gradient of a scalar field. Just like the gradient theorem itself, this converse has many striking consequences and applications in both pure and applied mathematics.

2. Divergence

In vector calculus, divergence is a vector operator that measures the magnitude of a vector field's source or sink at a given point, in terms of a signed scalar. More technically, *the divergence represents the volume density of the outward flux of a vector field from an infinitesimal volume around a given point*. It is a local measure of its "outgoingness"—the extent to which there is more exiting an infinitesimal region of space than entering it. If the divergence is nonzero at some point then there must be a source or sink at that position. More rigorously, the divergence of a vector field **F** at a point **p** is defined as the limit of the net flow of **F** across the smooth boundary of a three-dimensional region V divided by the volume of V as V shrinks to **p**. Formally,

$$\operatorname{div} \mathbf{F} = \lim_{V \to \{\mathbf{p}\}} \frac{1}{V} \oint_S \mathbf{F} \cdot \mathbf{n} \, da , \tag{2.1}$$

where S is the surface enclosing the volume V and the integral is a surface integral with **n** being the outward unit normal to that surface. The result, div **F**, is a function of **p**. From this definition it also becomes explicitly visible that div **F** can be seen as the source density of the flux of **F**. In light of the physical interpretation, a vector field with constant zero divergence is called *incompressible* or *solenoidal* – in this case, no net flow can occur across any closed surface. The intuition that the sum of all sources minus the sum of all sinks should give the net flow outwards of a region is made precise by the divergence theorem.

A formula for the divergence of a vector field can immediately be written down in Cartesian coordinates by constructing a hypothetical infinitesimal cubical box oriented along the coordinate axes around an infinitesimal region of space. There are six sides to this box, and the net "content" leaving the box is therefore simply the sum of differences in the values of the vector field along the three sets of parallel sides of the box. Writing $F = (F_x, F_y, F_z)$, it therefore following immediately that

$$\nabla \cdot F = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \quad . \tag{2.2}$$

This formula also provides the motivation behind the adoption of the symbol ∇ for the divergence. Interpreting ∇ as the gradient operator $\nabla = (\partial/\partial x, \partial/\partial y, \partial/\partial z)$, the "dot product" of this vector operator with the original vector field $F = (F_x, F_y, F_z)$ is precisely equation (2.2). While this derivative seems to in some way favor Cartesian coordinates, the general definition is completely free of the coordinates chosen. In fact, defining

$$\mathbf{F} = F_1\,\hat{\mathbf{q}}_1 + F_2\,\hat{\mathbf{q}}_2 + F_3\,\hat{\mathbf{q}}_3 \quad , \tag{2.3}$$

the divergence in arbitrary orthogonal curvilinear coordinates is simply given by

$$\nabla \cdot \mathbf{F} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial(h_2 h_3 F_1)}{\partial q_1} + \frac{\partial(h_3 h_1 F_2)}{\partial q_2} + \frac{\partial(h_1 h_2 F_3)}{\partial q_3}\right] \quad . \tag{2.4}$$

For a vector expressed in cylindrical coordinates as

$$\mathbf{F} = \mathbf{a}_{\boldsymbol{\rho}} F_\rho + \mathbf{a}_{\boldsymbol{\varphi}} F_\phi + \mathbf{a}_{\mathbf{z}} F_z \tag{2.5}$$

where $\mathbf{a}_i$ is the unit vector in direction *i*, the divergence is

$$\nabla \cdot \mathbf{F} = \frac{1}{\rho}\frac{\partial(\rho F_\rho)}{\partial \rho} + \frac{1}{\rho}\frac{\partial F_\phi}{\partial \phi} + \frac{\partial F_z}{\partial z} \quad . \tag{2.6}$$

In spherical coordinates, with θ the angle with the *z* axis and ϕ the rotation around the *z* axis, the divergence reads

$$\nabla \cdot \mathbf{F} = \frac{1}{r^2}\frac{\partial(r^2 F_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta\, F_\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial \phi} \quad . \tag{2.7}$$

There is a product rule of the following type: if Φ is a scalar valued function and **F** is a vector field, then

$$\nabla \cdot (\Phi \mathbf{F}) = (\nabla \Phi) \cdot \mathbf{F} + \Phi\,(\nabla \cdot \mathbf{F})\,. \tag{2.8}$$

Another product rule for the cross product of two vector fields **F** and **G** in three dimensions involves the curl and reads as follows:

$$\nabla \cdot (\mathbf{F} \times \mathbf{G}) = (\nabla \times \mathbf{F}) \cdot \mathbf{G} - \mathbf{F} \cdot (\nabla \times \mathbf{G}) \quad . \tag{2.9}$$

The Laplacian of a scalar field is the divergence of the field's gradient:

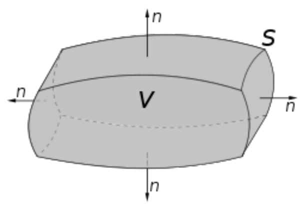
$$\nabla \cdot (\nabla \Phi) = \nabla^2 \Phi = \Delta \Phi \quad . \tag{2.10}$$

The divergence of the curl of any vector field (in three dimensions) is equal to zero:

$$\nabla \cdot (\nabla \times \mathbf{F}) = 0 \quad . \tag{2.11}$$

If a vector field **F** with zero divergence is defined on a ball in $\mathbf{R}^3$, then there exists some vector field **G** on the ball with **F** = curl(**G**).

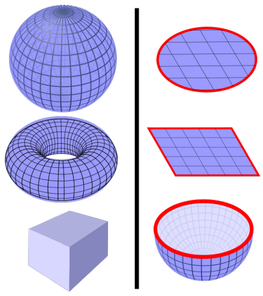
In vector calculus, the **divergence theorem**, also known as **Gauss's theorem** or **Ostrogradsky's theorem**, is a result that relates the flow (that is, flux) of a vector field through a surface to the behavior of the vector field inside the surface. More precisely, *the divergence theorem states that the outward flux of a vector field through a closed surface is equal to the volume integral of the divergence over the region inside the surface*. Suppose *V* is a subset of $\mathbf{R}^n$ (in the case of $n = 3$, *V* represents a volume in 3D space) which is compact and has a piecewise smooth boundary *S* (also indicated with $\partial V = S$). If **F** is a continuously differentiable vector field defined on a neighborhood of *V*, then we have:



$$\int_V \nabla \cdot \mathbf{F} \, d^3 r = \oint_S \mathbf{F} \cdot \mathbf{n} \, da$$

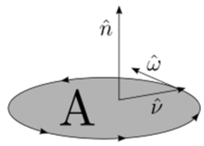
The left side is a volume integral over the volume *V*, the right side is the surface integral over the boundary of the volume *V*. The closed manifold ∂V is quite generally the boundary of *V* oriented by outward-pointing normals, and **n** is the outward pointing unit normal field of the boundary ∂V. ($d\mathbf{S}$ may be used as a shorthand for **n***da*.) The symbol within the two integrals stresses once more that ∂V is a *closed* surface. In terms of the intuitive description above, the left-hand side of the equation represents the total of the sources in the volume *V*, and the right-hand side represents the total flow across the boundary *S*.

The divergence theorem can be used to calculate a flux through a closed surface that fully encloses a volume, like any of the surfaces on the left. It can *not* directly be used to calculate the flux through surfaces with boundaries, like those on the right. (Surfaces are blue, boundaries are red.)



3. Curl

In vector calculus, the **curl** is a vector operator that describes the infinitesimal rotation of a 3-dimensional vector field. At every point in the field, the curl of that point is represented by a vector. The attributes of this vector (length and direction) characterize the rotation at that point. *The direction of the curl is the axis of rotation, as determined by the right-hand rule, and the magnitude of the curl is the magnitude of rotation*. If the vector field represents the flow velocity of a moving fluid, then the curl is the **circulation density** of the fluid. A vector field whose curl is zero is called **irrotational**. The curl is a form of differentiation for vector fields. The corresponding form of the fundamental theorem of calculus is Stokes' theorem, which *relates the surface integral of the curl of a vector field to the line integral of the vector field around the boundary curve*. The curl of a vector field **F**, denoted by curl **F**, or ∇ × **F**, or rot **F**, at a point is defined in terms of its projection onto various lines through the point. If **n** is any unit vector, the projection of the curl of **F** onto **n** is defined to be the limiting value of a closed line integral in a plane orthogonal to **n** as the path used in the integral becomes infinitesimally close to the point, divided by the area enclosed. Implicitly, curl is defined by:



Convention for vector orientation of the line integral

$$(\nabla \times \mathbf{F}) \cdot \mathbf{n} = \lim_{A \to 0} \frac{1}{A} \oint_C \mathbf{F} \cdot d\mathbf{r} \quad , \qquad (3.1)$$

where the right hand side is a line integral along the boundary of the area in question, and A is the magnitude of the area. If $\hat{\nu}$ is an outward pointing in-plane normal, whereas $\hat{n}$ is the unit vector perpendicular to the plane (see caption at right), then the orientation of C is chosen so that a tangent vector $\hat{\omega}$ to C is positively oriented if and only if $\{\hat{n}, \hat{\nu}, \hat{\omega}\}$ forms a positively oriented basis for $\mathbf{R}^3$ (right-hand rule). The above formula means that the curl of a vector field is defined as the infinitesimal *area density* of the *circulation* of that field.

The name "curl" was first suggested by James Clerk Maxwell in 1871 but the concept was apparently first used in the construction of an optical field theory by James MacCullagh in 1839. The curl can be defined in arbitrary orthogonal curvilinear coordinates using $\mathbf{F} = F_1\,\hat{\mathbf{q}}_1 + F_2\,\hat{\mathbf{q}}_2 + F_3\,\hat{\mathbf{q}}_3$ and defining $h_i \equiv |\partial \mathbf{r}/\partial q_i|$ as

$$\nabla\times\mathbf{F} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\hat{\mathbf{q}}_1 & h_2\hat{\mathbf{q}}_2 & h_3\hat{\mathbf{q}}_3 \\ \dfrac{\partial}{\partial q_1} & \dfrac{\partial}{\partial q_2} & \dfrac{\partial}{\partial q_3} \\ h_1F_1 & h_2F_2 & h_3F_3 \end{vmatrix} \quad . \tag{3.2}$$

Each component can be expressed as

$$(\nabla\times\mathbf{F})_1 = \frac{1}{h_2h_3}\left[\frac{\partial(h_3F_3)}{\partial q_2} - \frac{\partial(h_2F_2)}{\partial q_3}\right] \tag{3.3}$$

$$(\nabla\times\mathbf{F})_2 = \frac{1}{h_3h_1}\left[\frac{\partial(h_1F_1)}{\partial q_3} - \frac{\partial(h_3F_3)}{\partial q_1}\right] , \tag{3.4}$$

$$(\nabla\times\mathbf{F})_3 = \frac{1}{h_1h_2}\left[\frac{\partial(h_2F_2)}{\partial q_1} - \frac{\partial(h_1F_1)}{\partial q_2}\right] , \tag{3.5}$$

Note that the equation for each component, $(\nabla\times\mathbf{F})_k$ can be obtained by exchanging each occurrence of a subscript 1, 2, 3 in cyclic permutation: 1→2, 2→3, and 3→1 (where the subscripts represent the relevant indices).

In general curvilinear coordinates (not only in Cartesian coordinates), the curl of a cross product of vector fields **G** and **F** can be shown to be

$$\nabla\times(\mathbf{F}\times\mathbf{G}) = [(\nabla\cdot\mathbf{G}) + \mathbf{G}\cdot\nabla]\mathbf{F} - [(\nabla\cdot\mathbf{F}) + \mathbf{F}\cdot\nabla]\mathbf{G} . \tag{3.6}$$

Interchanging the vector field **F** and ∇ operator, we arrive at the cross product of a vector field with curl of a vector field:

$$\mathbf{F}\times(\nabla\times\mathbf{G}) = \nabla_G(\mathbf{F}\cdot\mathbf{G}) - (\mathbf{F}\cdot\nabla)\mathbf{G} \quad , \tag{3.7}$$

using the Feynman subscript notation, ∇_G, which operates only on the vector field **G**. Another example is the curl of a curl of a vector field. It can be shown that in general coordinates

$$\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F} \quad . \tag{3.8}$$

and this identity defines the vector Laplacian of **F**, symbolized as ∇^2**F**. The curl of the gradient of *any* scalar field Φ is always the zero vector field

$$\nabla \times (\nabla \Phi) = 0 \quad , \tag{3.9}$$

which follows from the antisymmetry in the definition of the curl, and the symmetry of second derivatives. If Φ is a scalar valued function and **F** is a vector field, then

$$\nabla \times (\Phi \mathbf{F}) = \nabla \Phi \times \mathbf{F} + \Phi \nabla \times \mathbf{F} \, . \tag{3.10}$$

Chapter Two: Electrostatics

1. Coulomb's Law & Gauss's Law

Coulomb's law for the electric field at the point $\boldsymbol{r}$ due to a point charge q_1 at the point $\boldsymbol{r}_1$ is given by

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_o} q_1 \frac{\mathbf{r}-\mathbf{r}_1}{|\mathbf{r}-\mathbf{r}_1|^3} \quad . \tag{1.1}$$

In the SI system, the constant $(4\pi\varepsilon_o)^{-1}$ is equal to $10^{-7}c^2$. The experimentally observed linear superposition of forces due to many charges means that the electric field at $\boldsymbol{r}$ due to a system of point charges q_i located at $\boldsymbol{r}_i$, i=1,2,….,n, is expressed as the vector sum

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_o} \sum_{i=1}^{n} q_i \frac{\mathbf{r}-\mathbf{r}_i}{|\mathbf{r}-\mathbf{r}_i|^3} \quad . \tag{1.2}$$

If the charges are so small and so numerous that they can be described by a charge density $\rho(\boldsymbol{r}')$, Eq. (1.2) is replaced by an integral:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_o} \int \rho(\mathbf{r}') \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} d^3r' \quad , \tag{1.3}$$

where $d^3r' = dx'\,dy'\,dz'$ is a three-dimensional volume element at $\boldsymbol{r}'$.

Gauss's law is sometimes more useful and furthermore leads to a differential equation for $\mathbf{E}(\mathbf{r})$. Considering a point charge q and a closed surface S, as shown in Fig. 1.2, the flux of the electric field produced by this charge through the surface element da is given by

$$\mathbf{E}\cdot\mathbf{n}\,da = \frac{q}{4\pi\varepsilon_o}\frac{\cos\theta}{r^2} da \quad , \tag{1.4}$$

where $\boldsymbol{r}$ is the distance from the charge to the surface element, n is the outwardly directed unit normal to the surface element, and θ is the angle between the $\mathbf{E}$ and $\mathbf{n}$. In terms of the solid angle, we have $\cos\theta\, da = r^2\, d\Omega$. Therefore,

$$\mathbf{E}\cdot\mathbf{n}\,da = \frac{q}{4\pi\varepsilon_o} d\Omega \quad . \tag{1.5}$$

Integrating the normal component of $\mathbf{E}$ over the whole surface, it can be shown that

$$\oint_S \mathbf{E}\cdot\mathbf{n}\,da = \begin{cases} q/\varepsilon_o & \text{if } q \text{ lies inside } S \\ 0 & \text{if } q \text{ lies outside } S \end{cases} \quad . \tag{1.6}$$

This result is Gauss's law for a single point charge. For a discrete set of charges, it is immediately apparent that

$$\oint_S \mathbf{E}\cdot\mathbf{n}\, da = \frac{1}{\varepsilon_o}\sum_i q_i \quad , \tag{1.7}$$

where the sum is over only those charge inside the surface *S*. For a continuous charge density $\rho(\mathbf{r})$, Gauss's law becomes

$$\oint_S \mathbf{E}\cdot\mathbf{n}\, da = \frac{1}{\varepsilon_o}\int_V \rho(\mathbf{r})\, d^3r \quad , \tag{1.8}$$

where *V* is the volume enclosed by *S*. The surface integral on the left-hand side is transformed into a volume integral with the help of the Gauss theorem:

$$\oint_S \mathbf{E}\cdot\mathbf{n}\, da = \int_V \nabla\cdot\mathbf{E}\, d^3r \quad . \tag{1.9}$$

Hence,

$$\int_V \nabla\cdot\mathbf{E}\, d^3r = \frac{1}{\varepsilon_o}\int_V \rho(\mathbf{r})\, d^3r \quad . \tag{1.10}$$

Since this is valid for an arbitrary volume, we obtain the relation

$$\nabla\cdot\mathbf{E} = \rho(\mathbf{r})/\varepsilon_o \quad . \tag{1.11}$$

This equation indicates that the charges in space are the sources (positive charges) and sinks (negative charges) of the electric field.

2. Electric Potential

Next the electric field can be written as the gradient of a potential. From

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_o}\int \rho(\mathbf{r}')\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}d^3r' \quad , \tag{2.1}$$

and using the fact that

$$\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} = -\nabla\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) \quad , \tag{2.2}$$

the electric field can be given by

$$\mathbf{E}(\mathbf{r}) = \frac{-1}{4\pi\varepsilon_o}\nabla\left(\int \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^3r'\right) \quad . \tag{2.3}$$

Thus, the field intensity can be derived as the gradient of a potential. The potential $\Phi(\mathbf{r})$ is obtained as the integral over the entire charge distribution:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_o}\int \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^3r' \, . \tag{2.4}$$

With this definition, we can write for the field intensity

$$\mathbf{E}(\mathbf{r}) = -\nabla\Phi(\mathbf{r}) . \tag{2.5}$$

Since the curl of the gradient of any well-behaved scalar function of position vanishes, we obtain

$$\nabla \times \mathbf{E}(\mathbf{r}) = 0 \quad . \tag{2.6}$$

This equation also implies that electrostatic forces are conservative forces. In other words, the electrostatic field is irrotational. Combining Gauss's law and Eq. (5), we have

$$\nabla^2\Phi(\mathbf{r}) = -\rho(\mathbf{r})/\varepsilon_o \quad . \tag{2.7}$$

This equation is called Poisson's equation; for a charge-free region $\rho = 0$, and Poisson's equation reduces to the Laplace equation

$$\nabla^2\Phi(\mathbf{r}) = 0 \quad . \tag{2.8}$$

3. Discontinuities in the Electric Field and Potential

The determination of electric field or potential due to a given surface distribution of charges is one of the common problems in electrostatics. Considering that a surface *S*, with a unit normal **n** directed from side 1 to side 2 of the surface, has a surface-charge density of $\sigma(\mathbf{r})$ and electric fields $\mathbf{E}_1$ and $\mathbf{E}_2$ on either side of the surface, as shown in Fig. 1.4, Gauss's law gives immediately that

$$(\mathbf{E}_2 - \mathbf{E}_1) \cdot \mathbf{n} = \sigma/\varepsilon_o \quad . \tag{3.1}$$

This equation indicates that there is a discontinuity of σ/ε_o in the normal component of electric field in crossing a surface with a surface-charge density σ, the crossing being made in the direction of **n**. On the other hand, the tangential component of electric field can be shown to be continuous across a boundary surface by using the line integral for **E** around a closed path. It is only necessary to take a rectangular path with negligible ends and one side on either side of the boundary.

Considering that a surface *S* has a surface-charge density of $\sigma(\boldsymbol{r})$, the potential at any point in space is given by

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_o}\int_S \frac{\sigma(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}da' \quad . \tag{3.2}$$

For volume or surface distributions of charge, the potential is everywhere continuous, even within the charge distribution. This can be shown from Eq. (2) or from the fact that **E** is bounded, even though discontinuous across a surface distribution of charge. However, the potential is no longer continuous with point or line charges, or dipole layers.

A dipole layer can be imagined as being formed by letting the surface S have a surface-charge density $\sigma(\boldsymbol{r})$ on it, and another surface S', lying close to S, have an equal and opposite surface-charge density on it at neighboring points, as shown in Fig. 1.5. The dipole-layer distribution of strength $D(\boldsymbol{r})$ is formed by letting S' approach infinitesimally close to S while the surface-charge density $\sigma(\boldsymbol{r})$ become infinite in such a manner that the product of $\sigma(\boldsymbol{r})$ and the local separation $d(\boldsymbol{r})$ of S and S' approached the limit $D(\boldsymbol{r})$:

$$\lim_{d(\mathbf{r})\to 0} \sigma(\mathbf{r})d(\mathbf{r}) = D(\mathbf{r})\,. \tag{3.3}$$

The direction of the dipole moment of the layer is normal to the surface S and in the direction going from negative to positive charge.

With $\mathbf{n}$, the unit normal to the surface S, directed away from S', as shown in Fig. 1.6, the potential due to the two close surfaces is

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_o}\int_S \frac{\sigma(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}da' - \frac{1}{4\pi\varepsilon_o}\int_{S'} \frac{\sigma(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'+\mathbf{n}d|}da' \quad . \tag{3.4}$$

With a Taylor series expansion in three dimensions, the term $|\mathbf{r}-\mathbf{r}'+\mathbf{n}d|^{-1}$ can be expressed as

$$\frac{1}{|\mathbf{r}-\mathbf{r}'+\mathbf{n}d|} = \frac{1}{|\mathbf{r}-\mathbf{r}'|} + (-\mathbf{n}d)\cdot\nabla'\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) + \cdots\,. \tag{3.5}$$

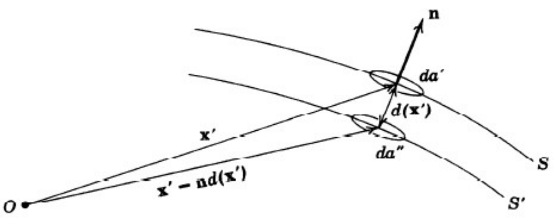
Using Eq. (5) and taking $d \to 0$, the potential becomes

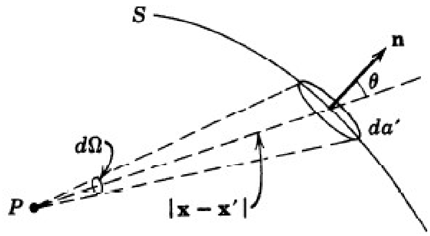
$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_o}\int_S D(\mathbf{r}')\mathbf{n}\cdot\nabla'\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)da' \quad . \tag{3.6}$$

The geometrical interpretation for Eq. (6) is given by

$$\mathbf{n}\cdot\nabla'\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)da' = -\frac{\cos\theta\, da'}{|\mathbf{r}-\mathbf{r}'|^2} = -d\Omega \quad , \tag{3.7}$$

where $d\Omega$ is the element of solid angle subtended at the observation point by the area element da', as indicated in Fig. 1.7.





Note that if θ is an acute angle, $d\Omega$ has a positive sign. Then the potential can be written as

$$\Phi(\mathbf{r}) = -\frac{1}{4\pi\varepsilon_o}\int_S D(\mathbf{r}')\, d\Omega \quad . \tag{3.8}$$

For a constant surface-dipole-moment density D, the potential is just the product of the moment divided by $4\pi\varepsilon_o$ and the solid angle subtended at the observation point by the surface, regardless of its shape.

There is a discontinuity in potential in crossing a double layer. This can be seen by letting the observation point come infinitesimally close to the double layer. The double layer is now imagined to consist of two parts, one being a small disc directly under the observation point. The disc is sufficiently small that it is sensibly flat and has constant surface-dipole-moment density D. Evidently the total potential can be obtained by linear superposition of the potential of the disc and that of the remainder. From Eq. (3.8) it is clear that the potential of the disc alone has a discontinuity of D/ε_o in crossing from the inner to the outer side, being $-D/2\varepsilon_o$ on the inner side and $D/2\varepsilon_o$ on the outer. The potential of the remainder alone, with its hole where the disc fits in, is continuous across the plane of the hole. As a result, the total potential jump in crossing the surface is

$$\Phi_2 - \Phi_1 = D/\varepsilon_o \quad . \tag{3.9}$$

4. The energy of a charge distribution

The potential energy of a point charge q in a scalar potential Φ is the product of both, $W = q\Phi$. The potential energy of a number of point charges can be calculated in the following way. It can be imagined that the charges q_i are infinitely far from each other to calculate the work required to bring them from infinitely to a certain separation $\boldsymbol{r}_i$. Now, the charge q_1 is shifted from infinity to $\mathbf{r}_1$. Since the space is still field-free, no work has to be done. However, the charge q_1 causes a potential

$$\Phi_1(\mathbf{r}) = \frac{1}{4\pi\varepsilon_o}\frac{q_1}{|\,\mathbf{r}-\mathbf{r}_1\,|} \quad . \tag{4.1}$$

One has to do work against this potential to bring the charge q_2 from infinity to $\mathbf{r}_2$. The work

is given by

$$W_2 = q_2 \Phi_1(\mathbf{r}_2) \quad . \tag{4.2}$$

For the charge q_3 one now has to expand work against the potentials $\Phi_1(\mathbf{r})$ and $\Phi_2(\mathbf{r})$:

$$W_3 = q_3 [\Phi_1(\mathbf{r}_3) + \Phi_2(\mathbf{r}_3)] \quad , \tag{4.3}$$

where $\Phi_2(\mathbf{r})$ is given by

$$\Phi_2(\mathbf{r}) = \frac{1}{4\pi\varepsilon_o} \frac{q_2}{|\mathbf{r} - \mathbf{r}_2|} \quad . \tag{4.4}$$

For other charges everything proceeds correspondingly. For the transport of the charge q_n, one has to spend the work

$$W_n = q_n [\Phi_1(\mathbf{r}_n) + \Phi_2(\mathbf{r}_n) + \cdots + \Phi_{n-1}(\mathbf{r}_n)] = q_n \sum_{k=1}^{n-1} \Phi_k(\mathbf{r}_n) \quad . \tag{4.5}$$

The total potential energy is given by the sum of all W_n:

$$W = \sum_{n=2}^{N} W_n = \sum_{n=2}^{N} q_n \sum_{k=1}^{n-1} \Phi_k(\mathbf{r}_n) \quad , \tag{4.6}$$

where $\Phi_k(\mathbf{r}_n)$ is given by

$$\Phi_k(\mathbf{r}_n) = \frac{1}{4\pi\varepsilon_o} \frac{q_k}{|\mathbf{r}_n - \mathbf{r}_k|} \quad . \tag{4.7}$$

Substituting Eq. (7) into Eq. (6), the total energy can be expressed as

$$W = \sum_{n=2}^{N} W_n = \frac{1}{4\pi\varepsilon_o} \sum_{n=2}^{N} \sum_{k=1}^{n-1} \frac{q_n q_k}{|\mathbf{r}_n - \mathbf{r}_k|} = \frac{1}{2} \left[\frac{1}{4\pi\varepsilon_o} \sum_{n=1}^{N} \sum_{\substack{k=1 \\ k \neq n}}^{N} \frac{q_n q_k}{|\mathbf{r}_n - \mathbf{r}_k|} \right] \quad . \tag{4.8}$$

where the terms $n = k$ need to be omitted.

For a continuous charge distribution, the point charge q_n is replaced by the charge element $\rho(\mathbf{r})\, d^3r$, and the summation becomes an integration

$$W = \frac{1}{2} \frac{1}{4\pi\varepsilon_o} \int \int \frac{\rho(\mathbf{r}')\rho(\mathbf{r})}{|\mathbf{r} - \mathbf{r}'|} d^3r' d^3r \quad . \tag{4.9}$$

Note that an important difference between the sum in Eq. (4.8) and the integral in Eq. (4.9) is that the integration extends over the point $\mathbf{r} = \mathbf{r}'$, so that equation (4.9) contains automatically *self-energy* parts which become infinitely large for point charge. An detailed description will be given in the following. Since the potential of a charge distribution is given by

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_o} \int \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3r' \quad , \tag{4.10}$$

Eq. (9) can be written in the form

$$W = \frac{1}{2}\int_V \rho(\mathbf{r})\Phi(\mathbf{r})d^3r \quad . \tag{4.11}$$

Now the interaction energy can be in terms of an integral over the field intensity. The charge density satisfies the Poisson equation

$$\nabla^2\Phi(\mathbf{r}) = -\rho(\mathbf{r})/\varepsilon_o \quad . \tag{4.12}$$

Substituting the charge density in Eq. (11) with Eq. (12), we obtain

$$W = \frac{-\varepsilon_o}{2}\int_V \Phi(\mathbf{r})\nabla^2\Phi(\mathbf{r})\, d^3r \quad . \tag{4.13}$$

Using the identity

$$\Phi\nabla^2\Phi = \nabla\cdot(\Phi\nabla\Phi) - (\nabla\Phi)^2 \quad , \tag{4.14}$$

Eq. (13) can be rewritten as

$$W = \frac{\varepsilon_o}{2}\int_V [\nabla\Phi(\mathbf{r})]^2\, d^3r - \frac{\varepsilon_o}{2}\int_V \nabla\cdot(\Phi\nabla\Phi)\, d^3r \quad . \tag{4.15}$$

The second volume integral can be converted to a surface integral by Gauss theorem:

$$\frac{\varepsilon_o}{2}\int_V \nabla\cdot(\Phi\nabla\Phi)\, d^3r = \frac{\varepsilon_o}{2}\oint_S (\Phi\nabla\Phi)\cdot\mathbf{n}\, da \quad . \tag{4.16}$$

This integral vanishes because approaching to the infinity we have $\Phi \sim 1/r$, $\nabla\Phi \sim 1/r^2$, so that the integrand with $1/r^3$ tends more rapidly to zero than the area element $a \sim r^2$ tends to infinity. Using $\mathbf{E}(\mathbf{r}) = -\nabla\Phi(\mathbf{r})$, the total energy in Eq. (15) can be expressed as

$$W = \frac{\varepsilon_o}{2}\int_V [\nabla\Phi(\mathbf{r})]^2\, d^3r = \frac{\varepsilon_o}{2}\int_V \mathbf{E}^2\, d^3r \quad . \tag{4.17}$$

This leads naturally to the identification of the integrand as an energy density *w*:

$$w = \frac{\varepsilon_o}{2}\mathbf{E}^2 \quad . \tag{4.18}$$

Chapter Three: Boundary Value Problems in Electrostatics (I)

1. Green's Theorem and Green function

When electrostatic problems involved localized discrete or continuous distributions of charge with no boundary surfaces, the most convenient and straightforward solution to any problem is given by

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_o}\int \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} d^3r' \quad .$$

There would be no need of the Poisson or Laplace equation. However, many of the problems of electrostatics involve finite regions of space, with or without charge inside, and with prescribed boundary conditions on the bounding surfaces. To deal with the boundary conditions it is necessary to exploit the mathematical method based on Green's theorem (George Green 1824). Two Green's identities follow as simple applications of the divergence theorem. For any well-behaved vector field **A** defined in the volume V bounded by the closed surface S, the divergence theorem is expressed as

$$\int_V \nabla' \cdot \mathbf{A}\, d^3r' = \oint_S \mathbf{A} \cdot \mathbf{n}'\, da' \tag{1.1}$$

where **n** is the outwardly directed unit vector normal to the surface and da be an element of surface area. Considering the vector filed A to be associated with two arbitrary scalar fields ϕ and ψ as $\mathbf{A} = \phi\, \nabla' \psi$, we have

$$\nabla' \cdot \mathbf{A} = \nabla' \cdot \left(\phi\, \nabla' \psi\right) = \phi\, \nabla'^2 \psi + \nabla' \phi \cdot \nabla' \psi \tag{1.2}$$

and

$$\mathbf{A} \cdot \mathbf{n}' = \phi\, \nabla' \psi \cdot \mathbf{n}' = \phi \frac{\partial \psi}{\partial n'} \quad , \tag{1.3}$$

where $\partial/\partial n'$ is the normal derivative at the surface S (directed outward from inside the volume V). Substituting Eqs. (2) and (3) into Eq. (1) leads to Green's first identity:

$$\int_V \left(\phi\, \nabla'^2 \psi + \nabla' \phi \cdot \nabla' \psi\right) d^3r = \oint_S \phi \frac{\partial \psi}{\partial n'}\, da \quad . \tag{1.4}$$

When Eq. (4) was written again with ϕ anf ψ interchanged, and then subtract it from Eq. (4), the terms $\nabla' \phi \cdot \nabla' \psi$ cancel. Green's second identity or Green's theorem can be obtained as

$$\int_V \left(\phi\, \nabla'^2 \psi - \psi\, \nabla'^2 \phi\right) d^3r' = \oint_S \left[\phi \frac{\partial \psi}{\partial n'} - \psi \frac{\partial \phi}{\partial n'}\right] da' \quad . \tag{1.5}$$

With the help of the second Green's theorem we can calculate the solution of the Poisson equation or the Laplace equation within a certain bounded volume with known Dirichlet or Neumann boundary conditions by means of so-called Green functions. The Green function is generally obtained from

$$\nabla'^2 G(\mathbf{r},\mathbf{r}') = -4\pi\delta(\mathbf{r}-\mathbf{r}') \tag{1.6}$$

with

$$G(\mathbf{r},\mathbf{r}') = \frac{1}{|\mathbf{r}-\mathbf{r}'|} + F(\mathbf{r},\mathbf{r}') \quad , \tag{1.7}$$

where F has to fulfill the Laplace equation $\nabla'^2 F(\mathbf{r},\mathbf{r}') = 0$. Note that although the function $1/|\mathbf{r}-\mathbf{r}'|$ obeys Poisson equation in Eq. (6), it does not satisfy Dirichlet or Neumann boundary conditions, except if the surface lies at infinity. For the Green function $G(\mathbf{r},\mathbf{r}')$ the boundary conditions can be taken into account via the functions $F(\mathbf{r},\mathbf{r}')$.

Considering the Poisson equation $\nabla'^2\Phi(\mathbf{r}') = -\rho(\mathbf{r}')/\varepsilon_o$, we can set $\psi = G(\mathbf{r},\mathbf{r}')$ and $\phi = \Phi(\mathbf{r}')$ in Eq. (5) and use Eq. (6) to obtain

$$\begin{aligned} &-4\pi\int_V \Phi(\mathbf{r}')\,\delta(\mathbf{r}-\mathbf{r}')\ d^3r' + \int_V G(\mathbf{r},\mathbf{r}')\,\rho(\mathbf{r}')/\varepsilon_o\ d^3r' \\ &= \oint_S \left[\Phi(\mathbf{r}')\frac{\partial G(\mathbf{r},\mathbf{r}')}{\partial n'} - G(\mathbf{r},\mathbf{r}')\frac{\partial \Phi(\mathbf{r}')}{\partial n'}\right] da' \end{aligned} \quad . \tag{1.8}$$

After some algebra, Eq. (8) can be given by

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_o}\int_V G(\mathbf{r},\mathbf{r}')\,\rho(\mathbf{r}')\ d^3r' + \frac{1}{4\pi}\oint_S \left[G(\mathbf{r},\mathbf{r}')\frac{\partial \Phi(\mathbf{r}')}{\partial n'} - \Phi(\mathbf{r}')\frac{\partial G(\mathbf{r},\mathbf{r}')}{\partial n'}\right] da' . \tag{1.9}$$

One or the other of surface integrals in Eq. (9) can be eliminated by appropriately choosing the boundary condition for $G(\mathbf{r},\mathbf{r}')$. For Dirichlet boundary conditions, we demand:

$$G_D(\mathbf{r},\mathbf{r}') = 0 \quad \text{for } \boldsymbol{r}' \text{ on } S. \tag{1.10}$$

Then Eq. (9) becomes

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_o}\int_V G_D(\mathbf{r},\mathbf{r}')\,\rho(\mathbf{r}')\ d^3r' - \frac{1}{4\pi}\oint_S \Phi(\mathbf{r}')\frac{\partial G_D(\mathbf{r},\mathbf{r}')}{\partial n'}\, da' \quad . \tag{1.11}$$

For Neumann boundary conditions the obvious ansatz $\partial G_N(\mathbf{r},\mathbf{r}')/\partial n' = 0$ leads to a wrong result since it does not fulfill the requirement of Gauss's law for a unit charge:

$$\oint_S \frac{\partial G_N(\mathbf{r},\mathbf{r}')}{\partial n'}\, da' = -4\pi \quad . \tag{1.12}$$

Therefore, the simple ansatz is

$$\frac{\partial G_N(\mathbf{r},\mathbf{r}')}{\partial n'} = -\frac{4\pi}{S} \tag{1.13}$$

if S is the entire surface. Then we obtain

$$\Phi(\mathbf{r}) = \langle\Phi\rangle_S + \frac{1}{4\pi\varepsilon_o}\int_V G_N(\mathbf{r},\mathbf{r}')\,\rho(\mathbf{r}')\; d^3r' + \frac{1}{4\pi}\oint_S G_N(\mathbf{r},\mathbf{r}')\frac{\partial\Phi(\mathbf{r}')}{\partial n'}\,da'\,, \tag{1.14}$$

where $\langle\Phi\rangle_S$ is the average value of the potential at the surface. This average value can be absorbed always into the additive constant in which the potential is arbitrary. The physical meaning of $F(\mathbf{r},\mathbf{r}')$ is that the function $F(\mathbf{r},\mathbf{r}')$ denotes the potential of a charge distribution outside the volume V so that, together with the potential $1/|\mathbf{r}-\mathbf{r}'|$ of the point charge at the point ***r***′, the Green function can just satisfy the boundary condition in Eq. (1.10) or Eq. (1.13). It is clear that the external charge distribution depends on the point charge in the volume whose potential or the normal derivative of whose potential it has to compensate at the boundary surface. This means that $F(\mathbf{r},\mathbf{r}')$ depends on the parameter ***r***′, which gives the position of the point charge distribution in the volume. The method of images is developed based on this requirement.

2. Expansion of arbitrary functions in terms of a complete set of functions

Expansion of arbitrary functions in terms of a complete set of orthogonal functions plays an important role in mathematical physics. Considering a finite or infinite system of real or complex functions $U_1(x)$, $U_2(x)$, $U_3(x)$,…. ,$U_n(x)$ in the interval [a,b], the system of functions is called to be orthogonal if the functions satisfy the orthogonality relation:

$$\int_a^b U_m^*(x)\,U_n(x)\,dx = s_n\;\delta_{n,m}\quad , \tag{2.1}$$

where $U_m^*(x)$ is the function complex conjugate to $U_m(x)$. If $s_n = 1$ for n=1,2,…, then the system is said to be normalized (orthonormal system). In terms of the norm, any function $U(x)$ different from the null function can be normalized by

$$U'(x) = \frac{U(x)}{\sqrt{\int_a^b U^*(x)\,U(x)\,dx}}\quad . \tag{2.2}$$

It is obvious that the orthonormality for the system of functions is analogous to the orthonormal vectors $\mathbf{e}_\nu$ with $\mathbf{e}_\nu\cdot\mathbf{e}_\mu = \delta_{\nu,\mu}$. All orthonormal sets of functions normally occurring in mathematical physics have been proven to be complete. In terms of the orthonormal system of functions, an arbitrary function $f(x)$, square integrable on the interval $[a, b]$, can be expanded as

$$f(x)=\sum_{n=1}^{\infty} a_n\, U_n(x) \tag{2.3}$$

with

$$a_n=\int_a^b U_n^*(x)\, f(x)\, dx \quad , \tag{2.4}$$

where the orthonormality condition has been used.

The expansion in Eq. (2.3) can be rewritten with the explicit form in Eq. (2.4) for the coefficients a_n:

$$f(x)=\int_a^b \left[\sum_{n=1}^{\infty} U_n(x')\, U_n^*(x)\right] f(x')\, dx' \quad . \tag{2.5}$$

This result implies that

$$\sum_{n=1}^{\infty} U_n(x')\, U_n^*(x)=\delta(x-x') \quad . \tag{2.6}$$

This is the so-called *completeness* or *closure relation*. It is analogous to the orthonormality condition in Eq. (2.1), except that the roles of the continuous variable x and the discrete index n have been interchanged.

For an arbitrary function $f(x,y)$ with two independent variables in the intervals: $x \in [a,b]$ and $x \in [a,b]$, the expansion can be generalize as

$$f(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{m,n}\, U_m(x)\; V_n(y) \tag{2.7}$$

with

$$a_{m,n}=\int_c^d\int_a^b U_m^*(x)\, V_n^*(y) f(x,y)\; dx\, dy \quad , \tag{2.8}$$

where $U_m(x)$ and $V_n(y)$ are orthonormal functions and form a complete set.

The most famous orthogonal functions are the sines and cosines, an expansion in terms of them being a Fourier series. If the intervals in x is $[-a/2, a/2]$, the orthonormal functions are

$$\sqrt{\frac{2}{a}}\sin\left(\frac{2\pi m}{a}x\right) \quad \text{and} \quad \sqrt{\frac{2}{a}}\cos\left(\frac{2\pi m}{a}x\right) \quad ,$$

where m is a non-negative integer and for $m = 0$ the cosine function is $1/\sqrt{a}$. Then arbitrary functions in the interval $[-a/2, a/2]$ can be represented by

$$f(x)=\frac{1}{2}A_0+\sum_{m=1}^{\infty}\left[A_m\cos\left(\frac{2\pi m}{a}x\right)+B_m\sin\left(\frac{2\pi m}{a}x\right)\right] \ , \tag{2.9}$$

where

$$A_m=\frac{2}{a}\int_{-a/2}^{a/2}\cos\left(\frac{2\pi m}{a}x\right)f(x)\,dx \tag{2.10}$$

and

$$B_m=\frac{2}{a}\int_{-a/2}^{a/2}\sin\left(\frac{2\pi m}{a}x\right)f(x)\,dx \ . \tag{2.11}$$

Substituting Eqs. (2.10) and (2.11) into Eq. (2.9), the expansion can be rewritten as

$$f(x)=\frac{1}{a}\int_{-a/2}^{a/2}f(x')\,dx'+\sum_{m=1}^{\infty}\left\{\frac{2}{a}\int_{-a/2}^{a/2}\cos\left[\frac{2\pi m}{a}(x'-x)\right]f(x')\,dx'\right\} \ . \tag{2.12}$$

Using

$$2\cos\left[\frac{2\pi m}{a}(x'-x)\right]=\exp\left[-i\frac{2\pi m}{a}(x'-x)\right]+\exp\left[i\frac{2\pi m}{a}(x'-x)\right], \tag{2.13}$$

Eq. (12) can be expressed as

$$f(x)=\sum_{m=-\infty}^{\infty}\left\{\frac{1}{a}\int_{-a/2}^{a/2}\exp\left[-i\frac{2\pi m}{a}(x'-x)\right]f(x')\,dx'\right\}. \tag{2.14}$$

Consequently the orthonormal set of complex exponentials

$$U_m(x)=\frac{1}{\sqrt{a}}e^{i(2\pi m x/a)} \tag{2.15}$$

can be defined to expand an arbitrary function $f(x)$ as

$$f(x)=\frac{1}{\sqrt{a}}\sum_{m=-\infty}^{\infty}A_m\,e^{i(2\pi m x/a)} \tag{2.16}$$

with

$$A_m=\frac{1}{\sqrt{a}}\int_{-a/2}^{a/2}e^{-(i2\pi m x'/a)}\,f(x')\,dx' . \tag{2.17}$$

Note that the $\delta(x-x')$ function with x in $[-a/2, a/2]$ can be given by

$$\delta(x-x')=\frac{1}{a}\sum_{m=-\infty}^{\infty}e^{i[2\pi m(x-x')/a]} \ . \tag{2.18}$$

Using $\Delta m=1$ and introducing $k_m=2\pi m/a$ with the limit $a\to\infty$, Eq. (2.18) can be converted to an integral with the continuous parameter k:

$$\begin{aligned}\delta(x-x')&=\frac{1}{a}\sum_{m=-\infty}^{\infty}e^{i[2\pi m(x-x')/a]}\Delta m=\sum_{m=-\infty}^{\infty}e^{ik_m(x-x')}\frac{1}{a}\frac{a}{2\pi}\Delta k_m \\ &\Rightarrow \lim_{a\to\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{ik(x-x')}dk\end{aligned} \ . \tag{2.19}$$

3. Separation of variables; Laplace equation in rectangular coordinates

First of all, we consider the two-dimensional (2D) problem, where the potential Φ is assumed to be independent of z. As shown in Fig. (a), the potential is specified on the four sides of the rectangle: $\Phi(x,0) = f_1(x)$, $\Phi(x,b) = f_2(x)$, $\Phi(0,y) = g_1(y)$, $\Phi(a,y) = g_2(y)$. This problem can be regarded as the sum of the two problems as shown in Figs. (b) and (c). The solutions to these two problems are denoted as Φ_1 and Φ_2, respectively. In Fig. (b), we have chosen $\Phi_1 = 0$ at $x = 0$ and $x = a$. In Fig. (c), we have chosen $\Phi_2 = 0$ at $y = 0$ and $y = b$. The 2D Laplace equation in rectangular coordinates is

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0 \quad . \tag{3.1}$$

With separation of variables, the potential $\Phi(x, y)$ can be represented by the product of two functions, one for each coordinate:

$$\Phi(x, y) = X(x)Y(y) \quad . \tag{3.2}$$

Substitution into Eq. (1) and division of the result by Eq. (2) yields

$$\frac{1}{X(x)}\frac{d^2X}{dx^2} + \frac{1}{Y(y)}\frac{d^2Y}{dy^2} = 0, \tag{3.3}$$

where total derivatives have replaced partial derivatives, since each term involves a function of one variable only. If Eq. (3.3) is to hold for arbitrary values of the independent coordinates, each of the two terms must be separately constant. For the boundary conditions shown in Fig. (b), we generally set

$$\frac{1}{X(x)}\frac{d^2X}{dx^2} = -\alpha^2 \quad ; \quad \frac{1}{Y(y)}\frac{d^2Y}{dy^2} = \alpha^2 . \tag{3.4}$$

The solutions of the two ordinary differential equations (3.4) are $e^{\pm i\alpha x}$ and $e^{\pm \alpha y}$. The potential Φ_1 can be thus built up from the product solutions:

$$\Phi_1 = e^{\pm i\alpha x}\, e^{\pm \alpha y} . \tag{3.5}$$

The boundary conditions $\Phi_1 = 0$ at $x = 0$ and $x = a$ imply that $X(x) = \sin \alpha x$ with $\alpha = n\pi / a$ and $n = 1,2,3,\cdots$. In general, Φ_1 can be the sum of the products and is given by

$$\Phi_1(x, y) = \sum_{n=1}^{\infty} \left[A_n\, e^{-n\pi y/a} + B_n\, e^{n\pi y/a} \right] \sin(n\pi\, x/a) \quad . \tag{3.6}$$

The Fourier coefficients A_n and B_n are determined from the boundary conditions $\Phi(x,0) = f_1(x)$ and $\Phi(x,b) = f_2(x)$:

$$f_1(x) = \sum_{n=1}^{\infty} \left[A_n + B_n \right] \sin(n\pi\, x/a) \tag{3.7}$$

and

$$f_2(x) = \sum_{n=1}^{\infty} \left[A_n \, e^{-n\pi b/a} + B_n \, e^{n\pi b/a} \right] \sin(n\pi \, x/a) \quad . \tag{3.8}$$

From the orthogonality condition, we get:

$$A_n + B_n = \frac{2}{a} \int_0^a f_1(x) \sin(n\pi \, x/a) \, dx \tag{3.9}$$

and

$$A_n \, e^{-n\pi b/a} + B_n \, e^{n\pi b/a} = \frac{2}{a} \int_0^a f_2(x) \sin(n\pi \, x/a) \, dx \quad . \tag{3.10}$$

The coefficients A_n and B_n can be determined by solving Eqs. (3.9) and (3.10) simultaneously.

For the boundary conditions shown in Fig. (c), we generally set

$$\frac{1}{X(x)} \frac{d^2 X}{dx^2} = \beta^2 \quad ; \quad \frac{1}{Y(y)} \frac{d^2 Y}{dy^2} = -\beta^2 \, . \tag{3.11}$$

The solutions of the two ordinary differential equations (3.11) are $e^{\pm \beta x}$ and $e^{\pm i \beta y}$. The boundary conditions $\Phi_2 = 0$ at $y = 0$ and $y = b$ imply that $Y(y) = \sin \beta y$ with $\beta = m\pi / b$ and $m = 1, 2, 3, \cdots$. In the place of Eq. (6), Φ_2 can be given by

$$\Phi_2(x, y) = \sum_{m=1}^{\infty} \left[C_m \, e^{-m\pi x/b} + D_m \, e^{m\pi x/b} \right] \sin(m\pi \, y/b) \quad . \tag{3.12}$$

The Fourier coefficients C_m and D_m are determined from the boundary conditions $\Phi(0, y) = g_1(y)$ and $\Phi(a, y) = g_2(y)$:

$$g_1(y) = \sum_{m=1}^{\infty} \left[C_m + D_m \right] \sin(m\pi \, y/b) \tag{3.13}$$

and

$$g_2(y) = \sum_{m=1}^{\infty} \left[C_m \, e^{-m\pi a/b} + D_m \, e^{m\pi a/b} \right] \sin(m\pi \, y/b) \quad . \tag{3.14}$$

From the orthogonality condition, we get:

$$C_m + D_m = \frac{2}{b} \int_0^b g_1(y) \sin(m\pi \, y/b) \, dy \tag{3.15}$$

and

$$C_m \, e^{-m\pi a/b} + D_m \, e^{m\pi a/b} = \frac{2}{b} \int_0^b g_2(y) \sin(m\pi \, y/b) \, dy \quad . \tag{3.16}$$

The coefficients C_m and D_m can be determined by solving Eqs. (3.15) and (3.16) simultaneously.

Example

Let us consider a simple case with $b \to \infty$ and the boundary conditions: $\Phi(x,0) = V$, $\Phi(y,0) = 0$, $\Phi(y,a) = 0$. Find the potential in terms of Fourier series and closed form.

Solution

As discussed above, only the coefficients A_n survive and can be solved as

$$A_n = \frac{2}{a}\int_0^a V \sin(n\pi\, x/a)\, dx = \frac{4V}{\pi n}\begin{cases} 1 & \text{for } n \text{ odd} \\ 0 & \text{for } n \text{ even} \end{cases} .$$

Consequently, the potential is therefore determined to be

$$\Phi(x,y) = \frac{4V}{\pi}\sum_{n\,\text{odd}} \frac{1}{n} e^{-n\pi y/a} \sin(n\pi\, x/a) . \tag{3.17}$$

There are many Fourier series that can be summed to give a result in closed form. The series in Eq. (3.17) is one of them. The derivation is as follows. With $\sin\theta = \text{Im}(e^{i\theta})$, where Im stands for the imaginary part, Eq. (3.17) can be written as

$$\Phi(x,y) = \frac{4V}{\pi}\,\text{Im}\sum_{n\,\text{odd}} \frac{Z^n}{n} \tag{3.18}$$

with

$$Z = e^{(i\pi/a)(x+iy)} . \tag{3.19}$$

Using the identities:

$$\ln(1+Z) = -\sum_{n=1}^{\infty}\frac{(-1)^n}{n}Z^n \qquad \text{and} \qquad -\ln(1-Z) = \sum_{n=1}^{\infty}\frac{1}{n}Z^n ,$$

Eq. (3.18) can be written as

$$\Phi(x,y) = \frac{2V}{\pi}\,\text{Im}\left[\ln\left(\frac{1+Z}{1-Z}\right)\right] . \tag{3.20}$$

Note that the imaginary part of a logarithm is equal to the phase of its argument. Accordingly, we can use

$$\arg\left(\frac{1+Z}{1-Z}\right) = \arg\left(\frac{1-|Z|^2 + 2i\,\text{Im}Z}{|1-Z|^2}\right) = \tan^{-1}\left[\frac{2\,\text{Im}Z}{1-|Z|^2}\right] \tag{3.21}$$

to obtain

$$\Phi(x,y) = \frac{2V}{\pi}\tan^{-1}\left(\frac{2e^{-\pi y/a}\sin\dfrac{\pi x}{a}}{1-e^{-2\pi y/a}}\right) = \frac{2V}{\pi}\tan^{-1}\left(\frac{\sin\dfrac{\pi x}{a}}{\sinh\dfrac{\pi y}{a}}\right) . \tag{3.22}$$

4. Expansion of Green functions in rectangular coordinates

A Green function for a Dirichlet potential problem in 2D rectangular coordinates satisfies the equation

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)G(x,y;x',y') = -4\pi\delta(x-x')\delta(y-y') \tag{4.1}$$

in a rectangular 2D region, $0 \le x \le a$ and $0 \le y \le b$. The completeness relation can be used to represent the function $\delta(x-x')$:

$$\delta(x-x') = \frac{2}{a}\sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi x'}{a}\right) \quad . \tag{4.2}$$

In terms of the same basis in the *x*-coordinate, the Green function can be expanded as

$$G(x,y;x',y') = \frac{2}{a}\sum_{n=1}^{\infty} g_n(y,y')\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi x'}{a}\right) \quad . \tag{4.3}$$

Substitution of Eqs. (4.2) and (4.3) into Eq. (4.1) leads to

$$\left(\frac{d^2}{dy^2}-\frac{n^2\pi^2}{a^2}\right)g_n(y,y') = -4\pi\,\delta(y-y') \,. \tag{4.4}$$

The Green function in the *y*-component is seen to satisfy the homogeneous equation for $y \ne y'$. Thus it can be written as

$$g_n(y,y') = \begin{cases} A(y')\sinh\left(\dfrac{n\pi y}{a}\right) & \text{for } y < y' \\ B(y')\sinh\left[\dfrac{n\pi(b-y)}{a}\right] & \text{for } y > y' \end{cases} \quad . \tag{4.5}$$

The symmetry in y and y' requires that the coefficients $A(y')$ and $B(y')$ be such that $g_n(y,y')$ can be written

$$g_n(y,y') = C\sinh\left(\frac{n\pi y_<}{a}\right)\sinh\left[\frac{n\pi(b-y_>)}{a}\right] \quad , \tag{4.6}$$

where $y_<$ ($y_>$) is the smaller (larger) of y and y'. To determine the constant C we must consider the effect of the delta function in Eq. (4.4). If we integrate both side of Eq. (4.4) over the interval from $y = y'-\varepsilon$ to $y = y'+\varepsilon$, where ε is very small, we obtain

$$\left[\frac{d}{dy}g_n(y,y')\right]_{y'+\varepsilon} - \left[\frac{d}{dy}g_n(y,y')\right]_{y'-\varepsilon} = -4\pi \quad . \tag{4.7}$$

With Eq. (4.6), we have

$$\left[\frac{d}{dy}g_n(y,y')\right]_{y'+\varepsilon} = -C\left(\frac{n\pi}{a}\right)\sinh\left(\frac{n\pi y'}{a}\right)\cosh\left[\frac{n\pi(b-y')}{a}\right] \tag{4.8}$$

and

$$\left[\frac{d}{dy}g_n(y,y')\right]_{y'-\varepsilon} = C\left(\frac{n\pi}{a}\right)\cosh\left(\frac{n\pi\, y'}{a}\right)\sinh\left[\frac{n\pi(b-y')}{a}\right] \quad . \tag{4.9}$$

Substituting Eqs. (4.8) and (4.9) into Eq. (4.7), we find

$$C = \frac{4a}{n\sinh\left(\frac{n\pi}{a}b\right)} \quad . \tag{4.10}$$

Combination of Eqs. (4.10), (4.6) and (4.3) yields the expansion of the Green function for a 2D rectangular region bounded by $0 \le x \le a$ and $0 \le y \le b$:

$$G(x,y;x',y') = 8\sum_{n=1}^{\infty}\frac{1}{n\sinh\left(\frac{n\pi}{a}b\right)}\sinh\left(\frac{n\pi\, y_<}{a}\right)\sinh\left[\frac{n\pi\,(b-y_>)}{a}\right]\sin\left(\frac{n\pi\, x}{a}\right)\sin\left(\frac{n\pi\, x'}{a}\right). \tag{4.11}$$

Example

With Eq. (4.11), let us consider the problem that all boundaries of the rectangular region hold at zero potential and there is a uniform charge density of strength ρ_o over the entire region.

<Solution>

In terms of the Green function, the potential is given by

$$\Phi(x,y) = \frac{\rho_o}{4\pi\varepsilon_o}\int_0^b dy'\int_0^a dx'\, G(x,y;x',y') \quad . \tag{4.12}$$

The integration in the x' variable can be in terms of

$$\int_0^a \sin\left(\frac{n\pi\, x'}{a}\right)dx' = \frac{2a}{\pi\, n}\begin{cases} 1 & \text{for } n \text{ odd} \\ 0 & \text{for } n \text{ even}\end{cases} \quad . \tag{4.13}$$

On the other hand, the integration in the y' variable can be in terms of

$$\begin{aligned}\int_0^b \sinh\left(\frac{n\pi\, y_<}{a}\right)\sinh\left[\frac{n\pi\,(b-y_>)}{a}\right]dy' &= \int_0^y \sinh\left(\frac{n\pi\, y'}{a}\right)\sinh\left[\frac{n\pi\,(b-y)}{a}\right]dy' \\ &\quad + \int_y^b \sinh\left(\frac{n\pi\, y}{a}\right)\sinh\left[\frac{n\pi\,(b-y')}{a}\right]dy'\end{aligned} \tag{4.14}$$

The integrated results in the right-hand side of Eq. (14) are given by

$$\int_0^y \sinh\left(\frac{n\pi\, y'}{a}\right)\sinh\left[\frac{n\pi\,(b-y)}{a}\right]dy' = \frac{a}{n\pi}\left\{\cosh\left(\frac{n\pi y}{a}\right)-1\right\}\sinh\left[\frac{n\pi\,(b-y)}{a}\right], \tag{4.15}$$

and

$$\int_y^b \sinh\left(\frac{n\pi\, y}{a}\right)\sinh\left[\frac{n\pi\,(b-y')}{a}\right]dy' = \frac{a}{n\pi}\left\{\cosh\left[\frac{n\pi\,(b-y)}{a}\right]-1\right\}\sinh\left(\frac{n\pi\, y}{a}\right) \quad . \tag{4.16}$$

Substituting Eqs. (4.15) and (4.16) into (4.14), we can obtain

$$\int_0^b \sinh\left(\frac{n\pi\, y_<}{a}\right)\sinh\left[\frac{n\pi\,(b-y_>)}{a}\right]dy'$$

$$=\frac{a}{n\pi}\left\{\sinh\left(\frac{n\pi}{a}b\right)-\left(\sinh\left[\frac{n\pi\,(b-y)}{a}\right]+\sinh\left(\frac{n\pi\, y}{a}\right)\right)\right\}$$

$$=\frac{a}{n\pi}\sinh\left(\frac{n\pi}{a}b\right)\left\{1-\left(\cosh(n\pi\, y/a)+\frac{[1-\cosh(n\pi b/a)]\sinh(n\pi\, y/a)}{\sinh(n\pi b/a)}\right)\right\}$$

$$=\frac{a}{n\pi}\sinh\left(\frac{n\pi}{a}b\right)\left\{1-\left(\cosh(n\pi\, y/a)+\frac{-2\sinh^2\left(\frac{n\pi}{2a}b\right)\sinh(n\pi\, y/a)}{2\sinh\left(\frac{n\pi}{2a}b\right)\cosh\left(\frac{n\pi}{2a}b\right)}\right)\right\}$$

$$=\frac{a}{n\pi}\sinh\left(\frac{n\pi}{a}b\right)\left\{1-\frac{\cosh\left[\frac{n\pi}{a}\left(y-\frac{b}{2}\right)\right]}{\cosh\left(\frac{n\pi}{2a}b\right)}\right\} \quad . \quad (4.17)$$

So the final result is given by

$$\Phi(x,y)=\frac{4\rho_o a^2}{\pi^3\varepsilon_o}\sum_{m=0}^{\infty}\frac{\sin\left[\frac{(2m+1)\pi\, x}{a}\right]}{(2m+1)^3}\left\{1-\frac{\cosh\left[\frac{(2m+1)\pi}{a}\left(y-\frac{b}{2}\right)\right]}{\cosh\left[\frac{(2m+1)\pi}{2a}b\right]}\right\} \quad . \quad (4.18)$$

5. Eigenfunction Expansions for Green functions

Another technique for obtaining expansions of Green functions is the use of eigenfunctions for some related problem. To represent the so-called eigenfunctions, we consider a differential equation of the form

$$\nabla^2\psi(\mathbf{r})+[f(\mathbf{r})+\lambda]\psi(\mathbf{r})=0 \quad . \quad (5.1)$$

If the solutions $\psi(\mathbf{r})$ are required to satisfy homogeneous boundary conditions on the surface S of the volume of interest V, then Eq. (5.1) will not in general have well-behaved solutions, except for certain values of λ. These values of λ, denoted by λ_n, are called eigenvalues (or characteristic values) and the solutions $\psi_n(\mathbf{r})$ are called eigenfunctions. The eigenvalue differential equation is written as

$$\nabla^2\psi_n(\mathbf{r})+[f(\mathbf{r})+\lambda_n]\psi_n(\mathbf{r})=0 \quad . \quad (5.2)$$

In general, the totality of eigenfunctions can form a complete orthonormal set.

Suppose that the equation for the Green function is given by

$$\nabla^2 G(\mathbf{r},\mathbf{r}') + \left[f(\mathbf{r}) + \lambda\right] G(\mathbf{r},\mathbf{r}') = -4\pi\,\delta(\mathbf{r}-\mathbf{r}') \quad , \tag{5.3}$$

where λ is not equal to one of the eigenvalues λ_n of Eq. (5.2). Furthermore, suppose that the Green function is to have the same boundary conditions as the eigenfunctions of Eq. (5.2). Then the Green function can be expanded in a series of eigenfunctions of the form:

$$G(\mathbf{r},\mathbf{r}') = \sum_n a_n(\mathbf{r}')\,\psi_n(\mathbf{r})\,. \tag{5.4}$$

Substitution into the differential equation for the Green function leads to the result:

$$\sum_m a_m(\mathbf{r}')\left(\lambda - \lambda_m\right)\psi_m(\mathbf{r}) = -4\pi\,\delta(\mathbf{r}-\mathbf{r}') \quad . \tag{5.5}$$

If we multiply both sides by $\psi_n^*(\mathbf{r})$ and integrate over the volume V, the orthogonality condition reduces the left-hand side to one term, and we obtain

$$a_n(\mathbf{r}') = 4\pi \frac{\psi_n^*(\mathbf{r}')}{\left(\lambda_n - \lambda\right)} \quad . \tag{5.6}$$

Consequently the eigenfunction expansion of the Green function is given by

$$G(\mathbf{r},\mathbf{r}') = 4\pi \sum_n \frac{\psi_n^*(\mathbf{r}')\,\psi_n(\mathbf{r})}{\left(\lambda_n - \lambda\right)} \quad . \tag{5.7}$$

Specializing the foregoing considerations to the Poisson equation, we place $f(\mathbf{r}) = 0$ and $\lambda = 0$ in Eq. (5.1). Consider the Green function for a Dirichlet problem inside a rectangular region defined by the $0 \le x \le a$ and $0 \le y \le b$. The expansion is to be made in terms of eigenfunctions of the wave equation:

$$\left(\nabla_{2D}^2 + k_{nm}^2\right)\psi_{nm}(x,y) = 0 \quad , \tag{5.8}$$

where the eigenfunctions which vanish on all the boundary sides are

$$\psi_{nm}(x,y) = \sqrt{\frac{4}{ab}}\,\sin\!\left(\frac{n\pi x}{a}\right)\sin\!\left(\frac{m\pi y}{b}\right) \tag{5.9}$$

and

$$k_{nm}^2 = \pi^2\left(\frac{n^2}{a^2} + \frac{m^2}{b^2}\right) \quad . \tag{5.10}$$

The expansion of the Green function is therefore given by

$$G(x,y;x',y') = \frac{16\pi}{ab} \sum_{n,m=1}^{\infty} \frac{\sin\!\left(\frac{n\pi x'}{a}\right)\sin\!\left(\frac{m\pi y'}{b}\right)}{\pi^2\left(\frac{n^2}{a^2} + \frac{m^2}{b^2}\right)} \sin\!\left(\frac{n\pi x}{a}\right)\sin\!\left(\frac{m\pi y}{b}\right) \quad . \tag{5.11}$$

As discussed in the previous section, the Green function can be obtained by singling out the y

coordinate for special treatment:

$$G(x,y;x',y')$$

$$=8\sum_{n=1}^{\infty}\frac{1}{n\sinh\left(\frac{n\pi}{a}b\right)}\sinh\left(\frac{n\pi\, y_<}{a}\right)\sinh\left[\frac{n\pi\,(b-y_>)}{a}\right]\sin\left(\frac{n\pi\, x}{a}\right)\sin\left(\frac{n\pi\, x'}{a}\right). \quad (5.12)$$

If Eqs. (5.11) and (5.12) are to be equal, it must be that the sum over m in Eq. (5.11) is just the Fourier series representation on the interval (0, b) of the one-dimensional Green function in y in Eq. (5.12):

$$\frac{\sinh\left(\frac{n\pi\, y_<}{a}\right)\sinh\left[\frac{n\pi\,(b-y_>)}{a}\right]}{\left(\frac{n\pi}{a}\right)\sinh\left(\frac{n\pi}{a}b\right)}=\frac{2}{b}\sum_{m=1}^{\infty}\frac{\sin\left(\frac{m\pi\, y'}{b}\right)}{\pi^2\left(\frac{n^2}{a^2}+\frac{m^2}{b^2}\right)}\sin\left(\frac{m\pi\, y}{b}\right)\ . \quad (5.13)$$

Equation (5.13) can be generally rewritten as

$$\frac{\sinh(\alpha\, y_<)\sinh[\alpha\,(b-y_>)]}{\alpha\,\sinh(\alpha\, b)}=\frac{2}{b}\sum_{m=1}^{\infty}\frac{\sin(m\pi\, y'/b)}{\alpha^2+(m\pi/b)^2}\sin(m\pi\, y/b)\ . \quad (5.14)$$

Verifying Eq. (5.14)

From the Fourier representation, verifying Eq. (5.14) is to show the following identity:

$$\int_0^b\frac{\sinh(\alpha\, y_<)\sinh[\alpha\,(b-y_>)]}{\alpha\,\sinh(\alpha\, b)}\sin(m\pi\, y/b)\,dy=\frac{\sin(m\pi\, y'/b)}{\alpha^2+(m\pi/b)^2}\ . \quad (5.\ 15)$$

The integration in the left-hand side of Eq. (5.15) can be explicitly given by

$$\begin{aligned}&\frac{\sinh[\alpha\,(b-y')]}{\alpha\,\sinh(\alpha\, b)}\int_0^{y'}\sinh(\alpha\, y)\sin(m\pi\, y/b)\,dy\\&+\frac{\sinh(\alpha\, y')}{\alpha\,\sinh(\alpha\, b)}\int_{y'}^{b}\sinh[\alpha\,(b-y)]\sin(m\pi\, y/b)\,dy\end{aligned}\ . \quad (5.16)$$

The integration can be further evaluated as

$$\int_0^{y'} \sinh(\alpha\, y)\sin(m\pi\, y/b)\, dy$$

$$= \frac{1}{(2i)}\left\{ e^{i\frac{m\pi\, y'}{b}}\left[\frac{\alpha\cosh(\alpha\, y') - i\left(\frac{m\pi}{b}\right)\sinh(\alpha\, y')}{\alpha^2 + \left(\frac{m\pi}{b}\right)^2}\right] - e^{-i\frac{m\pi\, y'}{b}}\left[\frac{\alpha\cosh(\alpha\, y') + i\left(\frac{m\pi}{b}\right)\sinh(\alpha\, y')}{\alpha^2 + \left(\frac{m\pi}{b}\right)^2}\right]\right\} \tag{5.17}$$

and

$$\int_{y'}^{b} \sinh[\alpha\,(b-y)]\sin(m\pi\, y/b)\, dy$$

$$= \frac{1}{(2i)}\left\{ e^{i\frac{m\pi\, y'}{b}}\left[\frac{\alpha\cosh[\alpha\,(b-y')] + i\left(\frac{m\pi}{b}\right)\sinh[\alpha\,(b-y')]}{\alpha^2 + \left(\frac{m\pi}{b}\right)^2}\right] - e^{-i\frac{m\pi\, y'}{b}}\left[\frac{\alpha\cosh[\alpha\,(b-y')] - i\left(\frac{m\pi}{b}\right)\sinh[\alpha\,(b-y')]}{\alpha^2 + \left(\frac{m\pi}{b}\right)^2}\right]\right\} . \tag{5.18}$$

Substitution of Eqs. (5.17) and (5.18) into Eq. (5.16) leads to

$$\frac{\sinh[\alpha\,(b-y')]}{\alpha\,\sinh(\alpha\, b)}\int_0^{y'} \sinh(\alpha\, y)\sin\left(\frac{m\pi\, y}{b}\right) dy + \frac{\sinh(\alpha\, y')}{\alpha\,\sinh(\alpha\, b)}\int_{y'}^{b} \sinh[\alpha\,(b-y)]\sin\left(\frac{m\pi\, y}{b}\right) dy$$

$$= \frac{1}{\alpha\,\sinh(\alpha\, b)}\frac{1}{(2i)}\left\{ e^{i\frac{m\pi\, y'}{b}}\left[\frac{\alpha\,\sinh(\alpha\, b)}{\alpha^2 + \left(\frac{m\pi}{b}\right)^2}\right] - e^{-i\frac{m\pi\, y'}{b}}\left[\frac{\alpha\,\sinh(\alpha\, b)}{\alpha^2 + \left(\frac{m\pi}{b}\right)^2}\right]\right\} . \tag{5.19}$$

$$= \frac{\sin\left(\frac{m\pi\, y'}{b}\right)}{\alpha^2 + \left(\frac{m\pi}{b}\right)^2}$$

Example

The expansion of the potential of a unit point charge in rectangular coordinates affords a useful example of Green function expansions. A Green function for a Dirichlet potential problem in rectangular coordinates satisfies the equation

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right]G(x,y,z;x',y',z')=-4\pi\delta(x-x')\delta(y-y')\delta(z-z') . \quad (1)$$

The completeness relation can be used to represent the functions $\delta(x-x')$ and $\delta(y-y')$:

$$\delta(x-x')=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i k_x (x-x')}dk_x \quad , \quad (2)$$

$$\delta(y-y')=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i k_y (y-y')}dk_y \, . \quad (3)$$

In terms of the basis in the *x* and *y* coordinates, the Green function can be expanded as

$$G(x,y,z;x',y',z')=\left(\frac{1}{2\pi}\right)^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(k_x,k_y,z,z')\, e^{i k_x (x-x')}\, e^{i k_y (y-y')}\, dk_x dk_y \quad . \quad (4)$$

Substitution of Eqs. (2), (3), and (4) into Eq. (1) leads to

$$\left(\frac{d^2}{dz^2}-k_t^2\right)g(k_t,z,z')=-4\pi\delta(z-z') \quad , \quad (5)$$

where $k_x^2+k_y^2=k_t^2$. For $z\neq z'$, the Green function in the *z*-component is seen to satisfy a simple differential equations. When there are no boundary surfaces, $g(k_t,z,z')$ vanishes at $z\to\pm\infty$. Consequently, $g(k_t,z,z')$ can be written as

$$g(k_t,z,z')=\begin{cases} A(k_t)\, e^{k_t z} & \text{for } z<z' \\ B(k_t)\, e^{-k_t z} & \text{for } z>z' \end{cases} \quad . \quad (6)$$

The symmetry in *z* and *z*′ requires that the coefficients $A(k_t)$ and $B(k_t)$ be such that $g(k_t,z,z')$ can be written

$$g(k_t,z,z')=C(k_t)\, e^{k_t z_<} e^{-k_t z_>} \quad , \quad (7)$$

where $z_<(z_>)$ is the smaller (larger) of *z* and *z*′. To determine the constant $C(k)$, we consider the effect of the delta function in Eq. (5). If we integrate both side of Eq. (5) over the interval from $z=z'-\varepsilon$ to $z=z'+\varepsilon$, where ε is very small, we obtain

$$\left[\frac{d}{dz}g(k_t,z,z')\right]_{z'+\varepsilon}-\left[\frac{d}{dz}g(k_t,z,z')\right]_{z'-\varepsilon}=-4\pi \quad . \quad (8)$$

Substituting Eq. (7) into Eq. (8), we can find

$$C(k_t) = 2\pi / k_t \quad . \tag{9}$$

Consequently, the free space Green function just for the expansion of $1/|\mathbf{r}-\mathbf{r}'|$ can be expressed as

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{2\pi}{k_t} e^{-k_t(z_> - z_<)} e^{i k_x (x-x')} e^{i k_y (y-y')} \, dk_x dk_y \quad , \tag{10}$$

where $k_x^2 + k_y^2 = k_t^2$ and $z_<(z_>)$ is the smaller (larger) of z and z'.

Expanding the free space Green function with the eigenfunctions of the Helmholtz equation, it can be shown that

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \left(\frac{1}{2\pi}\right)^3 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{4\pi}{k^2} e^{i k_x (x-x')} e^{i k_y (y-y')} e^{i k_z (z-z')} dk_x dk_y dk_z \quad , \tag{11}$$

where $k_x^2 + k_y^2 + k_z^2 = k^2$. If Eqs. (10) and (11) are to be equal, it must be that

$$e^{-k_t(z_> - z_<)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2k_t}{k_t^2 + k_z^2} e^{-i k_z z'} e^{i k_z z} dk_z \quad . \tag{12}$$

Eq. (12) can be verified by showing the following Fourier transformation:

$$\int_{-\infty}^{\infty} e^{-k_t(z_> - z_<)} e^{-i k_z z} dz = \frac{2k_t}{k_t^2 + k_z^2} e^{-i k_z z'} \quad . \tag{13}$$

The integration in Eq. (13) can be carried out by expressing as

$$\int_{-\infty}^{z'} e^{-k_t(z'-z)} e^{-i k_z z} dz + \int_{z'}^{\infty} e^{-k_t(z-z')} e^{-i k_z z} dz = \frac{2k_t}{k_t^2 + k_z^2} e^{-i k_z z'} \quad . \tag{14}$$

Chapter Four: Boundary Value Problems in Electrostatics (II)

1. Separation of variables; Laplace equation in Polar coordinates

In this section, we consider the two-dimensional (2D) problem in polar coordinates, where the potential Φ is assumed to be independent of *z*. As shown in Fig. , the potential may be specified on the surface of a cylinder of radius *b* as $\Phi_s(b,\phi)$. In terms of the polar coordinates (ρ,ϕ), the Laplace equation in two dimensions is

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\phi^2}=0 \quad . \tag{1.1}$$

Using the separation of variables approach, the potential $\Phi(\rho,\phi)$ can be represented by the product of two functions, one for each coordinate:

$$\Phi(\rho,\phi)=R(\rho)Q(\phi) \quad . \tag{1.2}$$

Substitution of Eq. (1.2) into Eq. (1.1) and multiplication by ρ^2/RQ yields

$$\frac{\rho}{R}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right)+\frac{1}{Q}\frac{\partial^2 Q}{\partial\phi^2}=0, \tag{1.3}$$

where total derivatives have replaced partial derivatives, since each term involves a function of one variable only. If Eq. (1.3) is to hold for arbitrary values of the independent coordinates, each of the two terms must be separately constant:

$$\frac{\rho}{R}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right)=m^2 \quad ; \quad \frac{1}{Q}\frac{\partial^2 Q}{\partial\phi^2}=-m^2. \tag{1.4}$$

The solutions of the two ordinary differential equations (1.4) are

$$\left.\begin{aligned} R(\rho)&=\rho^{\pm m} \\ Q(\phi)&=A\cos(m\phi)+B\sin(m\phi) \end{aligned}\right\}\text{ for } m\neq 0 \tag{1.5}$$

and

$$\left.\begin{aligned} R(\rho)&=a_0+b_0\ln\rho \\ Q(\phi)&=A_0+B_0\phi \end{aligned}\right\}\text{ for } m=0 \quad . \tag{1.6}$$

When the full azimuthal range is permitted, there is no restriction on ϕ and it is necessary that *m* must be a positive or negative integer or zero to ensure that the potential is single-valued. Furthermore, for $m = 0$, the constant B_0 in Eq. (1.6) must vanish for the same reason. Under this circumstance, the general solution is therefore of the form,

$$\Phi(\rho,\phi) = A_0 + B_0 \ln\rho + \sum_{m=1}^{\infty} \rho^m \left[A_m \cos(m\phi) + B_m \sin(m\phi) \right] \\ + \sum_{m=1}^{\infty} \rho^{-m} \left[C_m \cos(m\phi) + D_m \sin(m\phi) \right] \quad . \tag{1.7}$$

The Fourier coefficients A_m, B_m, C_m and D_m are determined from the boundary conditions. If the origin is included in the volume in which there is no charge, all C_m and D_m and B_0 are zero. Only a constant and positive powers of ρ appear. If the origin is excluded, the C_m and D_m and B_0 can be different from zero. In particular, the logarithmic term is equivalent to a line charge on the axis with charge density per unit length $\lambda = -2\pi\varepsilon_o B_0$, as is well known.

Example

Starting with the series solution Eq. (1.7) for the 2D potential problem with the potential specified on the surface of a cylinder of radius *b*, evaluate the coefficients formally, substitute them into the series, and sum it to obtain the potential inside the cylinder in the form of Poisson's integral:

$$\Phi(\rho,\phi) = \frac{1}{2\pi}\int_0^{2\pi} \Phi_s(b,\phi') \frac{b^2-\rho^2}{b^2+\rho^2-2b\rho\cos(\phi'-\phi)} d\phi' \quad . \tag{1.8}$$

What modification is necessary if the potential is desired in the region of space bounded by the cylinder and infinity?

<Solution>

Since the origin is included, the solution is given by

$$\Phi(\rho,\phi) = A_0 + \sum_{m=1}^{\infty} \rho^m \left[A_m \cos(m\phi) + B_m \sin(m\phi) \right] , \tag{1.9}$$

and the boundary condition leads to

$$\Phi_s(b,\phi) = A_0 + \sum_{m=1}^{\infty} b^m \left[A_m \cos(m\phi) + B_m \sin(m\phi) \right] . \tag{1.10}$$

From the orthogonality condition, we get:

$$A_0 = \frac{1}{2\pi}\int_0^{2\pi} \Phi_s(b,\phi') d\phi' \tag{1.11}$$

$$A_m = \frac{1}{\pi b^m}\int_0^{2\pi} \Phi_s(b,\phi') \cos(m\phi') d\phi' \tag{1.12}$$

$$B_m = \frac{1}{\pi b^m} \int_0^{2\pi} \Phi_s(b,\phi') \sin(m\phi') \, d\phi' \quad . \tag{1.13}$$

Substituting Eqs. (1.11)-(1.13) into Eq. (1.9), we obtain

$$\Phi(\rho,\phi) = \frac{1}{2\pi} \int_0^{2\pi} \Phi_s(b,\phi') \left\{ 1 + 2\sum_{m=1}^{\infty} \left(\frac{\rho}{b}\right)^m \cos[m(\phi'-\phi)] \right\} d\phi' \quad . \tag{1.14}$$

The term in the bracket of the integrand can be derived as

$$\begin{aligned} 1 + 2\sum_{m=1}^{\infty} \left(\frac{\rho}{b}\right)^m \cos[m(\phi'-\phi)] &= \sum_{m=0}^{\infty} \left(\frac{\rho}{b} e^{i(\phi'-\phi)}\right)^m + \sum_{m=0}^{\infty} \left(\frac{\rho}{b} e^{-i(\phi'-\phi)}\right)^m - 1 \\ &= \frac{1}{1 - \frac{\rho}{b} e^{i(\phi'-\phi)}} + \frac{1}{1 - \frac{\rho}{b} e^{-i(\phi'-\phi)}} - 1 \\ &= \frac{2 - 2(\rho/b)\cos(\phi'-\phi)}{1 + (\rho/b)^2 - 2(\rho/b)\cos(\phi'-\phi)} - 1 \\ &= \frac{b^2 - \rho^2}{b^2 + \rho^2 - 2b\rho\cos(\phi'-\phi)} \end{aligned} \quad . \tag{1.15}$$

Example

Two halves of a long hollow conducting cylinder of inner radius b are separated by small lengthwise gaps on each side, and are kept at different potentials V_1 and V_2. Show that the potential inside is given by

$$\Phi(\rho,\phi) = \frac{V_1 + V_2}{2} + \frac{V_1 - V_2}{\pi} \tan^{-1}\left(\frac{2b\rho \sin\phi}{b^2 - \rho^2} \right)$$

where ϕ is measured from a plane parallel to the plane through the gap.

<Solution>

The potential on the surface can be expressed as

$$\Phi_s(b,\phi) = \frac{V_1 + V_2}{2} + \begin{cases} \left(\dfrac{V_1 - V_2}{2} \right) & \text{for } 0 < \phi < \pi \\ -\left(\dfrac{V_1 - V_2}{2} \right) & \text{for } \pi < \phi < 2\pi \end{cases}$$

With Eq. (8), we can find

$$\Phi(\rho,\phi)=\frac{V_1+V_2}{2}+\frac{V_1-V_2}{4\pi}\int_0^{\pi}\left[\frac{b^2-\rho^2}{b^2+\rho^2-2b\rho\cos(\phi'-\phi)}-\frac{b^2-\rho^2}{b^2+\rho^2+2b\rho\cos(\phi'-\phi)}\right]d\phi'$$

$$=\frac{V_1+V_2}{2}+\frac{V_1-V_2}{4\pi}\int_0^{\pi}\left[\frac{4b\rho\left(b^2-\rho^2\right)\cos(\phi'-\phi)}{\left(b^2+\rho^2\right)^2-4b^2\rho^2\cos^2(\phi'-\phi)}\right]d\phi'$$

$$=\frac{V_1+V_2}{2}+\frac{V_1-V_2}{2\pi}\int_0^{\pi}\left[\frac{2b\rho\left(b^2-\rho^2\right)\cos(\phi'-\phi)}{\left(b^2-\rho^2\right)^2+4b^2\rho^2\sin^2(\phi'-\phi)}\right]d\phi'$$

$$=\frac{V_1+V_2}{2}+\frac{V_1-V_2}{2\pi}\int_{-2b\rho\sin\phi}^{2b\rho\sin\phi}\left[\frac{\left(b^2-\rho^2\right)}{\left(b^2-\rho^2\right)^2+x^2}\right]dx$$

$$=\frac{V_1+V_2}{2}+\frac{V_1-V_2}{\pi}\tan^{-1}\left[\frac{2b\rho\sin\phi}{b^2-\rho^2}\right]$$

2. Expansion of Green functions in polar coordinates

Example

(a) Consider the free-space Green function for two-dimensional electrostatics to show that the Green function can be written as

$$G(\rho,\phi;\rho',\phi')=-\ln\left[\rho^2+\rho'^2-2\rho\rho'\cos\left(\phi-\phi'\right)\right]$$

(b) By using separation of variables in polar coordinates to show that the Green functions can be expressed as a Fourier series in the azimuthal coordinates,

$$G(\rho,\phi;\rho',\phi')=\frac{1}{2\pi}\sum_{m=-\infty}^{\infty}e^{im(\phi-\phi')}g_m(\rho,\rho')$$

where the radial Green functions satisfy

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dg_m}{d\rho}\right)-\frac{m^2}{\rho^2}g_m=-4\pi\frac{\delta(\rho-\rho')}{\rho}$$

(c) Considering that $g_m(\rho,\rho')$ for fixed ρ is a different linear combination of the solutions of the homogeneous radial equation for $\rho'<\rho$ and for $\rho'>\rho$, show that the free space Green function has the expansion

$$G(\rho,\phi;\rho',\phi')=-\ln\left(\rho_>^2\right)+2\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{\rho_<}{\rho_>}\right)^m\cos\left[m(\phi'-\phi)\right]$$

where $\rho_<(\rho_>)$ is the smaller (larger) of ρ and ρ'.

(d) Show the identity

$$-\ln\left[\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')\right]=-\ln\left(\rho_>^2\right)+2\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{\rho_<}{\rho_>}\right)^m\cos[m(\phi'-\phi)]$$

<Solution>

(a) The Green function has to obey the following equation

$$\nabla_{2D}^2 G(\mathbf{r},\mathbf{r}')=-4\pi\delta^{(2)}(\mathbf{r}-\mathbf{r}').$$

Since the Laplace operator is invariant under translations and rotations, we expect the existence of a translational-invariant and rotational-invariant solution. Hence, we make the ansatz

$$G(\mathbf{r},\mathbf{r}')=G(\mathbf{r}-\mathbf{r}')=G(\mathbf{R})=G(R).$$

In polar coordinates, $R=\sqrt{\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')}$. The Green function satisfies the following equation

$$\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial G}{\partial R}\right)=-4\pi\frac{\delta(R)}{R}\delta(\phi).$$

Making an integration of ϕ for this equation and simplifying, we obtain

$$\frac{\partial}{\partial R}\left(R\frac{\partial G}{\partial R}\right)=-2\delta(R)\quad\Rightarrow\quad R\frac{\partial G}{\partial R}=-2\quad\Rightarrow\quad G=-2\ln R.$$

As a result, the Green function can be written as

$$G(\rho,\phi;\rho',\phi')=-\ln\left[\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')\right].$$

(b) A Green function for a Dirichlet potential problem in polar coordinates satisfies the equation

$$\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}\right]G(\rho,\phi;\rho',\phi')=-4\pi\frac{\delta(\rho-\rho')}{\rho}\delta(\phi-\phi') \quad . \tag{1}$$

The completeness relation can be used to represent the function $\delta(\phi-\phi')$:

$$\delta(\phi-\phi')=\frac{1}{2\pi}\sum_{m=-\infty}^{\infty}e^{im(\phi-\phi')} \quad . \tag{2}$$

In terms of the same basis in the ϕ-coordinate, the Green function can be expanded as

$$\begin{aligned}G(\rho,\phi;\rho',\phi')&=\frac{1}{2\pi}\sum_{m=-\infty}^{\infty}g_m(\rho,\rho')\,e^{im(\phi-\phi')}\\&=\frac{1}{2\pi}g_0(\rho,\rho')+\frac{1}{2\pi}\sum_{m=1}^{\infty}g_m(\rho,\rho')\,e^{im(\phi-\phi')}+\frac{1}{2\pi}\sum_{m=1}^{\infty}g_m(\rho,\rho')\,e^{-im(\phi-\phi')}\end{aligned} \quad .(3)$$

Here we have used the fact that $g_m(\rho,\rho')=g_{-m}(\rho,\rho')$. Substitution of Eqs. (2) and (3) into Eq. (1) leads to

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dg_m}{d\rho}\right)-\frac{m^2}{\rho^2}g_m=-4\pi\frac{\delta(\rho-\rho')}{\rho} \quad . \tag{4}$$

(c) The Green function in the ρ-component is seen to satisfy the homogeneous equation for $\rho \neq \rho'$. For $m \neq 0$, it can be written as

$$g_m(\rho,\rho')=\begin{cases} A_m\rho^m & \text{for } \rho<\rho' \\ B_m\rho^{-m} & \text{for } \rho>\rho' \end{cases} \quad . \tag{5}$$

For $m=0$, $g_0(\rho,\rho')$ can be written as

$$g_0(\rho,\rho')=\begin{cases} A_0 & \text{for } \rho<\rho' \\ B_0\ \ln\rho & \text{for } \rho>\rho' \end{cases} \quad . \tag{6}$$

The symmetry in ρ and ρ' requires that the coefficients A_m and B_m be such that $g_m(\rho,\rho')$ can be written

$$g_m(\rho,\rho')=\begin{cases} C_m\left(\dfrac{\rho_<}{\rho_>}\right)^m & \text{for } m\neq 0 \\ C_0\ln(\rho_>) & \text{for } m=0 \end{cases} \quad , \tag{7}$$

where $\rho_<(\rho_>)$ is the smaller (larger) of ρ and ρ'. To determine the constant C_m, we consider the effect of the delta function in Eq. (4). If we integrate both side of Eq. (4) for $m \neq 0$ over the interval from $\rho=\rho'-\varepsilon$ to $\rho=\rho'+\varepsilon$, where ε is very small, we obtain

$$\left[\rho'\frac{d}{d\rho}g_m(\rho,\rho')\right]_{\rho'+\varepsilon}-\left[\rho'\frac{d}{d\rho}g_m(\rho,\rho')\right]_{\rho'-\varepsilon}=-4\pi \quad . \tag{8}$$

With Eq. (7), we have

$$\left[\rho'\frac{d}{d\rho}g_m(\rho,\rho')\right]_{\rho'+\varepsilon}=-mC_m \tag{9}$$

and

$$\left[\rho'\frac{d}{d\rho}g_m(\rho,\rho')\right]_{\rho'-\varepsilon}=mC_m \quad . \tag{10}$$

Substituting Eqs. (9) and (10) into Eq. (8), we find

$$C_m=\frac{2\pi}{m} \quad . \tag{11}$$

Following the same procedure, we can obtain $C_0=-4\pi$. Combining all coefficients, the free space Green function has the expansion

$$G(\rho,\phi;\rho',\phi') = -\ln\left(\rho_>^2\right) + 2\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{\rho_<}{\rho_>}\right)^m \cos[m(\phi'-\phi)] \quad . \tag{12}$$

where $\rho_<(\rho_>)$ is the smaller (larger) of ρ and ρ'.

(d) The identity can be shown as

$$\begin{aligned}
&-\ln\left[\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')\right] \\
&= -\ln\left\{\left(\rho_>^2\right)\left[1+\left(\frac{\rho_<}{\rho_>}\right)^2-2\left(\frac{\rho_<}{\rho_>}\right)\cos(\phi-\phi')\right]\right\} \\
&= -\ln\left\{\left(\rho_>^2\right)\left[1-\left(\frac{\rho_<}{\rho_>}\right)e^{im(\phi-\phi')}\right]\left[1-\left(\frac{\rho_<}{\rho_>}\right)e^{-im(\phi-\phi')}\right]\right\} \quad . \\
&= -\ln\left(\rho_>^2\right) - \ln\left[1-\left(\frac{\rho_<}{\rho_>}\right)e^{im(\phi-\phi')}\right] - \ln\left[1-\left(\frac{\rho_<}{\rho_>}\right)e^{-im(\phi-\phi')}\right] \\
&= -\ln\left(\rho_>^2\right) + 2\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{\rho_<}{\rho_>}\right)^m \cos[m(\phi'-\phi)]
\end{aligned}$$

Here we have used the fact that

$$-\ln(1-Z) = \sum_{n=1}^{\infty}\frac{1}{n}Z^n \; .$$

Example

(a) Extend the previous example to find the Green function for the interior Dirichlet problem of a cylinder of radius *b*, i. e. $g_m(\rho,\rho'=b)=0$. First find the series expansion akin to the free-space Green function and then show that it can be written in closed form as

$$G(\rho,\phi;\rho',\phi') = -\ln\left[\frac{\rho^2\rho'^2+b^4-2\rho\rho' b^2\cos(\phi-\phi')}{b^2\left(\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')\right)}\right]$$

(b) Show that the solution of the Laplace equation with the potential given as $\Phi_s(b,\phi)$ on the cylinder can be expressed as Poisson's integral:

$$\Phi(\rho,\phi) = \frac{1}{2\pi}\int_0^{2\pi} \Phi_s(b,\phi')\frac{b^2-\rho^2}{b^2+\rho^2-2b\rho\cos(\phi'-\phi)}d\phi' \quad .$$

<Solution>

(a) In terms of the same basis in the ϕ-coordinate, the Green function can be expanded as

$$G(\rho,\phi;\rho',\phi') = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} g_m(\rho,\rho')\, e^{im(\phi-\phi')}$$
$$= \frac{1}{2\pi} g_0(\rho,\rho') + \frac{1}{2\pi}\sum_{m=1}^{\infty} g_m(\rho,\rho')\, e^{im(\phi-\phi')} + \frac{1}{2\pi}\sum_{m=1}^{\infty} g_m(\rho,\rho')\, e^{-im(\phi-\phi')} \quad . \tag{1}$$

The Green function in the ρ-component is seen to satisfy the homogeneous equation for $\rho \neq \rho'$. For $m \neq 0$, it can be written as

$$g_m(\rho,\rho') = \begin{cases} A_m \rho^m & \text{for } 0 < \rho < \rho' \\ B_m\left[\rho^{-m} - \dfrac{\rho^m}{b^{2m}}\right] & \text{for } \rho' < \rho < b \end{cases} \quad . \tag{2}$$

For $m = 0$, $g_0(\rho,\rho')$ can be written as

$$g_0(\rho,\rho') = \begin{cases} A_0 & \text{for } \rho < \rho' \\ B_0 \ln\left(\dfrac{\rho}{b}\right) & \text{for } \rho > \rho' \end{cases} \quad . \tag{3}$$

The symmetry in ρ and ρ' requires that the coefficients A_m and B_m be such that $g_m(\rho,\rho')$ can be written

$$g_m(\rho,\rho') = \begin{cases} C_m\, \rho_<^m\left(\dfrac{1}{\rho_>^m} - \dfrac{\rho_>^m}{b^{2m}}\right) & \text{for } m \neq 0 \\ C_0 \ln\left(\dfrac{\rho_>}{b}\right) & \text{for } m = 0 \end{cases} \quad , \tag{4}$$

where $\rho_<(\rho_>)$ is the smaller (larger) of ρ and ρ'. To determine the constant C_m, we consider the effect of the delta function:

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dg_m}{d\rho}\right) - \frac{m^2}{\rho^2} g_m = -4\pi\frac{\delta(\rho-\rho')}{\rho} \quad . \tag{5}$$

After integrating both side of Eq. (5) for $m \neq 0$ over the interval from $\rho = \rho' - \varepsilon$ to $\rho = \rho' + \varepsilon$, where ε is very small, we obtain

$$\left[\rho\frac{d}{d\rho} g_m(\rho,\rho')\right]_{\rho'+\varepsilon} - \left[\rho\frac{d}{d\rho} g_m(\rho,\rho')\right]_{\rho'-\varepsilon} = -4\pi \quad . \tag{6}$$

With Eq. (4), we have

$$\left[\rho'\frac{d}{d\rho} g_m(\rho,\rho')\right]_{\rho'+\varepsilon} = C_m(-2m) \tag{7}$$

and

$$\left[\rho' \frac{d}{d\rho} g_m(\rho,\rho')\right]_{\rho'-\varepsilon} = 0 \quad . \tag{8}$$

Substituting Eqs. (7) and (8) into Eq. (6), we find

$$C_m = \frac{2\pi}{m} \quad . \tag{9}$$

Following the same procedure, we can obtain $C_0 = -4\pi$. Combining all coefficients, the free space Green function has the expansion

$$G(\rho,\phi;\rho',\phi') = -\ln\left(\frac{\rho_>^2}{b^2}\right) + 2\sum_{m=1}^{\infty} \frac{1}{m} \rho_<^m \left(\frac{1}{\rho_>^m} - \frac{\rho_>^m}{b^{2m}}\right) \cos[m(\phi'-\phi)] \ . \tag{10}$$

where $\rho_<(\rho_>)$ is the smaller (larger) of ρ and ρ'. Eq. (10) can be rewritten as

$$\begin{aligned}
G(\rho,\phi;\rho',\phi') &= -\ln\left(\frac{\rho_>^2}{b^2}\right) + 2\sum_{m=1}^{\infty} \frac{1}{m} \rho_<^m \left(\frac{1}{\rho_>^m} - \frac{\rho_>^m}{b^{2m}}\right) \cos[m(\phi'-\phi)] \\
&= \left\{\ln(b^2) - 2\sum_{m=1}^{\infty} \frac{1}{m}\left(\frac{\rho_>\rho_<}{b^2}\right)^m \cos[m(\phi'-\phi)]\right\} - \left\{\ln(\rho_>^2) - 2\sum_{m=1}^{\infty} \frac{1}{m}\left(\frac{\rho_<}{\rho_>}\right)^m \cos[m(\phi'-\phi)]\right\} \\
&= \ln\left\{\frac{b^2\left[1 + \left(\frac{\rho_>\rho_<}{b^2}\right)^2 - 2\left(\frac{\rho_>\rho_<}{b^2}\right)\cos(\phi'-\phi)\right]}{\rho_>^2\left[1 + \left(\frac{\rho_<}{\rho_>}\right)^2 - 2\left(\frac{\rho_<}{\rho_>}\right)\cos(\phi'-\phi)\right]}\right\} \\
&= \ln\left[\frac{b^4 + \rho^2\rho'^2 - 2b^2\rho\rho'\cos(\phi'-\phi)}{b^2\left(\rho^2 + \rho'^2 - 2\rho\rho'\cos(\phi'-\phi)\right)}\right]
\end{aligned} \tag{11}$$

(b) In terms of the Green function, the solution of the Laplace equation with the potential given as $\Phi_s(b,\phi)$ on the cylinder is given by

$$\begin{aligned}
\Phi(\rho,\phi) &= -\frac{1}{4\pi}\int_0^{2\pi} \left.\frac{\partial G}{\partial \rho'}\right|_{\rho'=b} \Phi(b,\phi')\, b d\phi' \\
&= \frac{1}{2\pi}\int_0^{2\pi} \Phi_s(b,\phi') \frac{b^2-\rho^2}{b^2+\rho^2-2b\rho\cos(\phi'-\phi)} d\phi'
\end{aligned} \quad . \tag{12}$$

Example

Show that the two-dimensional Green function for Dirichlet boundary conditions for the annular region, $b \le \rho \le c$ (concentric cylinders) has the expansion

$$G(\rho,\phi;\rho',\phi') = \frac{\ln(\rho_<^2/b^2)\ln(c^2/\rho_>^2)}{\ln(c^2/b^2)} + 2\sum_{m=1}^{\infty}\frac{\cos[m(\phi'-\phi)]}{m[1-(b/c)^{2m}]}\left(\rho_<^m - \frac{b^{2m}}{\rho_<^m}\right)\left(\frac{1}{\rho_>^m} - \frac{\rho_>^m}{c^{2m}}\right).$$

<Solution>

In terms of the same basis in the ϕ-coordinate, the Green function can be expanded as

$$\begin{aligned} G(\rho,\phi;\rho',\phi') &= \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} g_m(\rho,\rho')\, e^{im(\phi-\phi')} \\ &= \frac{1}{2\pi}g_0(\rho,\rho') + \frac{1}{2\pi}\sum_{m=1}^{\infty} g_m(\rho,\rho')\, e^{im(\phi-\phi')} + \frac{1}{2\pi}\sum_{m=1}^{\infty} g_m(\rho,\rho')\, e^{-im(\phi-\phi')} \end{aligned} . \quad (1)$$

The Green function in the ρ-component is seen to satisfy the homogeneous equation for $\rho \neq \rho'$. For $m \neq 0$, it can be written as

$$g_m(\rho,\rho') = \begin{cases} A_m\left[\rho^m - \dfrac{b^m}{\rho^m}\right] & \text{for } b < \rho < \rho' \\ B_m\left[\rho^{-m} - \dfrac{\rho^m}{c^{2m}}\right] & \text{for } \rho' < \rho < c \end{cases} . \quad (1)$$

For $m = 0$, $g_0(\rho,\rho')$ can be written as

$$g_0(\rho,\rho') = \begin{cases} A_0 \ln\left(\dfrac{\rho}{b}\right) & \text{for } b < \rho < \rho' \\ B_0 \ln\left(\dfrac{c}{\rho}\right) & \text{for } \rho' < \rho < c \end{cases} . \quad (2)$$

The symmetry in ρ and ρ' requires that the coefficients A_m and B_m be such that $g_m(\rho,\rho')$ can be written

$$g_m(\rho,\rho') = \begin{cases} C_m\left(\rho_<^m - \dfrac{b^{2m}}{\rho_<^m}\right)\left(\dfrac{1}{\rho_>^m} - \dfrac{\rho_>^m}{c^{2m}}\right) & \text{for } m \neq 0 \\ C_0 \ln\left(\dfrac{\rho_<}{b}\right)\ln\left(\dfrac{c}{\rho_>}\right) & \text{for } m = 0 \end{cases} , \quad (3)$$

where $\rho_<(\rho_>)$ is the smaller (larger) of ρ and ρ'. To determine the constant C_m, we consider the effect of the delta function:

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial g_m}{\partial\rho}\right) - \frac{m^2}{\rho^2}g_m = -4\pi\frac{\delta(\rho-\rho')}{\rho} . \quad (4)$$

After integrating both side of Eq. (4) for $m \neq 0$ over the interval from $\rho = \rho' - \varepsilon$ to $\rho = \rho' + \varepsilon$, where ε is very small, we obtain

$$\left[\rho\frac{d}{d\rho}g_m(\rho,\rho')\right]_{\rho'+\varepsilon}-\left[\rho\frac{d}{d\rho}g_m(\rho,\rho')\right]_{\rho'-\varepsilon}=-4\pi \quad . \tag{5}$$

With Eq. (3), we have

$$\left[\rho\frac{d}{d\rho}g_m(\rho,\rho')\right]_{\rho'+\varepsilon}=C_m\left(\rho'^m-\frac{b^{2m}}{\rho'^m}\right)\left(\frac{-m}{\rho'^m}-\frac{m\rho'^m}{c^{2m}}\right) \tag{6}$$

and

$$\left[\rho\frac{d}{d\rho}g_m(\rho,\rho')\right]_{\rho'-\varepsilon}=C_m\left(m\rho'^m+m\frac{b^{2m}}{\rho'^m}\right)\left(\frac{1}{\rho'^m}-\frac{\rho'^m}{c^{2m}}\right) \quad . \tag{7}$$

Substituting Eqs. (9) and (10) into Eq. (8), we find

$$C_m=\frac{2\pi}{m\left[1-\left(\frac{b}{c}\right)^{2m}\right]} \quad . \tag{8}$$

Following the same procedure, we can obtain $C_0=4\pi/\ln(c/b)$. Combining all coefficients, the free space Green function has the expansion

$$G(\rho,\phi;\rho',\phi')=\frac{\ln\left(\rho_<^2/b^2\right)\ln\left(c^2/\rho_>^2\right)}{\ln\left(c^2/b^2\right)}+2\sum_{m=1}^{\infty}\frac{\cos[m(\phi'-\phi)]}{m\left[1-(b/c)^{2m}\right]}\left(\rho_<^m-\frac{b^{2m}}{\rho_<^m}\right)\left(\frac{1}{\rho_>^m}-\frac{\rho_>^m}{c^{2m}}\right). \tag{9}$$

Example

Two conducting panes at zero potential meet along the *z* axis and intersect at an angle *β*. A unit line charge parallel to the z axis is located between the planes at position (ρ',ϕ').

(a) Show that the completeness relation for the angular functions is

$$\delta(\phi-\phi')=\frac{2}{\beta}\sum_{m=1}^{\infty}\sin(m\pi\phi/\beta)\sin(m\pi\phi'/\beta) \quad \text{for} \quad 0\le\phi,\phi'\le\beta \quad .$$

(b) Show that $(4\pi\varepsilon_o)$ times the potential in the space between the planes, that is, the Dirichlet Green function $G(\rho,\phi;\rho',\phi')$, is given by

$$G(\rho,\phi;\rho',\phi')=4\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{\rho_<}{\rho_>}\right)^{m\pi/\beta}\sin(m\pi\phi/\beta)\sin(m\pi\phi'/\beta) \, .$$

(c) Show that the series can be summed to give a closed form

$$G(\rho,\phi;\rho',\phi')=\ln\left\{\frac{\rho^{2\pi/\beta}+\rho'^{2\pi/\beta}-2(\rho\rho')^{\pi/\beta}\cos[\pi(\phi+\phi')/\beta]}{\rho^{2\pi/\beta}+\rho'^{2\pi/\beta}-2(\rho\rho')^{\pi/\beta}\cos[\pi(\phi-\phi')/\beta]}\right\} \quad .$$

<Solution>

In terms of the same basis in the ϕ-coordinate, the Green function can be expanded as

$$\begin{aligned} G(\rho,\phi;\rho',\phi') &= \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} g_m(\rho,\rho')\, e^{im(\phi-\phi')} \\ &= \frac{1}{2\pi} g_0(\rho,\rho') + \frac{1}{2\pi}\sum_{m=1}^{\infty} g_m(\rho,\rho')\, e^{im(\phi-\phi')} + \frac{1}{2\pi}\sum_{m=1}^{\infty} g_m(\rho,\rho')\, e^{-im(\phi-\phi')} \end{aligned} \quad . (1)$$

3. Laplace Equation in Cylindrical Coordinates; Bessel Functions

The forms of the solutions of Laplace's equation in circular cylindrical coordinates can be given by

$$\Phi(\rho,\phi,z) = R(\rho)Q(\phi)Z(z) \quad . \tag{1}$$

Substitution of Eq. (1) into Laplace's equation and multiplication by ρ^2 / RQZ yields

$$\frac{1}{R\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{dR}{d\rho}\right) + \frac{1}{Q\rho^2}\frac{\partial^2 Q}{\partial\phi^2} + \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = 0\,, \tag{2}$$

where total derivatives have replaced partial derivatives, since each term involves a function of one variable only. The solutions can be divided into two categories:

$$\begin{cases} \dfrac{1}{Z}\dfrac{\partial^2 Z}{\partial z^2} = k^2 \\ \dfrac{1}{Q}\dfrac{\partial^2 Q}{\partial\phi^2} = -m^2 \\ \dfrac{1}{R\rho}\dfrac{d}{d\rho}\left(\rho\dfrac{dR}{d\rho}\right) + \left(k^2 - \dfrac{m^2}{\rho^2}\right) = 0 \end{cases} \tag{3}$$

and

$$\begin{cases} \dfrac{1}{Z}\dfrac{\partial^2 Z}{\partial z^2} = -k^2 \\ \dfrac{1}{Q}\dfrac{\partial^2 Q}{\partial\phi^2} = -m^2 \\ \dfrac{1}{R\rho}\dfrac{d}{d\rho}\left(\rho\dfrac{dR}{d\rho}\right) - \left(k^2 + \dfrac{m^2}{\rho^2}\right) = 0 \end{cases} \quad . \tag{4}$$

Consider the first case in Eq. (3), the solutions for z and ϕ variables are given by

$$Z(z) = e^{\pm kz} \quad ; \quad Q(\phi) = e^{\pm i\, m\, \phi} \quad . \tag{5}$$

On the other hand, the radial equation can be put in a standard form by the change of variable $x = k\rho$. The equation becomes

$$\frac{d^2R}{dx^2}+\frac{1}{x}\frac{dR}{dx}+\left(1-\frac{m^2}{x^2}\right)R=0 \quad . \tag{6}$$

This is the Bessel equation and the solutions are called Bessel functions of the order *m*. With the approach of power series solution, the Bessel function can be found to be given by

$$J_m(x)=\sum_{j=0}^{\infty}\frac{(-1)^j}{\Gamma(j+m+1)j!}\left(\frac{x}{2}\right)^{2j+m} \tag{7}$$

$$J_{-m}(x)=\sum_{j=0}^{\infty}\frac{(-1)^j}{\Gamma(j-m+1)j!}\left(\frac{x}{2}\right)^{2j-m} \quad . \tag{8}$$

These solutions are called Bessel functions of the first kind of order $\pm m$. The series converge for all finite values of *x*. If *m* is not an integer, these two solutions $J_{\pm m}(x)$ form a pair of linearly independent solutions to the second-order Bessel equation. For the potential to be single-valued when the full azimuthal is allowed, *m* must be an integer. Under this circumstance, it is well known that the solutions are linearly dependent. Actually it can be shown that

$$J_{-m}(x)=(-1)^m J_m(x)\,. \tag{9}$$

In general, no matter what *m* is, the second solution is replaced by the *Neumann function*:

$$N_m(x)=\frac{J_m(x)\cos m\pi - J_{-m}(x)}{\sin m\pi}\,. \tag{10}$$

The solutions $J_m(x)$ and $N_m(x)$ are called Bessel functions of the second kind.

The Bessel functions of the third kind, called *Hankel functions*, are defined as linear combinations of $J_m(x)$ and $N_m(x)$:

$$\begin{aligned} H_m^{(1)}(x)&=J_m(x)+iN_m(x)\\ H_m^{(2)}(x)&=J_m(x)-iN_m(x) \end{aligned} \quad . \tag{11}$$

The Hankel functions form a fundamental set of solutions to the Bessel equation, just as do $J_m(x)$ and $N_m(x)$.

The other solution in the separation of The Laplace equation is given by Eq. (4). The function $Z(z)$ would have been $\sin kz$ or $\cos kz$ and the equation for $R(\rho)$ would have been:

$$\frac{d^2R}{d\rho^2}+\frac{1}{\rho}\frac{dR}{d\rho}-\left(k^2+\frac{m^2}{\rho^2}\right)R=0 \quad . \tag{12}$$

With $x=k\rho$. The equation becomes

$$\frac{d^2R}{dx^2}+\frac{1}{x}\frac{dR}{dx}-\left(1+\frac{m^2}{x^2}\right)R=0 \quad . \tag{13}$$

The solutions of this equation are called modified Bessel functions. It is evident that they are just Bessel functions of pure imaginary argument. The usual choices of linearly independent solutions are denoted by $I_m(x)$ and $K_m(x)$. They are defined by

$$I_m(x)=i^{-m}J_m(ix) \quad , \tag{14}$$

$$K_m(x)=\frac{\pi}{2}i^{m+1}H_m^{(1)}(ix) \tag{15}$$

and are real functions for real x and m.

Generating Function for Bessel Functions

Another representation for the Bessel functions is based on the generating function. The generating function of the Bessel functions of integral order is given by

$$e^{i\rho\sin\phi}=\sum_{m=-\infty}^{\infty}J_m(\rho)e^{im\phi} . \tag{1}$$

Using $i\sin\phi=(e^{i\phi}-e^{-i\phi})/2$, the left-hand side of Eq. (1) can be derived as

$$e^{i\rho\sin\phi}=\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{\rho}{2}\right)^n\left(e^{i\phi}-e^{-i\phi}\right)^n=\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{\rho}{2}\right)^n\sum_{j=0}^{n}\frac{(-1)^j n!}{j!\,(n-j)!}e^{i(n-2j)\phi} , \tag{2}$$

where the binominal expansion has applied to the term $\left(e^{i\phi}-e^{-i\phi}\right)^n$ in the derivation. Changing the index n in Eq. (2) as $m=n-2j$, the expansion in Eq. (2) can be written as

$$e^{i\rho\sin\phi}=\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{\rho}{2}\right)^n\sum_{j=0}^{n}\frac{(-1)^j n!}{j!\,(n-j)!}e^{i(n-2j)\phi}=\sum_{m=-\infty}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^j}{j!\,(m+j)!}\left(\frac{\rho}{2}\right)^{m+2j}e^{im\phi} . \tag{3}$$

In comparison with Eq. (1), the Bessel function is given by

$$J_m(\rho)=\sum_{j=0}^{\infty}\frac{(-1)^j}{j!(m+j)!}\left(\frac{\rho}{2}\right)^{m+2j} \quad . \tag{4}$$

This result is the same as Eq. (7) in the above section. Differentiating Eq. (1) partially with respect to ϕ, we have

$$\frac{\partial}{\partial\phi}e^{i\rho\sin\phi}=\frac{\rho}{2}\left(e^{i\phi}+e^{-i\phi}\right)e^{i\rho\sin\phi}=\sum_{m=-\infty}^{\infty}mJ_m(\rho)e^{im\phi} \quad . \tag{5}$$

Substituting in Eq. (1) and equating the coefficients of like terms of $e^{im\phi}$, we obtain the result

$$\frac{\rho}{2}\left(e^{i\phi}+e^{-i\phi}\right)\sum_{m=-\infty}^{\infty} J_m(\rho)e^{im\phi} = \sum_{m=-\infty}^{\infty} \frac{\rho}{2}\left[J_{m-1}(\rho)+J_{m+1}(\rho)\right]e^{im\phi} = \sum_{m=-\infty}^{\infty} mJ_m(\rho)e^{im\phi} .$$

Consequently the first recurrence relation can be expressed as

$$J_{m-1}(\rho)+J_{m+1}(\rho)=\frac{2m}{\rho}J_m(\rho) \quad . \tag{6}$$

Similarly, differentiating Eq. (1) partially with respect to ρ, we have

$$\frac{d}{d\rho}e^{i\rho\sin\phi}=\frac{1}{2}\left(e^{i\phi}-e^{-i\phi}\right)e^{i\rho\sin\phi}=\sum_{m=-\infty}^{\infty} J'_m(\rho)e^{im\phi} \quad . \tag{7}$$

Substituting in Eq. (1) and equating the coefficients of like terms of $e^{im\phi}$, we obtain the result

$$\frac{1}{2}\left(e^{i\phi}-e^{-i\phi}\right)\sum_{m=-\infty}^{\infty} J_m(\rho)e^{im\phi} = \sum_{m=-\infty}^{\infty} \frac{1}{2}\left[J_{m-1}(\rho)-J_{m+1}(\rho)\right]e^{im\phi} = \sum_{m=-\infty}^{\infty} J'_m(\rho)e^{im\phi} .$$

Consequently the second recurrence relation can be expressed as

$$J_{m-1}(\rho)-J_{m+1}(\rho)=2J'_m(\rho) \quad . \tag{8}$$

Suppose we consider a set of function $\Omega_v(x)$ which satisfies the basic recurrence relations (Eqs. (6) and (8)), but with v not necessarily integer and $\Omega_v(x)$ not necessarily given by the series Eq. (4). Subtracting Eq. (8) from Eq. (6) and dividing by 2 yields

$$x\,\Omega'_v(x)-v\Omega_v(x)+x\Omega_{v+1}(x)=0 \quad , \tag{9}$$

where the index has been changed as $(m\to v)$. Adding Eq. (6) and Eq. (8) and dividing by 2, the result can be rewritten $(m\to v)$ as

$$x\,\Omega'_v(x)+v\Omega_v(x)-x\Omega_{v-1}(x)=0 \quad . \tag{10}$$

On differentiating with respect to x, we have

$$x\Omega''_v(x)+(v+1)\Omega'_v(x)-\Omega_{v-1}(x)-x\Omega'_{v-1}(x)=0 \quad . \tag{11}$$

Multiplying by x and then subtracting Eq. (10) multiplied by v gives us

$$x^2\,\Omega''_v(x)+x\Omega'_v(x)-v^2\,\Omega_v(x)+x(v-1)\Omega_{v-1}(x)-x^2\,\Omega'_{v-1}(x)=0 \quad . \tag{12}$$

Now we write Eq. (9) and replace v by v-1:

$$x\,\Omega'_{v-1}(x)-(v-1)\Omega_{v-1}(x)+x\Omega_v(x)=0 \; . \tag{13}$$

Adding Eqs. (12) and (13) for eliminating $\Omega'_{v-1}(x)$ and $\Omega_{v-1}(x)$, we finally get

$$x^2\,\Omega''_v(x)+x\Omega'_v(x)+\left(x^2-v^2\right)\Omega_v(x)=0 \; . \tag{14}$$

This is just Bessel's equation. Hence any functions, $\Omega_v(x)$, that satisfy the recurrence relations Eqs. (6) and (8) satisfy Bessel's equation. In other words, the unknown $\Omega_v(x)$ are Bessel functions. In particular, we have shown that the functions $J_m(\rho)$, defined by the generating function, satisfy Bessel's equation. If the argument is $k\rho$ rather than x, Eq. (14)

becomes

$$\rho^2 \frac{d^2}{d\rho^2}\Omega_v(k\rho) + \rho \frac{d}{d\rho}\Omega_v(k\rho) + \left(k^2\rho^2 - v^2\right)\Omega_v(k\rho) = 0 . \tag{15}$$

Closure equation and orthogonality

The generating function in Eq. (1) can be linked to the 2D plane wave:

$$e^{i\left(k_x x + k_y y\right)} = e^{i k\rho \cos(\phi-\theta)} = \sum_{m=-\infty}^{\infty} (i)^m J_m(k\rho) e^{i m(\phi-\theta)} . \tag{16}$$

From the Fourier transform, we have

$$\left(\frac{1}{2\pi}\right)^2 \iint e^{i\left[k_x(x-x')+k_y(y-y')\right]} dk_x dk_y = \delta(x-x')\delta(y-y') = \frac{\delta(\rho-\rho')}{\rho}\delta(\phi-\phi') . \tag{17}$$

Combining Eqs. (16) and (17), we have

$$\begin{aligned} &\left(\frac{1}{2\pi}\right)^2 \int_0^\infty \int_0^{2\pi} \sum_{m=-\infty}^{\infty} (i)^m J_m(k\rho) e^{i m(\phi-\theta)} \sum_{m'=-\infty}^{\infty} (i)^{-m'} J_{m'}(k\rho') e^{-i m'(\phi'-\theta)} d\theta \, kdk \\ &= \frac{\delta(\rho-\rho')}{\rho}\delta(\phi-\phi') \end{aligned} . \tag{18}$$

Using the orthonormal property

$$\left(\frac{1}{2\pi}\right) \int_0^{2\pi} e^{i\theta(m'-m)} d\theta = \delta_{m,m'} , \tag{19}$$

we can obtain

$$\left(\frac{1}{2\pi}\right) \int_0^\infty \sum_{m=-\infty}^{\infty} J_m(k\rho) J_m(k\rho') e^{i m(\phi-\phi')} \, kdk = \frac{\delta(\rho-\rho')}{\rho}\delta(\phi-\phi') . \tag{20}$$

Since the Fourier series gives

$$\left(\frac{1}{2\pi}\right) \sum_{m=-\infty}^{\infty} e^{i m(\phi-\phi')} = \delta(\phi-\phi') , \tag{21}$$

we have

$$\int_0^\infty J_m(k\rho) J_m(k\rho') \, k \, dk = \frac{\delta(\rho-\rho')}{\rho} . \tag{22}$$

Equivalently, we have

$$\int_0^\infty J_m(k\rho) J_m(k'\rho) \, \rho \, d\rho = \frac{\delta(k-k')}{k} . \tag{23}$$

If there is a boundary condition $J_m(ka) = 0$ for a finite interval $0 \le \rho \le a$, then the parameter k should be quantized as

$$k_{mn} = x_{mn} / a , \tag{24}$$

where x_{mn} is the nth zero of J_m. The solutions are expected to be orthogonal. The demonstration starts with the differential equation satisfied by $J_m(x_{mn}\rho/a)$:

$$\frac{1}{\rho}\frac{d}{d\rho}\left[\rho\frac{d\,J_m\left(x_{mn}\frac{\rho}{a}\right)}{d\rho}\right]+\left(\frac{x_{mn}^2}{a^2}-\frac{m^2}{\rho^2}\right)J_m\left(x_{mn}\frac{\rho}{a}\right)=0 \quad . \tag{25}$$

Changing the parameter x_{mn} to $x_{mn'}$, we find that $J_m(x_{mn'}\rho/a)$ satisfies

$$\frac{1}{\rho}\frac{d}{d\rho}\left[\rho\frac{d\,J_m\left(x_{mn'}\frac{\rho}{a}\right)}{d\rho}\right]+\left(\frac{x_{mn'}^2}{a^2}-\frac{m^2}{\rho^2}\right)J_m\left(x_{mn'}\frac{\rho}{a}\right)=0 \quad . \tag{26}$$

We multiply Eq. (25) by $\rho J_m(x_{mn'}\rho/a)$ and Eq. (26) by $\rho J_m(x_{mn}\rho/a)$ and subtract, obtaining

$$\begin{aligned}&J_m\left(x_{mn'}\frac{\rho}{a}\right)\frac{d}{d\rho}\left[\rho\frac{d\,J_m\left(x_{mn}\frac{\rho}{a}\right)}{d\rho}\right]-J_m\left(x_{mn}\frac{\rho}{a}\right)\frac{d}{d\rho}\left[\rho\frac{d\,J_m\left(x_{mn'}\frac{\rho}{a}\right)}{d\rho}\right]\\&=\left(\frac{x_{mn'}^2}{a^2}-\frac{x_{mn}^2}{a^2}\right)\rho J_m\left(x_{mn}\frac{\rho}{a}\right)J_m\left(x_{mn'}\frac{\rho}{a}\right)\end{aligned} \quad . \tag{27}$$

Integrating from $\rho=0$ to $\rho=a$, we obtain

$$\begin{aligned}&\frac{x_{mn'}^2-x_{mn}^2}{a^2}\int_0^a \rho\,J_m\left(x_{mn}\frac{\rho}{a}\right)J_m\left(x_{mn'}\frac{\rho}{a}\right)d\rho\\&=\int_0^a\left\{J_m\left(x_{mn'}\frac{\rho}{a}\right)\frac{d}{d\rho}\left[\rho\frac{d\,J_m\left(x_{mn}\frac{\rho}{a}\right)}{d\rho}\right]-J_m\left(x_{mn}\frac{\rho}{a}\right)\frac{d}{d\rho}\left[\rho\frac{d\,J_m\left(x_{mn'}\frac{\rho}{a}\right)}{d\rho}\right]\right\}d\rho\end{aligned} \quad . \tag{28}$$

Upon integrating by parts in the right-hand side of Eq. (28), we have

$$\begin{aligned}&\frac{x_{mn'}^2-x_{mn}^2}{a^2}\int_0^a \rho\,J_m\left(x_{mn}\frac{\rho}{a}\right)J_m\left(x_{mn'}\frac{\rho}{a}\right)d\rho\\&=\rho\frac{d\,J_m\left(x_{mn}\frac{\rho}{a}\right)}{d\rho}J_m\left(x_{mn'}\frac{\rho}{a}\right)\Bigg|_0^a-\rho\frac{d\,J_m\left(x_{mn'}\frac{\rho}{a}\right)}{d\rho}J_m\left(x_{mn}\frac{\rho}{a}\right)\Bigg|_0^a\end{aligned} \quad . \tag{29}$$

For $m \geq 0$ the factor ρ guarantees a zero at the lower limit, $\rho=0$. At $\rho=a$, each expression on the right-hand side of Eq. (29) vanishes because the parameters x_{mn} and $x_{mn'}$ are roots of J_m. Therefore, for $n \neq n'$

$$\int_0^a \rho\, J_m\left(x_{mn}\frac{\rho}{a}\right)J_m\left(x_{mn'}\frac{\rho}{a}\right)d\rho = 0 \quad . \tag{30}$$

This gives us orthogonality over the interval [0, *a*]. The normalization integral may be developed by rewriting Eq. (29) as

$$\begin{aligned}&\int_0^a \rho\, J_m\left(x_{mn}\frac{\rho}{a}\right)J_m\left(x_{mn'}\frac{\rho}{a}\right)d\rho\\ &=\left(\frac{a^2}{x_{mn'}^2-x_{mn}^2}\right)\left[x_{mn}\frac{d\,J_m(x_{mn})}{d\,x}J_m(x_{mn'})-x_{mn'}\frac{d\,J_m(x_{mn'})}{d\,x}J_m(x_{mn})\right]\end{aligned} \quad . \tag{30}$$

Setting $x_{mn'} = x_{mn} + \varepsilon$, and taking the limit $\varepsilon \to 0$, we have

$$\int_0^a \rho\, J_m\left(x_{mn}\frac{\rho}{a}\right)J_m\left(x_{mn}\frac{\rho}{a}\right)d\rho = \left(\frac{a^2}{2}\right)\left[\frac{d\,J_m(x_{mn})}{d\,x}\right]^2 \quad . \tag{31}$$

With the aid of the recurrence relation Eq. (9), this result can be also written as

$$\int_0^a \rho\, J_m\left(x_{mn}\frac{\rho}{a}\right)J_m\left(x_{mn}\frac{\rho}{a}\right)d\rho = \left(\frac{a^2}{2}\right)\left[J_{m+1}(x_{mn})\right]^2 \quad . \tag{32}$$

Example

Considering the potential in charge-free space between the planes at $z = 0$ and $z = L$, the potential in cylindrical coordinates is specified to be $V(\rho,\phi)$ at the plane $z = L$ and zero at the plane $z = 0$. Find the general form of the solution with the Fourier-Bessel integral and derive the expression for determining coefficients.

<Solution>

In order that $\Phi(\rho,\phi,z)$ be single valued and vanish at $z = 0$,

$$Q(\phi) = e^{\pm i m \phi} \quad ; \qquad Z(z) = \sinh kz \quad .$$

The radial factor is

$$R(\rho) = AJ_m(k\rho) + BN_m(k\rho) \quad .$$

Since the potential is finite at $\rho = 0$, the coefficient *B* needs to be zero. The general form of the solution for the space between the planes at $z = 0$ and $z = L$ is given by

$$\Phi(\rho,\phi,z) = \sum_{m=-\infty}^{\infty}\left[\int_0^\infty A_m(k)\sinh(kz)J_m(k\rho)dk\right]e^{i m \phi} \quad . \tag{1}$$

If the potential is specified over the whole plane $z = L$ to be $V(\rho,\phi)$, the coefficients

$A_m(k)$ are determined by

$$V(\rho',\phi') = \sum_{m=-\infty}^{\infty} \left[\int_0^\infty A_m(k')\sinh(k'L)J_m(k'\rho')dk'\right]e^{i\,m\,\phi'} \quad . \tag{2}$$

Using the orthogonal properties Eqs. (21) and (23), the coefficients $A_m(k)$ can be derived as

$$A_m(k) = \frac{k}{2\pi\sinh(kL)}\int_0^\infty \int_0^{2\pi} V(\rho',\phi')\,\rho' J_m(k\rho')\,e^{-i\,m\,\phi'}\,d\phi' d\rho' \quad . \tag{3}$$

Example

Consider the specific boundary-value problem in which the cylinder has a radius *a* and a height *L*, the top and bottom surfaces being at $z = L$ and $z = 0$. The potential on the side and the bottom of the cylinder is zero, while the top has a potential $V(\rho,\phi)$. Find the potential at any point inside the cylinder.

<Solution>

The requirement that the potential vanishes at $\rho = a$ means that the parameter *k* in the radial factor $R(\rho) = J_m(k\rho)$ can take on only those special values:

$$k_{mn} = x_{mn}/a \quad , \quad (n = 1, 2, 3, \ldots)$$

where x_{mn} are the roots of $J_m(x_{mn}) = 0$. Consequently the general form of the solution is given by

$$\Phi(\rho,\phi,z) = \sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty} A_{mn}\sinh(k_{mn}z)J_m(k_{mn}\rho)e^{i\,m\,\phi} \quad . \tag{1}$$

If the potential is specified over the whole plane $z = L$ to be $V(\rho,\phi)$, the coefficients A_{mn} are determined by

$$V(\rho',\phi') = \sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty} A_{mn}\sinh(k_{mn}L)J_m(k_{mn}\rho')e^{i\,m\,\phi'} \quad . \tag{2}$$

Using the orthogonal properties Eqs. (21) and (23), the coefficients $A_m(k)$ can be derived as

$$A_{mn} = \frac{\operatorname{csch}(k_{mn}L)}{\pi a^2 J_{m+1}^2(k_{mn}a)}\int_0^a \int_0^{2\pi} V(\rho',\phi')J_m(k_{mn}\rho')\,e^{-i\,m\,\phi'}\,d\phi'\rho' d\rho' \quad . \tag{3}$$

Example

A hollow right circular cylinder of radius *b* has its axis coincident with *z* axis and its ends at $z = 0$ and $z = L$. The potential on the end faces is zero, while the potential on the cylindrical surface is given as $V(\phi, z)$. Find a series solution for the potential anywhere inside the cylinder.

<Solution>

In order that $\Phi(\rho,\phi,z)$ be single valued and vanish at $z=0$ and $z=L$,

$$Q(\phi)=e^{\pm i m \phi} \quad ; \qquad Z(z)=\sin\left(\frac{n\pi z}{L}\right) \quad .$$

The radial factor is

$$R(\rho)=AI_m\left(\frac{n\pi\rho}{L}\right)+BK_m\left(\frac{n\pi\rho}{L}\right) \quad .$$

Since the potential is finite at $\rho=0$, the coefficient *B* needs to be zero. The general form of the solution for the space between the planes at $z=0$ and $z=L$ is given by

$$\Phi(\rho,\phi,z)=\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}A_{mn}\sin\left(\frac{n\pi}{L}z\right)I_m\left(\frac{n\pi}{L}\rho\right)e^{i m \phi} \quad . \tag{1}$$

If the potential is specified over the whole plane $\rho=b$ to be $V(\phi,z)$, the coefficients A_{mn} are determined by

$$V(\phi',z')=\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}A_{mn}\sin\left(\frac{n\pi}{L}z'\right)I_m\left(\frac{n\pi}{L}b\right)e^{i m \phi'} \,. \tag{2}$$

Using the orthogonal properties, the coefficients A_{mn} can be derived as

$$A_{mn}=\frac{1}{\pi L I_m\left(\frac{n\pi}{L}b\right)}\int_0^L\int_0^{2\pi} V(\phi',z')\sin\left(\frac{n\pi}{L}z'\right)e^{-i m \phi'}\, d\phi' dz' \quad . \tag{3}$$

Example

Following the above example, the cylindrical surface is made of two equal half cylinders, one at potential *V* and the other at potential –*V*, so that $V(\phi,z)=\begin{cases} V & \text{for } 0<\phi<\pi \\ -V & \text{for } \pi<\phi<2\pi \end{cases}$.

Find the potential inside the cylinder. Assuming *L*>>*b*, consider the potential at $z=L/2$ as a function of ρ and ϕ.

<Solution>

With the boundary condition, the coefficient A_{mn} is given by

$$A_{mn}=\frac{V}{\pi L I_m\left(\frac{n\pi}{L}b\right)}\int_0^L \sin\left(\frac{n\pi}{L}z'\right)dz'\left[\int_0^{\pi} e^{-i m \phi'}\, d\phi' - \int_{\pi}^{2\pi} e^{-i m \phi'}\, d\phi'\right] \quad .$$

The final integration yields

$$A_{mn}=\frac{V}{\pi^2 I_m\left(\frac{n\pi}{L}b\right)}\frac{1}{n}\left[1-(-1)^n\right]\left(\frac{-2i}{m}\right)\left[1-(-1)^m\right] \ .$$

As a result, only if both m and n are odd, A_{mn} can be nonzero. In terms of new indices, $m = 2k+1$ and $n = 2s+1$, the potential can be expressed as

$$\Phi(\rho,\phi,z) = \frac{16\,V}{\pi^2}\sum_{k=0}^{\infty}\sum_{s=0}^{\infty}\frac{I_{2k+1}\left(\frac{(2s+1)\pi}{L}\rho\right)}{I_{2k+1}\left(\frac{(2s+1)\pi}{L}b\right)}\left(\frac{1}{2s+1}\right)\left(\frac{1}{2k+1}\right)\sin\left(\frac{(2s+1)\pi}{L}z\right)\sin[(2k+1)\phi]$$

For L>>b, we have $\pi\rho/L << 1$ and $\pi b/L << 1$. Using the asymptotic form

$$I_m(x) \to \frac{1}{\Gamma(m+1)}\left(\frac{x}{2}\right)^m \quad \text{for } x << 1 \quad ,$$

we obtain

$$\frac{I_{2k+1}\left(\frac{(2s+1)\pi}{L}\rho\right)}{I_{2k+1}\left(\frac{(2s+1)\pi}{L}b\right)} = \left(\frac{\rho}{b}\right)^{2k+1} \quad .$$

Hence the potential at $z = L/2$ can be expressed as

$$\Phi\left(\rho,\phi,\frac{L}{2}\right) = \frac{16\,V}{\pi^2}\sum_{k=0}^{\infty}\sum_{s=0}^{\infty}\left(\frac{\rho}{b}\right)^{2k+1}\left(\frac{1}{2s+1}\right)\left(\frac{1}{2k+1}\right)(-1)^s\sin[(2k+1)\phi] \quad .$$

Using the Fourier series

$$1 = \frac{4}{\pi}\sum_{s=0}^{\infty}\left(\frac{1}{2s+1}\right)(-1)^s \quad ,$$

we obtain

$$\Phi\left(\rho,\phi,\frac{L}{2}\right) = \frac{4V}{\pi}\sum_{k=0}^{\infty}\left(\frac{1}{2k+1}\right)\left(\frac{\rho}{b}\right)^{2k+1}\sin[(2k+1)\phi] = \frac{4V}{\pi}\,\mathrm{Im}\left[\sum_{k=0}^{\infty}\left(\frac{1}{2k+1}\right)\left(\frac{\rho}{b}e^{i\phi}\right)^{2k+1}\right]$$

Using the fact that

$$\mathrm{Im}\sum_{n\,\mathrm{odd}}\frac{Z^n}{n} = \mathrm{Im}\left[\frac{1}{2}\ln\left(\frac{1+Z}{1-Z}\right)\right] = \frac{1}{2}\tan^{-1}\left[\frac{2\,\mathrm{Im}\,Z}{1-|Z|^2}\right] \quad ,$$

we can obtain

$$\Phi\left(\rho,\phi,\frac{L}{2}\right) = \frac{2V}{\pi}\tan^{-1}\left(\frac{2b\rho\sin\phi}{b^2-\rho^2}\right) \quad .$$

This result is the same as the case discussed in the 2D polar coordinates.

Self-adjoint differential equations

Linear second-order differential equations corresponding to linear second-order differential operators are generally given by

$$\hat{L}u(x) = p_0(x)\frac{d^2u(x)}{dx^2} + p_1(x)\frac{du(x)}{dx} + p_2(x)u(x) \quad . \tag{1}$$

The coefficients $p_0(x)$, $p_1(x)$, and $p_2(x)$ are real functions of x and over the region of interest, $a \le x \le b$, the first $2-n$ derivatives of $p_n(x)$ are continuous. Further, $p_0(x)$ does not vanish for $a < x < b$. Now, the zeros of $p_n(x)$ are singular points, and we choose the interval [a, b] so that there are no singular points in the interior of the interval.

In mathematics, the adjoint operator $\hat{L}^+$ corresponding to the operator $\hat{L}$ is defined as

$$\begin{aligned}\hat{L}^+u(x) &= \frac{d^2[p_0(x)u(x)]}{dx^2} - \frac{d[p_1(x)u(x)]}{dx} + p_2(x)u(x) \\ &= p_0(x)\frac{d^2u(x)}{dx^2} + \left[2p_0'(x) - p_1(x)\right]\frac{du(x)}{dx} + \left[p_0''(x) - p_1'(x) + p_2(x)\right]u(x)\end{aligned} \quad . \tag{2}$$

In comparison of Eqs. (1) and (2), the necessary and sufficient condition that $\hat{L} = \hat{L}^+$ is that

$$p_0'(x) = p_1(x) \quad . \tag{3}$$

When this condition is satisfied, we have the general form for the self-adjoint operator

$$\hat{L}^+u(x) = \hat{L}u(x) = \frac{d}{dx}\left[p(x)\frac{du(x)}{dx}\right] + q(x)u(x) \, . \tag{4}$$

Here $p_0(x)$ is replaced by $p(x)$ and $p_2(x)$ by $q(x)$ to avoid unnecessary subscripts.

From separation of variables or directly from a physical problem we have a linear second-order differential equation of the form

$$\hat{L}u(x) + \lambda w(x)u(x) = 0 \quad . \tag{6}$$

Here λ is a constant and $w(x)$ is a given function. For a given choice of the parameter λ, a function $u_\lambda(x)$, which satisfies Eq. (6) and the imposed boundary conditions, is called an eigenfunction corresponding to λ. The constant λ is then called an eigenvalue. The boundary conditions may take three forms: (a) Cauchy boundary conditions. The value of a function and normal derivative specified on the boundary. In electrostatics this would mean Φ, the potential, and E_n the normal components of the electric field. (b) Dirichlet boundary conditions. The value of a function specified on the boundary. (c) Neumann boundary conditions. The normal derivative (normal gradient) of a function specified on the boundary. Usually the form of the differential equation or the boundry conditions on the solutions will guarantee that at the ends of the interval [a,b] the following products will vanish:

$$\left. p(x)v^*(x)\frac{du(x)}{dx}\right|_{x=a} = 0$$
$$\left. p(x)v^*(x)\frac{du(x)}{dx}\right|_{x=b} = 0 \quad . \tag{7}$$

Here $u(x)$ and $v(x)$ are solutions of Eq. (6). Another important case for dealing with a periodic physical system is the cyclic boundary condition

$$\left. p(x)v^*(x)\frac{du(x)}{dx}\right|_{x=a} = \left. p(x)v^*(x)\frac{du(x)}{dx}\right|_{x=b} . \tag{8}$$

When $u(x)$ and $v(x)$ are solutions of Eq. (6), the Wronskian of $u(x)$ and $v(x)$ is equal to a constant divided by the coefficient $p(x)$. Since $u(x)$ and $v(x)$ are solutions of Eq. (6), we have

$$\frac{d}{dx}\left[p(x)\frac{du(x)}{dx}\right] + q(x)u(x) + \lambda w(x)u(x) = 0 \tag{9}$$

and

$$\frac{d}{dx}\left[p(x)\frac{dv(x)}{dx}\right] + q(x)v(x) + \lambda w(x)v(x) = 0 \tag{10}$$

Eq. (10) times $u(x)$ subtracting Eq. (9) times $v(x)$ yields

$$u(x)\frac{d}{dx}\left[p(x)\frac{dv(x)}{dx}\right] - v(x)\frac{d}{dx}\left[p(x)\frac{du(x)}{dx}\right] = 0 \quad . \tag{11}$$

Integrating Eq. (11) for both ends and using integration by parts, we can obtain

$$W[u,v] = u(x)\frac{dv(x)}{dx} - v(x)\frac{du(x)}{dx} = \frac{C}{p(x)} \quad . \tag{12}$$

For modified Bessel function $I_m(x)$ and $K_m(x)$,it can be found that

$$W\left[I_m(x), K_m(x)\right] = -\frac{1}{x} \quad . \tag{13}$$

4. Expansion of Green functions in cylindrical coordinates

The expansion of the potential of a unit point charge in cylindrical coordinates affords a useful example of Green function expansions. A Green function for a Dirichlet potential problem in cylindrical coordinates satisfies the equation

$$\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2}\right]G(\rho,\phi,z;\rho',\phi',z') = -4\pi\frac{\delta(\rho-\rho')}{\rho}\delta(\phi-\phi')\delta(z-z') . \tag{1}$$

The completeness relation can be used to represent the functions $\delta(\phi-\phi')$ and $\delta(z-z')$:

$$\delta(\phi-\phi')=\frac{1}{2\pi}\sum_{m=-\infty}^{\infty}e^{im(\phi-\phi')} \quad . \tag{2}$$

$$\delta(z-z')=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{ik(z-z')}dk=\frac{1}{\pi}\int_{0}^{\infty}\cos[k(z-z')]dk \quad . \tag{3}$$

In terms of the same basis in the ϕ and z coordinates, the Green function can be expanded as

$$G(\rho,\phi,z;\rho',\phi',z')=\frac{1}{2\pi^2}\sum_{m=-\infty}^{\infty}\int_0^{\infty}g_m(k,\rho,\rho')\,e^{im(\phi-\phi')}\cos[k(z-z')]dk \quad . \tag{4}$$

Substitution of Eqs. (2)-(4) into Eq. (1) leads to

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dg_m}{d\rho}\right)-\left(k^2+\frac{m^2}{\rho^2}\right)g_m=-4\pi\frac{\delta(\rho-\rho')}{\rho} \quad . \tag{5}$$

For $\rho\neq\rho'$, the Green function in the ρ-component is seen to satisfy the modified Bessel differential equations. When there are no boundary surfaces, $g_m(k,\rho,\rho')$ needs to be finite at $\rho=0$ and vanishes at $\rho\to\infty$. Consequently, $g_m(k,\rho,\rho')$ can be written as

$$g_m(k,\rho,\rho')=\begin{cases}A_m(k)I_m(k\rho) & \text{for } \rho<\rho' \\ B_m(k)K_m(k\rho) & \text{for } \rho>\rho'\end{cases} \quad . \tag{6}$$

The symmetry in ρ and ρ' requires that the coefficients $A_m(k)$ and $B_m(k)$ be such that $g_m(k,\rho,\rho')$ can be written

$$g_m(k,\rho,\rho')=C_m(k)\,I_m(k\rho_<)K_m(k\rho_>) \quad , \tag{7}$$

where $\rho_<(\rho_>)$ is the smaller (larger) of ρ and ρ'. To determine the constant C_m, we consider the effect of the delta function in Eq. (5). If we integrate both side of Eq. (5) over the interval from $\rho=\rho'-\varepsilon$ to $\rho=\rho'+\varepsilon$, where ε is very small, we obtain

$$\left[\frac{d}{d\rho}g_m(k,\rho,\rho')\right]_{\rho'+\varepsilon}-\left[\frac{d}{d\rho}g_m(k,\rho,\rho')\right]_{\rho'-\varepsilon}=-\frac{4\pi}{\rho'} \quad . \tag{8}$$

For modified Bessel function $I_m(x)$ and $K_m(x)$, it has been shown that

$$W[I_m(x),K_m(x)]=-\frac{1}{x} \quad . \tag{9}$$

Substituting Eq. (7) into Eq. (8) and using Eq. (9), we can find

$$C_m=4\pi \quad . \tag{10}$$

Combining all coefficients, the free space Green function just for the expansion of $1/|\mathbf{r}-\mathbf{r}'|$ therefore becomes

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|}=\frac{2}{\pi}\sum_{m=-\infty}^{\infty}\int_0^{\infty}I_m(k\rho_<)K_m(k\rho_>)\,e^{im(\phi-\phi')}\cos[k(z-z')]dk \quad . \tag{11}$$

where $\rho_<(\rho_>)$ is the smaller (larger) of ρ and ρ'. This can be written entirely in terms of real functions as

$$G(\rho,\phi,z;\rho',\phi',z') = \frac{1}{|\mathbf{r}-\mathbf{r}'|} =$$
$$\frac{4}{\pi}\int_0^\infty \left\{\frac{1}{2}I_0(k\rho_<)K_0(k\rho_>) + \sum_{m=1}^{\infty}\cos[m(\phi-\phi')]I_m(k\rho_<)K_m(k\rho_>)\right\}\cos[k(z-z')]dk \quad . \tag{12}$$

Alternatively, we can use the Bessel functions to represent the functions $\delta(\rho-\rho')/\rho$

$$\int_0^\infty J_m(k\rho)J_m(k\rho')\,k\,dk = \frac{\delta(\rho-\rho')}{\rho} \, . \tag{13}$$

In terms of the basis in the ϕ and ρ coordinates, the Green function can be expanded as

$$G(\rho,\phi,z;\rho',\phi',z') = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\int_0^\infty g(k,z,z')\,e^{im(\phi-\phi')}J_m(k\rho)J_m(k\rho')\,kdk \quad . \tag{14}$$

Substitution of Eqs. (2), (13), and (14) into Eq. (1) leads to

$$\left(\frac{d^2}{dz^2} - k^2\right)g(k,z,z') = -4\pi\delta(z-z') \quad . \tag{15}$$

For $z \neq z'$, the Green function in the z-component is seen to satisfy a simple differential equations. When there are no boundary surfaces, $g(k,z,z')$ vanishes at $z \to \pm\infty$. Consequently, $g(k,z,z')$ can be written as

$$g(k,z,z') = \begin{cases} A(k)\,e^{kz} & \text{for } z < z' \\ B(k)\,e^{-kz} & \text{for } z > z' \end{cases} \quad . \tag{16}$$

The symmetry in z and z' requires that the coefficients $A(k)$ and $B(k)$ be such that $g(k,z,z')$ can be written

$$g(k,z,z') = C(k)\,e^{kz_<}e^{-kz_>} \quad , \tag{17}$$

where $z_<(z_>)$ is the smaller (larger) of z and z'. To determine the constant $C(k)$, we consider the effect of the delta function in Eq. (15). If we integrate both side of Eq. (15) over the interval from $z = z' - \varepsilon$ to $z = z' + \varepsilon$, where ε is very small, we obtain

$$\left[\frac{d}{dz}g(k,z,z')\right]_{z'+\varepsilon} - \left[\frac{d}{dz}g(k,z,z')\right]_{z'-\varepsilon} = -4\pi \quad . \tag{18}$$

Substituting Eq. (17) into Eq. (18), we can find

$$C(k) = 2\pi / k \quad . \tag{19}$$

Consequently, the free space Green function just for the expansion of $1/|\mathbf{r}-\mathbf{r}'|$ can be expressed as

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \sum_{m=-\infty}^{\infty}\int_0^\infty e^{-k(z_>-z_<)}\,e^{im(\phi-\phi')}J_m(k\rho)J_m(k\rho')\,dk \quad . \tag{20}$$

where $z_<(z_>)$ is the smaller (larger) of z and z'.

Next we consider the Dirichlet Green function for the unbounded space between the planes at $z=0$ and $z=L$. One form of the Green function can be found by using the completeness relation to represent the functions $\delta(\phi-\phi')$ and $\delta(z-z')$:

$$\delta(\phi-\phi')=\frac{1}{2\pi}\sum_{m=-\infty}^{\infty}e^{im(\phi-\phi')} \quad . \tag{21}$$

$$\delta(z-z')=\frac{2}{L}\sum_{n=1}^{\infty}\sin\left(\frac{n\pi z}{L}\right)\sin\left(\frac{n\pi z'}{L}\right) \quad . \tag{22}$$

In terms of the same basis in the ϕ and z coordinates, the Green function can be expanded as

$$G(\rho,\phi,z;\rho',\phi',z')=\frac{1}{\pi L}\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty}g_{mn}(\rho,\rho')\sin\left(\frac{n\pi z}{L}\right)\sin\left(\frac{n\pi z'}{L}\right)e^{im(\phi-\phi')} \quad . \tag{23}$$

Substitution of Eqs. (21)-(23) into Eq. (1) leads to

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dg_m}{d\rho}\right)-\left(\left(\frac{n\pi}{L}\right)^2+\frac{m^2}{\rho^2}\right)g_{mn}=-4\pi\frac{\delta(\rho-\rho')}{\rho} \quad . \tag{24}$$

For $\rho\neq\rho'$, the Green function in the ρ-component is seen to satisfy the modified Bessel differential equations. When there are no boundary surfaces in the lateral direction, $g_{mn}(\rho,\rho')$ needs to be finite at $\rho=0$ and vanishes at $\rho\to 0$. Consequently, $g_{mn}(\rho,\rho')$ can be written as

$$g_{mn}(\rho,\rho')=\begin{cases}A_{mn}I_m\left(\dfrac{n\pi\rho}{L}\right) & \text{for } \rho<\rho' \\ B_{mn}K_m\left(\dfrac{n\pi\rho}{L}\right) & \text{for } \rho>\rho'\end{cases} \quad . \tag{25}$$

The symmetry in ρ and ρ' requires that the coefficients A_{mn} and B_{mn} be such that $g_{mn}(\rho,\rho')$ can be written

$$g_{mn}(\rho,\rho')=C_{mn}\,I_m\left(\frac{n\pi\rho_<}{L}\right)K_m\left(\frac{n\pi\rho_>}{L}\right) \quad , \tag{26}$$

where $\rho_<(\rho_>)$ is the smaller (larger) of ρ and ρ'. To determine the constant C_{mn}, we consider the effect of the delta function in Eq. (24). If we integrate both side of Eq. (24) over the interval from $\rho=\rho'-\varepsilon$ to $\rho=\rho'+\varepsilon$, where ε is very small, we obtain

$$\left[\frac{d}{d\rho}g_{mn}(\rho,\rho')\right]_{\rho'+\varepsilon}-\left[\frac{d}{d\rho}g_{mn}(\rho,\rho')\right]_{\rho'-\varepsilon}=-\frac{4\pi}{\rho'} \quad . \tag{27}$$

For modified Bessel function $I_m(x)$ and $K_m(x)$, it has been shown that

$$W[I_m(x), K_m(x)] = -\frac{1}{x} \quad . \tag{28}$$

Substituting Eq. (26) into Eq. (27) and using Eq. (28), we can find

$$C_{mn} = 4\pi \quad . \tag{29}$$

Combining all coefficients, the Dirichlet Green function for the unbounded space between the planes at $z = 0$ and $z = L$ therefore becomes

$$G(\rho,\phi,z;\rho',\phi',z') = \frac{4}{L}\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty} I_m\left(\frac{n\pi\,\rho_<}{L}\right)K_m\left(\frac{n\pi\,\rho_>}{L}\right)\sin\left(\frac{n\pi\,z}{L}\right)\sin\left(\frac{n\pi\,z'}{L}\right)e^{im(\phi-\phi')} . \tag{30}$$

where $\rho_<(\rho_>)$ is the smaller (larger) of ρ and ρ'.

An alternative form of the Dirichlet Green function for the unbounded space between the planes at $z = 0$ and $z = L$ can be found by using the completeness relation to represent the functions $\delta(\phi-\phi')$ and $\delta(\rho-\rho')/\rho$:

$$\delta(\phi-\phi') = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')} \quad . \tag{31}$$

$$\int_0^{\infty} J_m(k\rho)J_m(k\rho')\,k\,dk = \frac{\delta(\rho-\rho')}{\rho} . \tag{32}$$

In terms of the basis in the ϕ and ρ coordinates, the Green function can be expanded as

$$G(\rho,\phi,z;\rho',\phi',z') = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\int_0^{\infty} g(k,z,z')\,e^{im(\phi-\phi')}J_m(k\rho)J_m(k\rho')\,kdk \quad . \tag{33}$$

Substitution of Eqs. (31)-(33) into Eq. (1) leads to

$$\left(\frac{d^2}{dz^2} - k^2\right)g(k,z,z') = -4\pi\delta(z-z') \quad . \tag{34}$$

For $z \neq z'$, the Green function in the z-component is seen to satisfy a simple differential equations. Since there are two boundary surfaces at $z = 0$ and $z = L$, $g(k,z,z')$ vanishes at these two surfaces. Consequently, $g(k,z,z')$ can be written as

$$g(k,z,z') = \begin{cases} A(k)\,\sinh(kz) & \text{for } z < z' \\ B(k)\,\sinh[k(L-z)] & \text{for } z > z' \end{cases} \quad . \tag{35}$$

The symmetry in z and z' requires that the coefficients $A(k)$ and $B(k)$ be such that $g(k,z,z')$ can be written

$$g(k,z,z') = C(k)\,\sinh(kz_<)\sinh[k(L-z_>)] \quad , \tag{36}$$

where $z_<(z_>)$ is the smaller (larger) of z and z'. To determine the constant $C(k)$, we consider the effect of the delta function in Eq. (34). If we integrate both side of Eq. (34) over the interval from $z = z' - \varepsilon$ to $z = z' + \varepsilon$, where ε is very small, we obtain

$$\left[\frac{d}{dz}g(k,z,z')\right]_{z'+\varepsilon} - \left[\frac{d}{dz}g(k,z,z')\right]_{z'-\varepsilon} = -4\pi \quad . \tag{37}$$

Substituting Eq. (36) into Eq. (37), we can find

$$C(k) = 4\pi / [k \sinh(kL)] \quad . \tag{38}$$

Consequently, the Dirichlet Green function for the unbounded space between the planes at $z = 0$ and $z = L$ can be expressed as

$$G(\rho,\phi,z;\rho',\phi',z') = 2\sum_{m=-\infty}^{\infty} \int_0^{\infty} e^{im(\phi-\phi')} J_m(k\rho) J_m(k\rho') \frac{\sinh(kz_<)\sinh[k(L-z_>)]}{\sinh(kL)} dk \quad . \tag{39}$$

where $z_< (z_>)$ is the smaller (larger) of z and z'.

Example (Jackson Prob. 3.18)

Considering the potential in charge-free space between the planes at $z = 0$ and $z = L$, the potential at the plane $z = L$ is specified to be a fixed potential V inside a circle of radius a and zero outside the circle. The other plane at $z = 0$ is grounded. Find the potential between the planes in cylindrical coordinates.

<Solution>

The general form of the solution for the space between the planes at $z = 0$ and $z = L$ is given by

$$\Phi(\rho,\phi,z) = \sum_{m=-\infty}^{\infty} \left[\int_0^{\infty} A_m(k)\sinh(kz) J_m(k\rho) dk\right] e^{im\phi} \quad . \tag{1}$$

Due to the symmetry, only the term with $m = 0$ survives, so

$$\Phi(\rho,z) = \int_0^{\infty} A_0(k)\sinh(kz) J_0(k\rho) dk$$

Since the potential at the plane $z = L$ is specified to be a fixed potential V inside a circle of radius a and zero outside the circle, using the orthogonal properties Eqs. (21) and (23), the coefficient $A_0(k)$ is given by

$$A_0(k) = \frac{k}{\sinh(kL)} \int_0^a V \rho J_0(k\rho) d\rho = \frac{V}{k\sinh(kL)} \int_0^{ka} x J_0(x) dx \,. \tag{2}$$

Using the property

$$\frac{d}{dx}\left(x^m J_m\right) = x^m J_{m-1} \quad ,$$

we have

$$\int_0^{ka} x J_0(x) dx = ka J_1(ka) \quad .$$

Therefore,

$$A_0(k) = \frac{Va}{\sinh(kL)} J_1(ka) \quad , \tag{3}$$

and

$$\Phi(\rho, z) = Va\int_0^\infty \frac{\sinh(kz)}{\sinh(kL)} J_1(ka) J_0(k\rho) dk = V\int_0^\infty \frac{\sinh(\lambda z/a)}{\sinh(\lambda L/a)} J_1(\lambda) J_0(\lambda\rho/a) d\lambda \quad . \tag{4}$$

Another method for finding the solution is based on the Green function. The Green function is given by

$$G(\rho,\phi,z;\rho',\phi',z') = 2\sum_{m=-\infty}^{\infty}\int_0^\infty e^{im(\phi-\phi')} J_m(k\rho) J_m(k\rho') \frac{\sinh(kz_<)\sinh[k(L-z_>)]}{\sinh(kL)} dk \quad .$$

In terms of the Green function, the potential is given by

$$\Phi(\rho,\phi,z) = -\frac{1}{4\pi}\int_0^\infty\int_0^{2\pi} \left.\frac{\partial G}{\partial z'}\right|_{z'=L} \Phi(\rho',\phi',L)\, d\phi' \rho' d\rho' \quad .$$

Due to the symmetry, only the term with $m = 0$ survives, so

$$\Phi(\rho, z) = \int_0^\infty\int_0^a V J_0(k\rho) J_0(k\rho')\, k \frac{\sinh(kz)}{\sinh(kL)} \rho' d\rho' dk = Va\int_0^\infty \frac{\sinh(kz)}{\sinh(kL)} J_1(ka) J_0(k\rho) dk \, .$$

Example (Jackson Prob. 3.19)

Consider a point charge q between two infinite parallel conducting planes held at zero potential. The planes are located at $z = 0$ and $z = L$ in a cylindrical coordinate system with the charge on the z axis at $z = z_o$ (a) Using Green's reciprocation theorem and the above example, show that the amount of induced charge on the plate at $z = L$ inside a circle of radius a whose center is on the z axis is given by

$$Q_L(a) = -\frac{q}{V}\Phi(0, z_o) \quad .$$

(b) Show that the induced charge density on the upper plate can be written as

$$\sigma(\rho) = -\frac{q}{2\pi}\int_0^\infty \frac{\sinh(kz_o)}{\sinh(kL)} J_0(k\rho) k dk \quad .$$

<Solution>

The Green's reciprocation theorem is stated as: If Φ is the potential due to a volume-charge density ρ within a volume V and a surface-charge density σ on the conducting surface S bounding the volume V, while Φ' is the potential due to another charge distribution ρ' and σ', then

$$\int_V \rho\Phi'\, dv + \int_S \sigma\Phi'\, da = \int_V \rho'\Phi\, dv + \int_S \sigma'\Phi\, da \quad .$$

The Green's second identity or Green's theorem is given by

$$\int_V \left(\phi\,\nabla^2\psi - \psi\,\nabla^2\phi\right) d^3r = \oint_S \left[\phi\,\frac{\partial\psi}{\partial n} - \psi\,\frac{\partial\phi}{\partial n}\right] da \quad .$$

Assume the two Poisson eqautions to be given by

$$\nabla^2\Phi = -\frac{\rho}{\varepsilon_o}, \quad \nabla^2\Phi' = -\frac{\rho'}{\varepsilon_o} \quad .$$

Based on the boundary conditions, we have

$$\frac{\partial\Phi}{\partial n} = \frac{\sigma}{\varepsilon_o}, \quad \frac{\partial\Phi'}{\partial n} = \frac{\sigma'}{\varepsilon_o} \quad .$$

Therefore, assigning $\phi = \Phi$ and $\psi = \Phi'$ into the Green's second identity leads to

$$\int_V \rho\Phi'\,dv + \int_S \sigma\Phi'\,da = \int_V \rho'\Phi\,dv + \int_S \sigma'\Phi\,da \quad .$$

Applying the Green's reciprocation theorem to the case (a), we have

$$\int_V \rho'\Phi\,dv + \int_S \sigma'\Phi\,da = 0 \quad ,$$

with

$$\rho' = q\delta(x)\delta(y)\delta(z - z_o) \quad ,$$

$$\Phi(\rho, z) = Va\int_0^\infty \frac{\sinh(kz)}{\sinh(kL)} J_1(ka) J_0(k\rho) dk \quad .$$

Therefore,

$$Q_L = -\frac{q}{V}\Phi(0, z_o) = -qa\int_0^\infty \frac{\sinh(kz_o)}{\sinh(kL)} J_1(ka) dk \quad .$$

(b) The potential due to the point charge q is given by

$$\Phi(\rho,\phi,z;\rho',\phi',z') = \frac{q}{2\pi\varepsilon_o}\sum_{m=-\infty}^{\infty}\int_0^\infty e^{im(\phi-\phi')} J_m(k\rho) J_m(k\rho') \frac{\sinh(kz_<)\sinh[k(L - z_>)]}{\sinh(kL)} dk \quad .$$

For the location $(\rho',\phi',z') = (0,0,z_o)$, the potential is then given by

$$\Phi(\rho,\phi,z,z_o) = \frac{q}{2\pi\varepsilon_o}\int_0^\infty J_0(k\rho)\,\frac{\sinh(kz_<)\sinh[k(L - z_>)]}{\sinh(kL)} dk \quad .$$

The induced charge density on the upper plate is given by

$$\sigma(\rho) = \varepsilon_o \left.\frac{\partial\Phi(\rho,\phi,z,z_o)}{\partial z}\right|_{z=L} = -\frac{q}{2\pi}\int_0^\infty J_0(k\rho)\,\frac{\sinh(kz_o)}{\sinh(kL)} dk \quad .$$

As a result, the total induced charge on the upper plate is given by

$$\int_0^a\int_0^{2\pi} \sigma(\rho)\rho\,d\phi d\rho = -q\int_0^\infty\int_0^a J_0(k\rho)\,k\,\frac{\sinh(kz_o)}{\sinh(kL)} dk\,\rho d\rho = -qa\int_0^\infty J_1(ka)\,\frac{\sinh(kz_o)}{\sinh(kL)} dk$$

Note that

$$\int_0^a J_0(k\rho)\rho d\rho = \frac{a}{k} J_1(ka)$$

Now we consider the Dirichlet Green function for a grounded cylindrical box defined by the surfaces $z=0$, $z=L$, and $\rho=a$. One form of the Green function can be found by using the completeness relation to represent the functions $\delta(\phi-\phi')$ and $\delta(z-z')$:

$$\delta(\phi-\phi') = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')} \quad . \tag{40}$$

$$\delta(z-z') = \frac{2}{L}\sum_{n=1}^{\infty} \sin\left(\frac{n\pi z}{L}\right)\sin\left(\frac{n\pi z'}{L}\right) \quad . \tag{41}$$

In terms of the same basis in the ϕ and z coordinates, the Green function can be expanded as

$$G(\rho,\phi,z;\rho',\phi',z') = \frac{1}{\pi L}\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty} g_{mn}(\rho,\rho') \sin\left(\frac{n\pi z}{L}\right)\sin\left(\frac{n\pi z'}{L}\right)e^{im(\phi-\phi')} \quad . \tag{42}$$

Substitution of Eqs. (40)-(42) into Eq. (1) leads to

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dg_m}{d\rho}\right) - \left(\left(\frac{n\pi}{L}\right)^2 + \frac{m^2}{\rho^2}\right)g_{mn} = -4\pi\frac{\delta(\rho-\rho')}{\rho} \quad . \tag{43}$$

For $\rho \neq \rho'$, the Green function in the ρ-component is seen to satisfy the modified Bessel differential equations. When there is a boundary surface in the lateral direction $\rho = a$, $g_{mn}(\rho,\rho')$ needs to be finite at $\rho = 0$ and vanishes at $\rho = a$. Consequently, $g_{mn}(\rho,\rho')$ can be written as

$$g_{mn}(\rho,\rho') = \begin{cases} A_{mn} I_m\left(\frac{n\pi\rho}{L}\right) & \text{for } \rho < \rho' \\ B_{mn}\left[K_m\left(\frac{n\pi\rho}{L}\right)I_m\left(\frac{n\pi a}{L}\right) - I_m\left(\frac{n\pi\rho}{L}\right)K_m\left(\frac{n\pi a}{L}\right)\right] & \text{for } \rho > \rho' \end{cases} \quad . \tag{44}$$

The symmetry in ρ and ρ' requires that the coefficients A_{mn} and B_{mn} be such that $g_{mn}(\rho,\rho')$ can be written

$$g_{mn}(\rho,\rho') = C_{mn}\, I_m\left(\frac{n\pi\rho_<}{L}\right)\left[K_m\left(\frac{n\pi\rho_>}{L}\right)I_m\left(\frac{n\pi a}{L}\right) - I_m\left(\frac{n\pi\rho_>}{L}\right)K_m\left(\frac{n\pi a}{L}\right)\right] \quad , \tag{45}$$

where $\rho_<(\rho_>)$ is the smaller (larger) of ρ and ρ'. To determine the constant C_{mn}, we consider the effect of the delta function in Eq. (43). If we integrate both side of Eq. (43) over the interval from $\rho = \rho' - \varepsilon$ to $\rho = \rho' + \varepsilon$, where ε is very small, we obtain

$$\left[\frac{d}{d\rho} g_{mn}(\rho,\rho')\right]_{\rho'+\varepsilon} - \left[\frac{d}{d\rho} g_{mn}(\rho,\rho')\right]_{\rho'-\varepsilon} = -\frac{4\pi}{\rho'} \quad . \tag{46}$$

Substituting Eq. (45) into Eq. (46) yeilds

$$\begin{aligned}
&C_{mn}\left(\frac{n\pi}{L}\right)\left\{I_m\left(\frac{n\pi\rho'}{L}\right)\left[K'_m\left(\frac{n\pi\rho'}{L}\right)I_m\left(\frac{n\pi a}{L}\right)-I'_m\left(\frac{n\pi\rho'}{L}\right)K_m\left(\frac{n\pi a}{L}\right)\right]\right.\\
&\left.-I'_m\left(\frac{n\pi\rho'}{L}\right)\left[K_m\left(\frac{n\pi\rho'}{L}\right)I_m\left(\frac{n\pi a}{L}\right)-I_m\left(\frac{n\pi\rho'}{L}\right)K_m\left(\frac{n\pi a}{L}\right)\right]\right\}\\
&=C_{mn}\left(\frac{n\pi}{L}\right)I_m\left(\frac{n\pi a}{L}\right)\left[K'_m\left(\frac{n\pi\rho'}{L}\right)I_m\left(\frac{n\pi\rho'}{L}\right)-I'_m\left(\frac{n\pi\rho'}{L}\right)K_m\left(\frac{n\pi\rho'}{L}\right)\right]\\
&=-C_{mn}I_m\left(\frac{n\pi a}{L}\right)\frac{1}{\rho'}=-\frac{4\pi}{\rho'}
\end{aligned} \quad . \tag{47}$$

Here we have used the property that

$$W\left[I_m(x),K_m(x)\right]=-\frac{1}{x} \quad . \tag{48}$$

Consequently, we obtain

$$C_{mn}=\frac{4\pi}{I_m\left(\frac{n\pi a}{L}\right)} \quad . \tag{49}$$

Combining all coefficients, the Dirichlet Green function for the unbounded space between the planes at $z=0$ and $z=L$ therefore becomes

$$\begin{aligned}
G(\rho,\phi,z;\rho',\phi',z')=\frac{4}{L}\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty}&\left\{\frac{I_m\left(\frac{n\pi\rho_<}{L}\right)}{I_m\left(\frac{n\pi a}{L}\right)}\sin\left(\frac{n\pi z}{L}\right)\sin\left(\frac{n\pi z'}{L}\right)e^{im(\phi-\phi')}\right.\\
&\left.\times\left[K_m\left(\frac{n\pi\rho_>}{L}\right)I_m\left(\frac{n\pi a}{L}\right)-I_m\left(\frac{n\pi\rho_>}{L}\right)K_m\left(\frac{n\pi a}{L}\right)\right]\right\}
\end{aligned} \quad . \tag{50}$$

where $\rho_<(\rho_>)$ is the smaller (larger) of ρ and ρ'.

An alternative form of the Dirichlet Green function for a grounded cylindrical box defined by the surfaces $z=0$, $z=L$, and $\rho=a$ can be found by using the completeness relation to represent the functions $\delta(\phi-\phi')$ and $\delta(\rho-\rho')/\rho$:

$$\delta(\phi-\phi')=\frac{1}{2\pi}\sum_{m=-\infty}^{\infty}e^{im(\phi-\phi')} \quad . \tag{51}$$

$$\frac{\delta(\rho-\rho')}{\rho}=\frac{2}{a^2}\sum_{s=1}^{\infty}\frac{1}{\left[J_{m+1}(x_{ms})\right]^2}J_m\left(x_{ms}\frac{\rho'}{a}\right)J_m\left(x_{ms}\frac{\rho}{a}\right) \quad . \tag{52}$$

In terms of the basis in the ϕ and ρ coordinates, the Green function can be expanded as

$$G(\rho,\phi,z;\rho',\phi',z')=\frac{1}{\pi a^2}\sum_{m=-\infty}^{\infty}\sum_{s=1}^{\infty}g_{ms}(z,z')\frac{e^{im(\phi-\phi')}}{\left[J_{m+1}(x_{ms})\right]^2}J_m\left(x_{ms}\frac{\rho'}{a}\right)J_m\left(x_{ms}\frac{\rho}{a}\right) \quad . \tag{53}$$

Substitution of Eqs. (31)-(33) into Eq. (1) leads to

$$\left(\frac{d^2}{dz^2}-\frac{x_{ms}{}^2}{a^2}\right)g_{ms}(z,z')=-4\pi\delta(z-z') \quad . \tag{54}$$

For $z\neq z'$, the Green function in the z-component is seen to satisfy a simple differential equations. Since there are two boundary surfaces at $z=0$ and $z=L$, $g_{ms}(z,z')$ vanishes at these two surfaces. Consequently, $g_{ms}(z,z')$ can be written as

$$g_{ms}(z,z')=\begin{cases} A_{ms}\sinh\left(\frac{x_{ms}}{a}z\right) & \text{for } z<z' \\ B_{ms}\sinh\left[\frac{x_{ms}}{a}(L-z)\right] & \text{for } z>z' \end{cases} \quad . \tag{55}$$

The symmetry in z and z' requires that the coefficients A_{ms} and B_{ms} be such that $g_{ms}(z,z')$ can be written

$$g_{ms}(z,z')=C_{ms}\sinh\left(\frac{x_{ms}}{a}z_<\right)\sinh\left[\frac{x_{ms}}{a}(L-z_>)\right] \quad , \tag{56}$$

where $z_<(z_>)$ is the smaller (larger) of z and z'. To determine the constant C_{ms}, we consider the effect of the delta function in Eq. (54). If we integrate both side of Eq. (54) over the interval from $z=z'-\varepsilon$ to $z=z'+\varepsilon$, where ε is very small, we obtain

$$\left[\frac{d}{dz}g_{ms}(z,z')\right]_{z'+\varepsilon}-\left[\frac{d}{dz}g_{ms}(z,z')\right]_{z'-\varepsilon}=-4\pi \quad . \tag{57}$$

Substituting Eq. (56) into Eq. (57), we can find

$$C_{ms}=\frac{4\pi a}{x_{ms}\sinh\left(\frac{x_{ms}}{a}L\right)} \quad . \tag{58}$$

Consequently, the Dirichlet Green function for the unbounded space between the planes at $z=0$ and $z=L$ can be expressed as

$$G(\rho,\phi,z;\rho',\phi',z') = \frac{4}{a}\sum_{m=-\infty}^{\infty}\sum_{s=1}^{\infty}\frac{\sinh\left(\frac{x_{ms}}{a}z_<\right)\sinh\left[\frac{x_{ms}}{a}(L-z_>)\right]}{x_{ms}\sinh\left(\frac{x_{ms}}{a}L\right)}\frac{e^{im(\phi-\phi')}J_m\left(x_{ms}\frac{\rho'}{a}\right)J_m\left(x_{ms}\frac{\rho}{a}\right)}{\left[J_{m+1}(x_{ms})\right]^2} \quad . \tag{59}$$

where $z_<(z_>)$ is the smaller (larger) of z and z'.

Finally, we use the eigenfunction expansion to find the form of the Dirichlet Green function for a grounded cylindrical box defined by the surfaces $z=0$, $z=L$, and $\rho=a$. Using the completeness relation, the functions $\delta(\phi-\phi')$, $\delta(z-z')$ and $\delta(\rho-\rho')/\rho$ can be expressed as

$$\delta(\phi-\phi') = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')} \quad . \tag{60}$$

$$\delta(z-z') = \frac{2}{L}\sum_{n=1}^{\infty} \sin\left(\frac{n\pi z}{L}\right)\sin\left(\frac{n\pi z'}{L}\right) \quad . \tag{61}$$

$$\frac{\delta(\rho-\rho')}{\rho} = \frac{2}{a^2}\sum_{s=1}^{\infty}\frac{1}{[J_{m+1}(x_{ms})]^2} J_m\left(x_{ms}\frac{\rho'}{a}\right)J_m\left(x_{ms}\frac{\rho}{a}\right) \quad . \tag{62}$$

In terms of the basis in the ϕ, z and ρ coordinates, the Green function can be expanded as

$$G(\rho,\phi,z;\rho',\phi',z') = \frac{2}{\pi a^2 L}\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}\sum_{s=1}^{\infty} \left\{ A_{mns}\frac{J_m\left(x_{ms}\frac{\rho'}{a}\right)J_m\left(x_{ms}\frac{\rho}{a}\right)}{[J_{m+1}(x_{ms})]^2} e^{im(\phi-\phi')}\sin\left(\frac{n\pi z}{L}\right)\sin\left(\frac{n\pi z'}{L}\right)\right\} \quad . \tag{63}$$

Substitution of Eqs. (60)-(63) into Eq. (1) leads to

$$-A_{mns}\left[\left(\frac{x_{ms}}{a}\right)^2 + \left(\frac{n\pi}{L}\right)^2\right] = -4\pi \quad . \tag{64}$$

Consequently,

$$A_{mns} = \frac{4\pi}{\left[\left(\frac{x_{ms}}{a}\right)^2 + \left(\frac{n\pi}{L}\right)^2\right]} \quad . \tag{65}$$

Substituting Eq. (65) into Eq. (63) yields

$$G(\rho,\phi,z;\rho',\phi',z') = \frac{8}{a^2 L}\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}\sum_{s=1}^{\infty} \left\{ \frac{J_m\left(x_{ms}\frac{\rho'}{a}\right)J_m\left(x_{ms}\frac{\rho}{a}\right) e^{im(\phi-\phi')}}{\left[\left(\frac{x_{ms}}{a}\right)^2 + \left(\frac{n\pi}{L}\right)^2\right][J_{m+1}(x_{ms})]^2}\sin\left(\frac{n\pi z}{L}\right)\sin\left(\frac{n\pi z'}{L}\right)\right\} \quad . \tag{66}$$

Making a comparison between Eq. (50) and Eq. (66), we can obtain

$$\begin{aligned} &\frac{I_m\left(\frac{n\pi\rho_<}{L}\right)}{I_m\left(\frac{n\pi a}{L}\right)}\left[K_m\left(\frac{n\pi\rho_>}{L}\right)I_m\left(\frac{n\pi a}{L}\right) - I_m\left(\frac{n\pi\rho_>}{L}\right)K_m\left(\frac{n\pi a}{L}\right)\right] \\ &= \frac{2}{a^2}\sum_{s=1}^{\infty}\frac{J_m\left(x_{ms}\frac{\rho'}{a}\right)J_m\left(x_{ms}\frac{\rho}{a}\right)}{\left[\left(\frac{x_{ms}}{a}\right)^2 + \left(\frac{n\pi}{L}\right)^2\right][J_{m+1}(x_{ms})]^2} \end{aligned} \quad . \tag{67}$$

On the other hand, making a comparison between Eq. (59) and Eq. (66) can yield

$$\frac{\sinh\left(\frac{x_{ms}}{a}z_<\right)\sinh\left[\frac{x_{ms}}{a}(L-z_>)\right]}{\frac{x_{ms}}{a}\sinh\left(\frac{x_{ms}}{a}L\right)} = \frac{2}{L}\sum_{n=1}^{\infty}\frac{1}{\left[\left(\frac{x_{ms}}{a}\right)^2+\left(\frac{n\pi}{L}\right)^2\right]}\sin\left(\frac{n\pi z}{L}\right)\sin\left(\frac{n\pi z'}{L}\right) \quad . \tag{68}$$

5. Electrostatics in Spherical Coordinates

In spherical coordinates r, θ, and ϕ, the Laplace equation can be written in the form

$$\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)\right]\Phi=0$$

or

$$\left[\frac{1}{r}\frac{\partial^2}{\partial r^2}r+\frac{1}{r^2}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)\right]\Phi=0 \quad . \tag{1}$$

Since Eq. (1) is a sum of radial part and an angular part, the solution $\Phi(\vec{r})$ can be expressed as the product of a radial part and an angular part,

$$\Phi(\vec{r})=\frac{U(r)}{r}P(\theta)Q(\phi) \quad . \tag{2}$$

Substituting Eq. (2) into Eq. (1) yields

$$PQ\frac{d^2U}{dr^2}+\frac{UQ}{r^2}\frac{1}{\sin\theta}\frac{d}{d\theta}\sin\theta\frac{dP}{d\theta}+\frac{UP}{r^2\sin^2\theta}\frac{d^2Q}{d\phi^2}=0 \quad . \tag{3}$$

Multiplying Eq. (3) by r^2/UPQ can result in

$$\frac{r^2}{U}\frac{d^2U}{dr^2}+\frac{1}{P}\frac{1}{\sin\theta}\frac{d}{d\theta}\sin\theta\frac{dP}{d\theta}+\frac{1}{\sin^2\theta}\frac{1}{Q}\frac{d^2Q}{d\phi^2}=0 \quad . \tag{4}$$

If Eq. (4) is to hold for arbitrary values of the independent coordinates, each of the terms must be separately constant:

$$\begin{aligned}
&\frac{r^2}{U}\frac{d^2U}{dr^2}=l(l+1)\\
&\frac{1}{Q}\frac{\partial^2 Q}{\partial\phi^2}=-m^2\\
&\frac{1}{P}\frac{1}{\sin\theta}\frac{d}{d\theta}\sin\theta\frac{dP}{d\theta}-\frac{m^2}{\sin^2\theta}=-l(l+1)
\end{aligned} \quad . \tag{5}$$

The solutions of the first two ordinary differential equations in (5) are

$$R(r) = \frac{U}{r} = Ar^l + Br^{-l-1}$$
$$Q(\phi) = e^{\pm im\phi} \quad . \tag{6}$$

In terms of $x = \cos\theta$, the θ equation for $P(\theta)$ is usually expressed as

$$\frac{d}{dx}\left[(1-x^2)\frac{dP_l^m(x)}{dx}\right] + \left[l(l+1) - \frac{m^2}{1-x^2}\right]P_l^m(x) = 0 \quad . \tag{7}$$

This equation is called the generalized Legendre equation and its solutions are the associated Legendre functions. For the case of $m = 0$, Eq. (7) is the ordinary Legendre differential equation

$$\frac{d}{dx}\left[(1-x^2)\frac{dP_l(x)}{dx}\right] + l(l+1)P_l(x) = 0 \quad . \tag{8}$$

Legendre Functions

The Legendre polynomial $P_l(x)$ is defined as the coefficient of the nth power in the generating function $g(t, x)$ as

$$g(t,x) = (1-2xt+t^2)^{-1/2} = \sum_{l=0}^{\infty} P_l(x)\, t^l \quad . \tag{1}$$

With Eq. (1), the electrostatic potential at $\boldsymbol{r}$ for a unit charge at $\boldsymbol{r}'$ is given by

$$\Phi(\mathbf{r},\mathbf{r}') = \frac{1}{4\pi\varepsilon_o}\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{4\pi\varepsilon_o}\frac{1}{r_>}\sum_{l=0}^{\infty}\left(\frac{r_<}{r_>}\right)^l P_l(\cos\gamma) \quad , \tag{2}$$

where $r_<(r_>)$ is the smaller (larger) of r and r' and γ is the angle between vectors $\boldsymbol{r}$ and $\boldsymbol{r}'$, explicitly, $\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')$. Using the binomial theorem, the generating function can be expanded as

$$g(t,x) = (1-2xt+t^2)^{-1/2} = \sum_{l=0}^{\infty}\frac{(2l)!}{2^{2l}(l!)^2}(2xt-t^2)^l = 1 + \sum_{l=1}^{\infty}\frac{(2l-1)!!}{(2l)!!}(2xt-t^2)^l \, . \tag{3}$$

The first three Legendre polynomials are then given by

$$P_0(x) = 1 \, , \quad P_1(x) = x \, , \quad P_2(x) = \frac{3}{2}x^2 - \frac{1}{2} \, .$$

With Eqs. (1) and (3), we can set $x = 0$ to obtain

$$P_{2n}(0) = (-1)^n \frac{(2n)!}{2^{2n}(n!)^2} = (-1)^n \frac{(2n-1)!!}{(2n)!!} \tag{4}$$

and $P_{2n+1}(0) = 0$. Furthermore, it can be shown that $P_n(1) = 1$ and $P_n(-1) = (-1)^n$. The binomial expansion of the $(2xt-t^2)^l$ factor in Eq. (3) yields the double series

$$\begin{aligned}(1-2xt+t^2)^{-1/2} &= \sum_{l=0}^{\infty}\frac{(2l)!}{2^{2l}(l!)^2}(2xt-t^2)^l = \sum_{l=0}^{\infty}\frac{(2l)!}{2^{2l}(l!)^2}t^l\sum_{k=0}^{l}\frac{(-1)^k l!}{k!(l-k)!}(2x)^{l-k}t^k \\ &= \sum_{l=0}^{\infty}\sum_{k=0}^{l}\frac{(-1)^k}{2^{2l}}\frac{(2l)!}{l!k!(l-k)!}(2x)^{l-k}t^{l+k}\end{aligned} \quad . \tag{5}$$

Rearranging the order of summation, Eq. (5) becomes

$$(1-2xt+t^2)^{-1/2} = \sum_{l=0}^{\infty}\sum_{k=0}^{[l/2]}\frac{(-1)^k}{2^{2l-2k}}\frac{(2l-2k)!}{k!(l-k)!(l-2k)!}(2x)^{l-2k}t^l \quad . \tag{6}$$

With t^l independent of the index k. Note that $[l/2] = l/2$ for l even, $(l-1)/2$ for l odd. Now, equating two power series in Eq. (1) and Eq. (6) term by term, we have

$$P_l(x) = \sum_{k=0}^{[l/2]}\frac{(-1)^k}{2^l}\frac{(2l-2k)!}{k!(l-k)!(l-2k)!}x^{l-2k} \quad . \tag{7}$$

Hence, for l even, P_l has only even powers of x and even parity, and odd powers and odd parity for odd l.

Recurrence relations and special properties

Differentiating Eq. (1) with respect to t yields

$$\frac{\partial g(t,x)}{\partial t} = (x-t)(1-2xt+t^2)^{-3/2} = \sum_{l=0}^{\infty} l\, P_l(x)\, t^{l-1} \quad . \tag{8}$$

Substituting Eq. (1) into this and rearranging terms, we obtain

$$(1-2xt+t^2)\sum_{l=0}^{\infty} l\, P_l(x)\, t^{l-1} + (t-x)\sum_{l=0}^{\infty} P_l(x)\, t^l = 0 \quad . \tag{9}$$

The left-hand side is a power series in t. Using distinctive summation indices, we have

$$\sum_{l=0}^{\infty} l\, P_l(x)\, t^{l-1} - \sum_{l=0}^{\infty}(2l+1)xP_l(x)\, t^l + \sum_{l=0}^{\infty}(l+1)\, P_l(x)\, t^{l+1} = 0 \,. \tag{10}$$

Since the power series vanishes for all value of t, we can find

$$(l+1)\, P_{l+1}(x) + l\, P_{l-1}(x) = (2l+1)xP_l(x) \quad . \tag{11}$$

Equation (11) can be used to obtain an identity

$$P_{l-1}(0) = -\frac{l+1}{l}P_{l+1}(0) \quad . \tag{12}$$

Now differentiating Eq. (1) with respect to x yields

$$\frac{\partial g(t,x)}{\partial x} = t(1-2xt+t^2)^{-3/2} = \sum_{l=0}^{\infty} P_l'(x)\, t^l \quad . \tag{13}$$

Substituting Eq. (1) into this and rearranging terms, we obtain

$$(1-2xt+t^2)\sum_{l=0}^{\infty} P_l'(x)\, t^l - t\sum_{l=0}^{\infty} P_l(x)\, t^l = 0 \quad . \tag{14}$$

The left-hand side is a power series in *t*. Using distinctive summation indices, we have

$$\sum_{l=0}^{\infty} P_l'(x)\, t^l - \sum_{l=0}^{\infty}\left[2x\, P_l'(x) + P_l(x)\right] t^{l+1} + \sum_{l=0}^{\infty} P_l'(x)\, t^{l+2} = 0\,. \tag{15}$$

Since the power series vanishes for all value of *t*, we can find

$$P_{l+1}'(x) + P_{l-1}'(x) = 2x\, P_l'(x) + P_l(x)\,. \tag{16}$$

A more useful relation can be found by differentiating Eq. (11) with respect to *x*:

$$(l+1)\, P_{l+1}'(x) + l\, P_{l-1}'(x) = (2l+1)xP_l'(x) + (2l+1)P_l(x) \quad . \tag{17}$$

With Eqs. (16) and (17), we can cancel $P_l'(x)$ term to obtain

$$P_{l+1}'(x) - P_{l-1}'(x) = (2l+1)P_l(x)\,. \tag{18}$$

From Eqs. (16) and (18), numerous additional equations can be developed, including

$$P_{l+1}'(x) = x\, P_l'(x) + (l+1)P_l(x) \quad , \tag{19}$$

$$P_{l-1}'(x) = x\, P_l'(x) - lP_l(x) \quad . \tag{20}$$

Using Eq. (19) and the result by replacing $l-1$ with *l* in Eq. (20), the term with $P_{l+1}'(x)$ can be eliminated to lead to

$$(1-x^2)P_l'(x) = x(l+1)P_l(x) - (l+1)P_{l+1}(x) \quad . \tag{21}$$

Differentiating Eq. (21) with respect to *x* yields

$$\left[(1-x^2)P_l'(x)\right]' = (l+1)P_l(x) + (l+1)\left[xP_l'(x) - P_{l+1}'(x)\right]. \tag{22}$$

Using Eq. (19) to replace the term $\left[xP_l'(x) - P_{l+1}'(x)\right]$ with $-(l+1)P_l(x)$, then Eq. (22) can be expressed as

$$\left[(1-x^2)P_l'(x)\right]' + l(l+1)P_l(x) = 0 \quad . \tag{23}$$

Equation (23) is Legendre's differential equation. In other words, the polynomials $P_l(x)$ generated by the expansion of $(1-2xt+t^2)^{-1/2}$ satisfy Legendre's equation which, of course, is why they are called Legendre polynomials. In terms of $x = \cos\theta$, we encounter Legendre's equation expressed in terms of differentiation with respect to θ:

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP_l(\cos\theta)}{d\theta}\right) + l(l+1)P_l(\cos\theta) = 0 \quad . \tag{24}$$

Orthogonality

Repeating the Strum-Liouville analysis, we multiply Eq. (23) by $P_{l'}(x)$ and subtract the corresponding equation with l and l' interchanged. Integrating from -1 to 1, we obtain

$$\begin{aligned}&\int_{-1}^{1}\left\{P_{l'}(x)\frac{d}{dx}\left[(1-x^2)P_l'(x)\right]-P_l(x)\frac{d}{dx}\left[(1-x^2)P_{l'}'(x)\right]\right\}dx\\&=\left[l'(l'+1)-l(l+1)\right]\int_{-1}^{1}P_l(x)P_{l'}(x)dx\end{aligned} \quad . \tag{25}$$

Integrating by parts, the integrated part vanishing because of the factor $(1-x^2)$, we have

$$\left[l'(l'+1)-l(l+1)\right]\int_{-1}^{1}P_l(x)P_{l'}(x)dx=0 \quad . \tag{26}$$

Then for $l\neq l'$, it can be found that

$$\int_{-1}^{1}P_l(x)P_{l'}(x)dx=0 \quad \Rightarrow \quad \int_{0}^{\pi}P_l(\cos\theta)P_{l'}(\cos\theta)\sin\theta\, d\theta=0 . \tag{27}$$

for $l=l'$, we shall need to evaluate the integral $\int_{-1}^{1}\left[P_l(x)\right]^2dx$. From the recurrence relation in Eq. (11),

$$(l+1)\,P_{l+1}(x)+l\,P_{l-1}(x)=(2l+1)xP_l(x) \quad , \tag{28}$$

we replace l with l-1 to obtain

$$l\,P_l(x)+(l-1)\,P_{l-2}(x)=(2l-1)xP_{l-1}(x) . \tag{29}$$

We multiply Eq. (28) by $(2l-1)P_{l-1}(x)$ and subtract Eq. (29) multiplied by $(2l+1)P_l(x)$ to obtain

$$(2l-1)\left\{(l+1)\,P_{l+1}(x)P_{l-1}(x)+l\left[P_{l-1}(x)\right]^2\right\}=(2l+1)\left\{l\left[P_l(x)\right]^2+(l-1)P_l(x)P_{l-2}(x)\right\} \quad . \tag{30}$$

Integrating Eq. (30) from -1 to 1 and using the orthogonality in Eq. (27), we obtain

$$\int_{-1}^{1}\left[P_l(x)\right]^2dx=\frac{(2l-1)}{(2l+1)}\int_{-1}^{1}\left[P_{l-1}(x)\right]^2dx \quad . \tag{31}$$

Consequently, we have

$$\int_{-1}^{1}\left[P_l(x)\right]^2dx=\frac{1}{(2l+1)}\int_{-1}^{1}\left[P_0(x)\right]^2dx=\frac{2}{(2l+1)} \quad . \tag{32}$$

In addition to orthogonality, the Sturm-Liouville theory shows that the Legendre polynomials form a complete set. Therefore, it is able to express a given function $f(x)$ in the form of Legendre series,

$$f(x)=\sum_{l=0}^{\infty}A_l\,P_l(x), \quad -1<x<1 \quad . \tag{33}$$

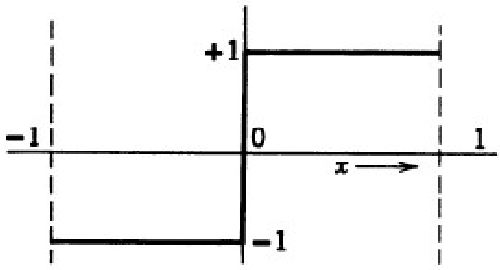
From the orthogonality relation, the coefficient in the series is given by

$$A_l=\frac{2l+1}{2}\int_{-1}^{1}f(x)P_l(x)\,dx \quad . \tag{34}$$

From Eq. (34) for the coefficient of a Legendre series, and from the fact that the Legendre polynomials are odd or even, we see that an odd function will have only odd-indexed coefficients that are nonzero, and an even function will have only even-indexed coefficients that are nonzero.

Example

Let $f(x)=\begin{cases}-1, & -1<x<0\\ 1, & 0<x<1\end{cases}$. The Legendre series will contain only odd-indexed polynomials. Find the coefficients.



<Solution>

The coefficient is given by

$$A_l=\frac{2l+1}{2}\int_{-1}^{1} f(x)P_l(x)\,dx=(2l+1)\int_0^1 P_l(x)\,dx$$

Using Eq. (18), we obtain

$$A_l=(2l+1)\int_0^1 P_l(x)\,dx=\int_0^1\left[P'_{l+1}(x)-P'_{l-1}(x)\right]dx=P_{l-1}(0)-P_{l+1}(0)$$

With Eq. (12), the coefficient is given by

$$A_l=P_{l-1}(0)+\frac{l}{l+1}P_{l-1}(0)=\frac{2l+1}{l+1}P_{l-1}(0)$$

Let $l=2n+1$. The coefficient can be explicitly given by

$$A_{2n+1}=\frac{4n+3}{2n+2}P_{2n}(0)=(-1)^n\left(\frac{4n+3}{2n+2}\right)\frac{(2n)!}{2^{2n}(n!)^2}=(-1)^n\left(\frac{4n+3}{2n+2}\right)\frac{(2n-1)!!}{(2n)!!}\ .$$

As a result, we obtain

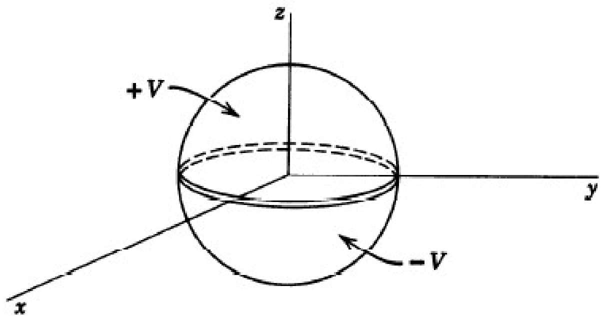
$$f(x)=\sum_{n=0}^{\infty}(-1)^n\left(\frac{4n+3}{2n+2}\right)\frac{(2n)!}{2^{2n}(n!)^2}P_{2n+1}(x)=\sum_{n=0}^{\infty}(-1)^n(4n+3)\frac{(2n-1)!!}{(2n+2)!!}P_{2n+1}(x)$$
$$=\frac{3}{2}P_1(x)-\frac{7}{8}P_3(x)+\frac{11}{16}P_5(x)+\cdots$$

Boundary-Value Problems with Azimuthal Symmetry

From the form of the solution of the Laplace equation in spherical coordinates, the general solution for a problem possessing azimuthal symmetry $m = 0$ is given by

$$\Phi(r,\theta) = \sum_{l=0}^{\infty} \left(A_l\, r^l + B_l\, r^{-l-1}\right) P_l(\cos\theta) \quad . \tag{1}$$

Now we consider the problem that the hemisphere defined by $r = a$, $0 \le \theta < \pi/2$ has an electrostatic potential V_o and the other hemisphere defined by $r = a$, $\pi/2 \le \theta < \pi$ has an electrostatic potential -V_o.



The potential at interior points can be expressed as

$$\Phi(r,\theta) = \sum_{l=0}^{\infty} A_l\, r^l P_l(\cos\theta) \quad . \tag{2}$$

The coefficient A_l can be evaluated by

$$A_l\, a^l = \frac{2l+1}{2}\int_0^{\pi} \Phi(a,\theta) P_l(\cos\theta)\, \sin\theta d\theta = (2l+1)\, V_o \int_0^1 P_l(x)\, dx = V_o \frac{2l+1}{l+1} P_{l-1}(0) \, . \tag{3}$$

Let $l = 2n+1$. The coefficient can be explicitly given by

$$\begin{aligned} A_{2n+1} &= \frac{V_o}{a^{2n+1}}\left(\frac{4n+3}{2n+2}\right) P_{2n}(0) = \frac{V_o}{a^{2n+1}}(-1)^n \left(\frac{4n+3}{2n+2}\right)\frac{(2n)!}{2^{2n}(n!)^2} \\ &= \frac{V_o}{a^{2n+1}}(-1)^n (4n+3)\frac{(2n-1)!!}{(2n+2)!!} \end{aligned} \quad . \tag{4}$$

Thus the potential inside the sphere is

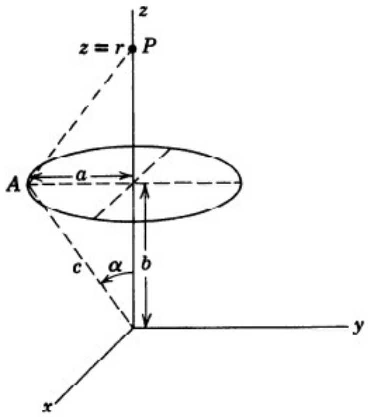
$$\begin{aligned} \Phi(r,\theta) &= V_o \sum_{n=0}^{\infty}\left(\frac{r}{a}\right)^{2n+1}\left(\frac{4n+3}{2n+2}\right) P_{2n}(0)\, P_{2n+1}(\cos\theta) \\ &= V_o \sum_{n=0}^{\infty}\left(\frac{r}{a}\right)^{2n+1}(-1)^n (4n+3)\frac{(2n-1)!!}{(2n+2)!!} P_{2n+1}(\cos\theta) \end{aligned} \quad . \tag{5}$$

The potential outside the sphere can be found by merely replacing $(r/a)^{2n+1}$ by $(a/r)^{2n+2}$. Since the solution in Eq. (1) with its coefficients determined by the boundary conditions is a unique expansion of the potential, this uniqueness provides a means of obtaining the solution of potential problems from the expression of the potential on the symmetry axis. The potential on the symmetry axis with $z = r$ can be given by

$$\Phi(z=r)=\sum_{l=0}^{\infty}\left(A_l r^l + B_l r^{-l-1}\right) \quad , \tag{6}$$

where the value of z is positive. For negative z each term must be multiplied by $(-1)^l$. Suppose that the potential on the symmetry axis can be evaluated and be expanded in a power series of the form in Eq. (6), the solution for the potential at any point in space can be obtained by multiplying each power of r^l and $r^{-(l+1)}$ by $P_l(\cos\theta)$.

An important example is the potential due to a total charge q uniformly distributed around a circular ring of radius a, located as shown in Fig. , with its axis the z axis and its center at $z=b$.



The potential at a point on the axis of symmetry with $z=r$ can be given by

$$\Phi(z=r)=\frac{q}{4\pi\varepsilon_o}\frac{1}{(r^2+c^2-2rc\cos\alpha)^{1/2}} \quad , \tag{7}$$

where $c^2=a^2+b^2$ and $\alpha=\tan^{-1}(a/b)$. In terms of Legendre polynomials, Eq. (7) can be expanded as

$$\Phi(z=r)=\frac{q}{4\pi\varepsilon_o}\sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}P_l(\cos\alpha) \quad , \tag{8}$$

where $r_<(r_>)$ is the smaller (larger) of r and c. The potential at any point in space is now obtained by multiplying each term of these series by $P_l(\cos\theta)$:

$$\Phi(r,\theta)=\frac{q}{4\pi\varepsilon_o}\sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}P_l(\cos\alpha)P_l(\cos\theta) \quad . \tag{9}$$

When the ring is in the plane of $\alpha=\pi/2$, i.e. $b=0$, the result can be given by

$$\Phi(r,\theta)=\frac{q}{4\pi\varepsilon_o}\sum_{n=0}^{\infty}\frac{r_<^{2n}}{r_>^{2n+1}}P_{2n}(0)P_{2n}(\cos\theta)=\frac{q}{4\pi\varepsilon_o}\sum_{n=0}^{\infty}\frac{r_<^{2n}}{r_>^{2n+1}}(-1)^n\frac{(2n-1)!!}{(2n)!!}P_{2n}(\cos\theta), \tag{10}$$

where $r_<(r_>)$ is the smaller (larger) of r and a.

Rodrigues' Formula and Associated Legendre Functions

From the following expression

$$P_l(x) = \sum_{k=0}^{[l/2]} \frac{(-1)^k}{2^l} \frac{(2l-2k)!}{k!(l-k)!(l-2k)!} x^{l-2k} \quad ,$$

we can have

$$P_l(x) = \sum_{k=0}^{[l/2]} \frac{(-1)^k}{2^l} \frac{1}{k!(l-k)!} \frac{d^l}{dx^l} x^{2l-2k} \quad . \tag{1}$$

The upper limit in Eq. (1) can be changed from $[l/2]$ to l because the additional terms $[l/2]+1$ to l in the summation contribute nothing. With this change, the expression in Eq. (1) can be in terms of the binomial expansion to be given by

$$P_l(x) = \sum_{k=0}^{l} \frac{(-1)^k}{2^l} \frac{1}{k!(l-k)!} \frac{d^l}{dx^l} x^{2l-2k} = \frac{1}{2^l l!} \frac{d^l}{dx^l} (x^2-1)^l \quad . \tag{2}$$

This is Rodrigues' formula.

When the Laplacian operator is separated in spherical coordinates, one of the separated ordinary differential equations is the associated Legendre equation

$$\frac{d}{dx}\left[(1-x^2)\frac{dP_l^m(x)}{dx}\right] + \left[l(l+1) - \frac{m^2}{1-x^2}\right]P_l^m(x) = 0 \quad . \tag{3}$$

One way of developing the solution of the associated Legendre equation is to start with the regular Legendre equation and convert it into the associated Legendre equation by using multiple differentiation. We use the Leibnitz's formula to differentiate the regular Legendre equation m times to obtain

$$\begin{aligned} &\frac{d^m}{dx^m}\left[(1-x^2)\frac{d^2P_l(x)}{dx^2} - 2x\frac{dP_l(x)}{dx} + l(l+1)P_l(x)\right] = 0 \\ &\Rightarrow \quad (1-x^2)\frac{d^2}{dx^2}\frac{d^mP_l(x)}{dx^m} - 2x(m+1)\frac{d}{dx}\frac{d^mP_l(x)}{dx^m} + [l(l+1) - m(m+1)]\frac{d^mP_l(x)}{dx^m} = 0 \end{aligned} \quad . \tag{4}$$

We take

$$u(x) \equiv \frac{d^mP_l(x)}{dx^m} \tag{5}$$

and replace $u(x)$ by

$$v(x) = (1-x^2)^{m/2}u(x) = (1-x^2)^{m/2}\frac{d^mP_l(x)}{dx^m} \quad . \tag{6}$$

Solving for $u(x)$ and differentiating, we have

$$u'(x)=\frac{d}{dx}(1-x^2)^{-m/2}v(x)=(1-x^2)^{-m/2}\left[v'(x)+\frac{mxv(x)}{1-x^2}\right] \quad , \tag{7}$$

$$\begin{aligned} u''(x)&=\frac{d}{dx}(1-x^2)^{-m/2}\left[v'(x)+\frac{mxv(x)}{1-x^2}\right] \\ &=(1-x^2)^{-m/2}\left[v''(x)+\frac{mv(x)}{1-x^2}+\frac{2mxv'(x)}{1-x^2}+\frac{m(m+2)x^2v(x)}{(1-x^2)^2}\right] \end{aligned} \quad . \tag{8}$$

Substituting Eqs. (7) and (8) into Eq. (4), we find that the new function $v(x)$ satisfies the differential equation

$$(1-x^2)v''(x)-2xv'(x)+\left[l(l+1)-\frac{m^2}{1-x^2}\right]v(x)=0 \quad . \tag{9}$$

Which is the associated Legendre equation reducing to Legendre's equation, as it must when *m* is set equal to zero. As a result, the regular solutions are given by

$$v(x)=P_l^m(x)=(1-x^2)^{m/2}\frac{d^mP_l\ (x)}{dx^m}. \tag{10}$$

Occasionally, we may find the associated Legendre functions defined with an additional factor of $(-1)^m$. This $(-1)^m$ seems to be an unnecessary complication at this point. It will be included in the definition of the spherical harmonics in the following section.

The form in Eq. (10) seems to imply that *m* must be nonnegative because of differentiating a negative number of times not having been defined. Even so, if $P_l(x)$ is expressed by Rodrigues' formula, this limitation on *m* is relaxed and we may have $-l\le m\le l$, negative as well as positive values of *m* being permitted. Furthermore, we can use Leibnitz'z differentiation to show that

$$P_l^{-m}(x)=(-1)^m\frac{(l-m)!}{(l+m)!}P_l^m(x) \quad . \tag{11}$$

To show Eq. (11), we expand

$$\begin{aligned} \frac{d^{l+m}}{dx^{\,l+m}}(1-x^2)^l &= \frac{d^{l+m}}{dx^{\,l+m}}(1-x)^l(1+x)^l \\ &=\sum_{k=0}^{l+m}\binom{l+m}{k}\left[\frac{d^k}{dx^{\,k}}(1-x)^l\right]\left[\frac{d^{l+m-k}}{dx^{\,l+m-k}}(1+x)^l\right] \end{aligned} \quad . \tag{12}$$

Although the sum extends for $0\le k\le l+m$, the term in the first square parentheses vanishes for $k>l$ and the term in the second square parentheses vanishes for $l+m-k>l$. Therefore, the sum is taken only for $m\le k\le l$. Then

$$\begin{aligned}
\frac{d^{l+m}}{dx^{l+m}}(1-x^2)^l &= \sum_{k=0}^{l+m}\binom{l+m}{k}\left[\frac{d^k}{dx^k}(1-x)^l\right]\left[\frac{d^{l+m-k}}{dx^{l+m-k}}(1+x)^l\right] \\
&= \sum_{k=m}^{l}\binom{l+m}{k}\left[\frac{(-1)^k l!}{(l-k)!}(1-x)^{l-k}\right]\left[\frac{l!}{(k-m)!}(1+x)^{k-m}\right] \\
&= \sum_{s=0}^{l-m}\binom{l+m}{m+s}\left[\frac{(-1)^{m+s} l!}{(l-m-s)!}(1-x)^{l-m-s}\right]\left[\frac{l!}{s!}(1+x)^s\right] \\
&= \frac{1}{\left(1-x^2\right)^m}\sum_{s=0}^{l-m}\binom{l+m}{m+s}\left[\frac{(-1)^{m+s} l!}{(l-m-s)!}(1-x)^{l-s}\right]\left[\frac{l!}{s!}(1+x)^{s+m}\right] \\
&= \frac{1}{\left(1-x^2\right)^m}\sum_{s=0}^{l-m}\frac{(l+m)!}{(l-s)!(m+s)!}\left[\frac{(-1)^{m+s} l!}{(l-m-s)!}(1-x)^{l-s}\right]\left[\frac{l!}{s!}(1+x)^{s+m}\right] \\
&= \frac{(-1)^m}{\left(1-x^2\right)^m}\frac{(l+m)!}{(l-m)!}\sum_{s=0}^{l-m}\frac{(l-m)!}{(l-m-s)!\,s!}\left[\frac{(-1)^s l!}{(l-s)!}(1-x)^{l-s}\right]\left[\frac{l!}{(m+s)!}(1+x)^{s+m}\right] \\
&= \frac{(-1)^m}{\left(1-x^2\right)^m}\frac{(l+m)!}{(l-m)!}\sum_{s=0}^{l-m}\binom{l-m}{s}\left[\frac{d^s}{dx^s}(1-x)^l\right]\left[\frac{d^{l-m-s}}{dx^{l-m-s}}(1+x)^l\right] \\
&= \frac{(-1)^m}{\left(1-x^2\right)^m}\frac{(l+m)!}{(l-m)!}\frac{d^{l-m}}{dx^{l-m}}(1-x^2)^l
\end{aligned} \quad . \quad (13)$$

With Eq. (13), Eq. (11) can be clearly obtained. As expected, the associated Legendre functions satisfy recurrence relations. Differentiating the equation

$$(l+1)\, P_{l+1}(x) + l\, P_{l-1}(x) = (2l+1)xP_l(x)$$

m times, we have

$$(l+1)\,\frac{d^m P_{l+1}(x)}{dx^m} + l\,\frac{d^m P_{l-1}(x)}{dx^m} = (2l+1)m\frac{d^{m-1}P_l(x)}{dx^{m-1}} + (2l+1)x\frac{d^m P_l(x)}{dx^m} \quad . \quad (14)$$

On the other hand, differentiating the equation

$$P'_{l+1}(x) - P'_{l-1}(x) = (2l+1)P_l(x)$$

m-1 times, we have

$$\frac{d^m P_{l+1}(x)}{dx^m} - \frac{d^m P_{l-1}(x)}{dx^m} = (2l+1)\frac{d^{m-1}P_l(x)}{dx^{m-1}} \quad . \quad (15)$$

Using Eqs. (14) and (15) to eliminate the term $d^{m-1}P_l(x)/dx^{m-1}$, we obtain

$$(l-m+1)\,\frac{d^m P_{l+1}(x)}{dx^m}+(l+m)\frac{d^m P_{l-1}(x)}{dx^m}=(2l+1)x\frac{d^m P_l(x)}{dx^m} \quad . \tag{16}$$

Consequently, we have

$$(l-m+1)\;P_{l+1}^m(x)+(l+m)\;P_{l-1}^m(x)=(2l+1)x\;P_l^m(x) \quad . \tag{17}$$

Repeating the Strum-Liouville analysis, we can show that for $l \neq l'$

$$\left[l'(l'+1)-l(l+1)\right]\int_{-1}^{1}P_l^m(x)\;P_{l'}^m(x)dx=0 \qquad . \tag{18}$$

for $l = l'$, we shall need to evaluate the integral $\int_{-1}^{1}\left[P_l^m(x)\right]^2 dx$. From the recurrence relation in Eq. (17), we replace *l* with *l*-1 to obtain

$$(l-m)\;P_l^m(x)+(l+m-1)\;P_{l-2}^m(x)=(2l-1)x\;P_{l-1}^m(x)\,. \tag{19}$$

We multiply Eq. (17) by $(2l-1)P_{l-1}^m(x)$ and subtract Eq. (19) multiplied by $(2l+1)P_l^m(x)$ to obtain

$$\begin{aligned}&(2l-1)\left\{(l-m+1)\;P_{l+1}^m(x)P_{l-1}^m(x)+(l+m)\left[P_{l-1}^m(x)\right]^2\right\}\\&=(2l+1)\left\{(l-m)\;\left[P_l^m(x)\right]^2+(l+m-1)P_l^m(x)P_{l-2}^m(x)\right\}\end{aligned} \quad . \tag{20}$$

Integrating Eq. (30) from -1 to 1 and using the orthogonality in Eq. (18), we obtain

$$\int_{-1}^{1}\left[P_l^m(x)\right]^2 dx=\frac{(2l-1)}{(2l+1)}\frac{(l+m)}{(l-m)}\int_{-1}^{1}\left[P_{l-1}^m(x)\right]^2 dx \quad . \tag{21}$$

Consequently, we have

$$\begin{aligned}\int_{-1}^{1}\left[P_l^m(x)\right]^2 dx&=\frac{(2l-1)}{(2l+1)}\frac{(l+m)}{(l-m)}\int_{-1}^{1}\left[P_{l-1}^m(x)\right]^2 dx\\&=\frac{(2l-1)(2l-3)}{(2l+1)(2l-1)}\frac{(l+m)(l+m-1)}{(l-m)(l-m-1)}\int_{-1}^{1}\left[P_{l-2}^m(x)\right]^2 dx \quad .\\&=\frac{(2m+1)}{(2l+1)}\frac{(l+m)!}{(l-m)!}\frac{1}{(2m)!}\int_{-1}^{1}\left[P_m^m(x)\right]^2 dx\end{aligned} \tag{22}$$

Since

$$\begin{aligned}\int_{-1}^{1}\left[P_m^m(x)\right]^2 dx&=\left[\frac{(2m)!}{(2^m m!)}\right]^2\int_{-1}^{1}(1-x^2)^m dx=\left[\frac{(2m)!}{(2^m m!)}\right]^2\frac{2^m m!}{(2m-1)!!}\int_{-1}^{1}\;x^{2m}dx\\&=\frac{\left[(2m)!\right]^2}{(2^m m!)}\frac{2}{(2m+1)!!}=\frac{2}{(2m+1)}(2m)!\end{aligned} \quad , \tag{23}$$

we have

$$\int_{-1}^{1}\left[P_l^m(x)\right]^2 dx = \frac{2}{(2l+1)}\frac{(l+m)!}{(l-m)!} \quad . \tag{24}$$

With this normalization condition, the expression for the spherical harmonics with is given by

$$\begin{aligned} Y_{lm}(\theta,\phi) &= (-1)^m\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}e^{im\phi}P_l^m(\cos\theta) \\ &= \frac{(-1)^{l+m}}{2^l l!}\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}e^{im\phi}\sin^m\theta\frac{d^{l+m}}{d(\cos\theta)^{l+m}}(1-\cos^2\theta)^l \end{aligned} \quad . \tag{25}$$

With Eq. (25), we can show that

$$Y_{l-m}(\theta,\phi) = (-1)^m\, Y_{lm}^*(\theta,\phi) \quad . \tag{26}$$

The normalization and orthogonality conditions are given by

$$\int_0^{2\pi}\int_0^{\pi} Y_{l'm'}^*(\theta,\phi)Y_{lm}(\theta,\phi)\sin\theta\, d\theta\, d\phi = \delta_{l'l}\,\delta_{m'm} \quad . \tag{27}$$

On the other hand, the completeness relation is given by

$$\sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi) = \delta(\phi-\phi')\delta(\cos\theta-\cos\theta') \quad . \tag{28}$$

6. Expansion of Green Functions in Spherical Coordinates

The expansion of the potential of a unit point charge in spherical coordinates provides a pedagogical example of Green function expansions. A Green function for a Dirichlet potential problem in spherical coordinates satisfies the equation

$$\begin{aligned} &\left[\frac{1}{r}\frac{\partial^2}{\partial r^2}r + \frac{1}{r^2}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)\right]G(r,\theta,\phi;r',\theta',\phi') \\ &= -4\pi\frac{\delta(r-r')}{r^2}\delta(\phi-\phi')\delta(\cos\theta-\cos\theta') \end{aligned} \quad . \tag{1}$$

The completeness relation can be used to represent the functions $\delta(\phi-\phi')\delta(\cos\theta-\cos\theta')$:

$$\sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi) = \delta(\phi-\phi')\delta(\cos\theta-\cos\theta'). \tag{2}$$

In terms of the same basis in the ϕ and θ coordinates, the Green function can be expanded as

$$G(r,\theta,\phi;r',\theta',\phi') = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} g_l(r,r')Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi) \quad . \tag{3}$$

Substitution of Eqs. (2) and (3) into Eq. (1) leads to

$$\frac{1}{r}\frac{d^2}{dr^2}\left(r\frac{dg_l}{dr}\right) - \frac{l(l+1)}{r^2}g_l = -4\pi\frac{\delta(r-r')}{r^2} \quad . \tag{4}$$

For $r \neq r'$, the Green function in the *r*-component is seen to satisfy the radial differential equations. Thus it can be written as

$$g_l(r,r') = \begin{cases} A_l\, r^l + B_l\, r^{-(l+1)} & \text{for } r < r' \\ A_l'\, r^l + B_l'\, r^{-(l+1)} & \text{for } r > r' \end{cases} \quad . \tag{5}$$

When there are no boundary surfaces, $g_l(r,r')$ needs to be finite at $r = 0$ and vanish at $r \to \infty$. Consequently, $g_l(r,r')$ can be written as

$$g_l(r,r') = \begin{cases} A_l\, r^l & \text{for } r < r' \\ B_l'\, r^{-(l+1)} & \text{for } r > r' \end{cases} \quad . \tag{6}$$

The symmetry in *r* and *r′* requires that the coefficients A_l and B_l' be such that $g_l(r,r')$ can be written

$$g_l(r,r') = C_l\, \frac{r_<^l}{r_>^{l+1}} \quad , \tag{6}$$

where $r_<(r_>)$ is the smaller (larger) of *r* and *r′*. To determine the constant C_l, we consider the effect of the delta function in Eq. (4). If we integrate both side of Eq. (4) over the interval from $r = r' - \varepsilon$ to $r = r' + \varepsilon$, where ε is very small, we obtain

$$\left[\frac{d}{dr} r g_l(r,r')\right]_{r'+\varepsilon} - \left[\frac{d}{dr} r g_l(r,r')\right]_{r'-\varepsilon} = -\frac{4\pi}{r'} \quad . \tag{7}$$

Substituting Eq. (6) into Eq. (7), we can find

$$C_l = \frac{4\pi}{2l+1} \quad . \tag{8}$$

Combining all coefficients, the free space Green function just for the expansion of $1/|\mathbf{r} - \mathbf{r}'|$ therefore becomes

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y_{lm}^*(\theta',\phi')\, Y_{lm}(\theta,\phi) . \tag{9}$$

where $r_<(r_>)$ is the smaller (larger) of *r* and *r′*. Equation (9) gives the potential in a completely factorized form in the coordinates ***r*** and ***r′***. This is useful in any integration over charge densities, etc., where one variable is the variable of integration and the other is the coordinate of the observation point. With the generating function of Legendre polynomials, we have

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \frac{1}{r_>} \sum_{l=0}^{\infty} \left(\frac{r_<}{r_>}\right)^l P_l(\cos\gamma) , \tag{10}$$

where $\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')$. Now, equating two power series in Eq. (9) and Eq. (10) term by term, we have

$$P_l(\cos\gamma) = \frac{4\pi}{2l+1}\sum_{m=-l}^{l} Y_{lm}^{*}(\theta',\phi')\, Y_{lm}(\theta,\phi) \quad . \tag{11}$$

The result of Eq. (11) is called the addition theorem for spherical harmonics. In other words, the addition theorem expresses a Legendre polynomials of order l in the angle γ in terms of products of the spherical harmonics of the angles θ, ϕ and θ, ϕ.

Now we consider a Green function for a Dirichlet potential problem with the boundary surfaces to be concentric spheres at $r = a$ and $r = b$. Since the radial Green function must vanish at $r = a$ and $r = b$, from Eq. (5) it becomes

$$g_l(r,r') = \begin{cases} A_l\left(r^l - \dfrac{a^{2l+1}}{r^{l+1}}\right) & \text{for } r < r' \\ B_l'\left(\dfrac{1}{r^{l+1}} - \dfrac{r^l}{b^{2l+1}}\right) & \text{for } r > r' \end{cases} \quad . \tag{12}$$

The symmetry in r and r' requires that the coefficients A_l and B_l' be such that $g_l(r,r')$ can be written

$$g_l(r,r') = C_l\left(r_<^l - \frac{a^{2l+1}}{r_<^{l+1}}\right)\left(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}}\right) \quad , \tag{13}$$

where $r_<(r_>)$ is the smaller (larger) of r and r'. To determine the constant C_l, we consider the effect of the delta function in Eq. (4). If we integrate both side of Eq. (4) over the interval from $r = r' - \varepsilon$ to $r = r' + \varepsilon$, where ε is very small, we obtain

$$\left[\frac{d}{dr} r g_l(r,r')\right]_{r'+\varepsilon} - \left[\frac{d}{dr} r g_l(r,r')\right]_{r'-\varepsilon} = -\frac{4\pi}{r'} \quad . \tag{14}$$

Substituting Eq. (13) into Eq. (14), we find

$$\begin{aligned} &C_l\left[\left(r'^l - \frac{a^{2l+1}}{r'^{l+1}}\right)\left(\frac{-l}{r'^{l+1}} - \frac{(l+1)r'^l}{b^{2l+1}}\right) - C_l\left((l+1)r'^l - \frac{(-l)a^{2l+1}}{r'^{l+1}}\right)\left(\frac{1}{r'^{l+1}} - \frac{r'^l}{b^{2l+1}}\right)\right] \\ &= -C_l(2l+1)\left[\frac{1}{r'}\left(1-\left(\frac{a}{b}\right)^{2l+1}\right)\right] = -\frac{4\pi}{r'} \\ &\Rightarrow \quad C_l = \frac{4\pi}{(2l+1)\left[1-\left(\dfrac{a}{b}\right)^{2l+1}\right]} \end{aligned} \quad . \tag{15}$$

Combining all coefficients, the Green function for a spherical shell bounded by $r = a$ and $r = b$ is given by

$$G(r,\theta,\phi;r',\theta',\phi') = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{(4\pi)\, Y_{lm}^{*}(\theta',\phi')\, Y_{lm}(\theta,\phi)}{(2l+1)\left[1-\left(\frac{a}{b}\right)^{2l+1}\right]} \left(r_<^{l} - \frac{a^{2l+1}}{r_<^{l+1}} \right) \left(\frac{1}{r_>^{l+1}} - \frac{r_>^{l}}{b^{2l+1}} \right) . \tag{16}$$

where $r_<(r_>)$ is the smaller (larger) of r and r'. We can take some limits to obtain the Green function for the special cases. For $b \to \infty$, Eq. (16) can be reduced to be the Green function for the "exterior" problem with a sphere of radius a:

$$G(r,\theta,\phi;r',\theta',\phi') = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{4\pi}{(2l+1)} Y_{lm}^{*}(\theta',\phi')\, Y_{lm}(\theta,\phi) \frac{1}{r_>^{l+1}} \left(r_<^{l} - \frac{a^{2l+1}}{r_<^{l+1}} \right) . \tag{17}$$

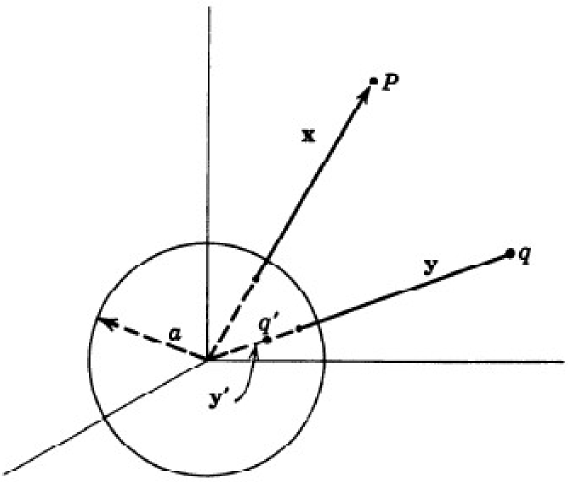
Using the addition theorem in Eq. (11), Eq. (17) can be expressed as

$$G(r,\theta,\phi;r',\theta',\phi') = \sum_{l=0}^{\infty} \frac{1}{r_>^{l+1}} \left(r_<^{l} - \frac{a^{2l+1}}{r_<^{l+1}} \right) P_l(\cos\gamma) . \tag{18}$$

In terms of the generating function for Legendre polynomials, the Green function in Eq. (18) can be rewritten as

$$\begin{aligned} G(r,\theta,\phi;r',\theta',\phi') &= \frac{1}{\sqrt{r^2 + r'^2 - 2rr'\cos\gamma}} - \frac{a}{rr'} \frac{1}{\sqrt{1^2 + \left(\frac{a^2}{rr'}\right)^2 - 2\frac{a^2}{rr'}\cos\gamma}} \\ &= \frac{1}{\sqrt{r^2 + r'^2 - 2rr'\cos\gamma}} - \frac{a}{r'} \frac{1}{\sqrt{r^2 + \left(\frac{a^2}{r'}\right)^2 - 2\frac{ra^2}{r'}\cos\gamma}} \\ &= \frac{1}{|\, r\mathbf{n} - r'\mathbf{n}'\,|} - \frac{a}{r'} \frac{1}{\left| r\mathbf{n} - \frac{a^2}{r'}\mathbf{n}' \right|} \end{aligned} \tag{19}$$

where $\mathbf{n}$ is the unit vector for $\mathbf{r}$ and $\mathbf{n}'$ is the unit vector for $\mathbf{r}'$. The final expression in Eq. (19) implies that the Green function can be expressed as a superposition of the potentials produced by the original charge q and the so-called image charge with the magnitude $-(a/r')q$. When the original charge is outside the sphere, the position of the image charge must be inside the sphere and is given by $(a^2/r')\mathbf{n}'$. It is worthwhile to note that when the charge q is brought closer to the sphere, the image charge grows in magnitude and moves out from the center to the sphere. When q is just outside the surface of the sphere, the image charge is equal and opposite in magnitude and lies just beneath the surface.



The general solution of the Poisson equation with specified values of the potential on the boundary surface is given by

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_o}\int_V G_D(\mathbf{r},\mathbf{r}')\,\rho(\mathbf{r}')\;d^3r' - \frac{1}{4\pi}\oint_S \Phi(\mathbf{r}')\,\frac{\partial G_D(\mathbf{r},\mathbf{r}')}{\partial n'}\,da' \quad . \tag{20}$$

Using Eq. (19), we have

$$\begin{aligned}\left.\frac{\partial G}{\partial n'}\right|_{r'=a} = -\left.\frac{\partial G}{\partial r'}\right|_{r'=a} &= \frac{a - r\cos\gamma}{\left(r^2 + a^2 - 2ra\cos\gamma\right)^{3/2}} - \frac{r^2 - ra\cos\gamma}{a\left(r^2 + a^2 - 2ra\cos\gamma\right)^{3/2}} \\ &= \frac{a^2 - r^2}{a\left(r^2 + a^2 - 2ra\cos\gamma\right)^{3/2}}\end{aligned} \quad . \tag{21}$$

Consequently, the solution of the Laplace equation outside a sphere with the potential specified on its surface is given by

$$\Phi(\mathbf{r}) = \frac{1}{4\pi}\int \Phi(a,\theta',\phi')\frac{a\left(r^2 - a^2\right)}{\left(r^2 + a^2 - 2ra\cos\gamma\right)^{3/2}}\,d\Omega' \;, \tag{22}$$

where $d\Omega'$ is the element of solid angle at the point (a,θ',ϕ') and $\cos\gamma$ = $\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi')$.

For $a \to 0$, Eq. (16) can be reduced to be the Green function for the "interior" problem with a sphere of radius *b*:

$$G(r,\theta,\phi;r',\theta',\phi') = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{(2l+1)}Y^*_{lm}(\theta',\phi')\,Y_{lm}(\theta,\phi)\,r_<^l\left(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}}\right) . \tag{23}$$

Using the addition theorem in Eq. (11), Eq. (23) can be expressed as

$$G(r,\theta,\phi;r',\theta',\phi') = \sum_{l=0}^{\infty} r_<^l\left(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}}\right)P_l(\cos\gamma)\,. \tag{24}$$

In terms of the generating function for Legendre polynomials, the Green function in Eq. (24) can be rewritten as

$$
\begin{aligned}
G(r,\theta,\phi;r',\theta',\phi') &= \frac{1}{\sqrt{r^2+r'^2-2rr'\cos\gamma}} - \frac{1}{b}\frac{1}{\sqrt{1^2+\left(\frac{rr'}{b^2}\right)^2-2\frac{rr'}{b^2}\cos\gamma}} \\
&= \frac{1}{\sqrt{r^2+r'^2-2rr'\cos\gamma}} - \frac{b}{r'}\frac{1}{\sqrt{r^2+\left(\frac{b^2}{r'}\right)^2-2\frac{rb^2}{r'}\cos\gamma}} \\
&= \frac{1}{|\,r\mathbf{n}-r'\mathbf{n}'\,|} - \frac{b}{r'}\frac{1}{\left|\,r\mathbf{n}-\frac{b^2}{r'}\mathbf{n}'\,\right|}
\end{aligned}
\tag{25}
$$

where $\mathbf{n}$ is the unit vector for $\mathbf{r}$ and $\mathbf{n}'$ is the unit vector for $\mathbf{r}'$. Using Eq. (25), we have

$$
\begin{aligned}
\left.\frac{\partial G}{\partial n'}\right|_{r'=b} = \left.\frac{\partial G}{\partial r'}\right|_{r'=b} &= \frac{r\cos\gamma-b}{\left(r^2+b^2-2rb\cos\gamma\right)^{3/2}} - \frac{rb\cos\gamma-r^2}{b\left(r^2+b^2-2rb\cos\gamma\right)^{3/2}} \\
&= \frac{r^2-b^2}{b\left(r^2+b^2-2rb\cos\gamma\right)^{3/2}}
\end{aligned}
\quad . \tag{26}
$$

Consequently, the solution of the Laplace equation inside a sphere with the potential specified on its surface is given by

$$
\Phi(\mathbf{r}) = \frac{1}{4\pi}\int \Phi(b,\theta',\phi')\frac{b\left(b^2-r^2\right)}{\left(r^2+b^2-2rb\cos\gamma\right)^{3/2}}\,d\Omega' \quad . \tag{27}
$$

Alternatively, we can directly use Eq. (23) to derive $\left.\partial G/\partial n'\right|_{r'=b}$:

$$
\begin{aligned}
\left.\frac{\partial G}{\partial r'}\right|_{r'=b} &= \frac{\partial}{\partial r'}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{(2l+1)}Y^*_{lm}(\theta',\phi')\,Y_{lm}(\theta,\phi)\,r_<^l\left(\frac{1}{r_>^{l+1}}-\frac{r_>^l}{b^{2l+1}}\right) \\
&= -\frac{4\pi}{b^2}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(\frac{r}{b}\right)^l Y^*_{lm}(\theta',\phi')\,Y_{lm}(\theta,\phi)
\end{aligned}
\quad . \tag{28}
$$

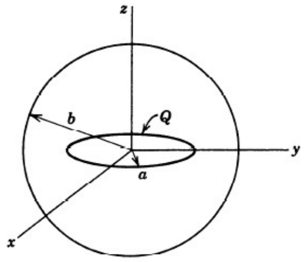
Consequently the solution of the Laplace equation inside $r=b$ with $\Phi=V(\theta',\phi')$ on the surface is given by

$$
\Phi(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(\frac{r}{b}\right)^l\left[\int V(\theta',\phi')Y^*_{lm}(\theta',\phi')\,d\Omega'\,\right]Y_{lm}(\theta,\phi) \quad . \tag{29}
$$

Now let us turn to the problems with charge distributed in the volume, so that the volume integral in Eq. (20) is involved. It is sufficient to consider problems in which the potential vanishes on the boundary surfaces. By linear superposition of a solution of the Laplace equation, the general solution can be obtained.

Example

Find the potential for a hollow grounded sphere of radius b with a concentric ring of charge of radius a and total charge Q.



<Solution>

The charge density of the ring can be written with the help of delta functions in angle and radius as

$$\rho(\mathbf{r}') = \frac{Q}{2\pi a^2}\delta(r'-a)\,\delta(\cos\theta') \ . \tag{30}$$

Substituting Eq. (23) and Eq. (30) into Eq. (20) yields

$$\Phi(\mathbf{r}) = \frac{Q}{4\pi\varepsilon_o}\sum_{l=0}^{\infty}\frac{4\pi}{(2l+1)}Y_{l0}^{*}(\pi/2,\phi')\,Y_{l0}(\theta,\phi)\,r_<^l\left(\frac{1}{r_>^{l+1}}-\frac{r_>^l}{b^{2l+1}}\right) \ , \tag{31}$$

where $r_<(r_>)$ is the smaller (larger) of r and a, and we have used the fact that only terms with $m=0$ will survive because of azimuthal symmetry. Then, using

$$Y_{l0}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}}\,P_l(\cos\theta)\,, \tag{32}$$

Eq. (31) can be simplified as

$$\Phi(\mathbf{r}) = \frac{Q}{4\pi\varepsilon_o}\sum_{l=0}^{\infty}P_l\ (0)\ P_l(\cos\theta)\,r_<^l\left(\frac{1}{r_>^{l+1}}-\frac{r_>^l}{b^{2l+1}}\right) \ . \tag{33}$$

Using the fact that $P_{2n+1}(0)=0$ and $P_{2n}(0)=(-1)^n(2n-1)!!/(2n)!!$, Eq. (33) can be written as

$$\Phi(r,\theta) = \frac{Q}{4\pi\varepsilon_o}\sum_{n=0}^{\infty}(-1)^n\frac{(2n-1)!!}{(2n)!!}P_{2n}(\cos\theta)\,r_<^{2n}\left(\frac{1}{r_>^{2n+1}}-\frac{r_>^{2n}}{b^{4n+1}}\right). \tag{34}$$

In the limit $b\to\infty$, it will be seen that Eq. (34) reduces to the expression for a ring of charge in free space. The present result can be obtained alternatively by using that result and the images for a sphere.

Chapter Five: Multipoles and Dielectrics

1. Multipole Expansions

If there is a bounded charge distribution vanishing outside a sphere of radius a about the origin, then the potential in the external region (without boundary surfaces with boundary conditions in finiteness, the Dirichlet surface term vanishes) for $r > a$ is given by

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_o}\int \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}dV' = \frac{1}{\varepsilon_o}\sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{1}{2l+1}\frac{1}{r^{l+1}}\left[\int \rho(\mathbf{r}')(r')^l\, Y_{lm}^*(\theta',\phi')\, dV'\right] Y_{lm}(\theta,\phi). \quad (1)$$

The integrals occurring in the sum depend only on the particular charge distribution. They describe the outward action of the charge distribution completely. The expressions

$$q_{lm} = \int \rho(\mathbf{r}')(r')^l\, Y_{lm}^*(\theta',\phi')\, dV' \quad . \quad (2)$$

are called the multipole moments of the charge distribution $\rho(\mathbf{r}')$. In particular, the most important q_{lm} are the monopole moment for $l=0$, the dipole moment for $l=1$, the quadrupole moment for $l=2$, the octupole moment for $l=3$, and the hexadecupole moment for $l=4$. For each l the multipole moments q_{lm} form a tensor of rank l with $2l$+1 components. To achieve a unique representation the origin of the coordinate system is set at the center of gravity of the charge distribution.

Multipole Expansion in Cartesian coordinates

The expansion of a function $f(\mathbf{r}')$ in a Taylor series about the point $\mathbf{r}'=0$ is

$$\begin{aligned} f(x_1',x_2',x_3') &= f(0,0,0) + \sum_{i=1}^{3}\frac{\partial f}{\partial x_i'}(0,0,0)x_i' + \frac{1}{2!}\sum_{i,j=1}^{3}\frac{\partial^2 f}{\partial x_i'\partial x_j'}(0,0,0)x_i'x_j' + \cdots \\ &= \sum_{n=0}^{\infty}\frac{1}{n!}\left(\sum_{i=1}^{3} x_i'\frac{\partial}{\partial x_i'}\right)^n f(x_1',x_2',x_3')\Big|_{x_1'=x_2'=x_3'=0} = \sum_{n=0}^{\infty}\frac{(\mathbf{r}'\cdot\nabla')^n}{n!} f(\mathbf{r}')\Big|_{\mathbf{r}'=0} \end{aligned} . \quad (3)$$

For the special function

$$f(\mathbf{r}') = \frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{\sqrt{(x_1-x_1')^2+(x_2-x_2')^2+(x_3-x_3')^2}} \quad , \quad (4)$$

the first three components for the expansion are given by

$$f(\mathbf{r}')\Big|_{\mathbf{r}'=0} = \frac{1}{\sqrt{(x_1)^2+(x_2)^2+(x_3)^2}} = \frac{1}{r} \quad , \quad (5)$$

$$\left.\frac{\partial f(\mathbf{r}')}{\partial x_i'}\right|_{\mathbf{r}'=0} = \left.\frac{x_i - x_i'}{\left[\sqrt{(x_1-x_1')^2+(x_2-x_2')^2+(x_3-x_3')^2}\right]^3}\right|_{\mathbf{r}'=0} = \frac{x_i}{r^3} \quad , \tag{6}$$

$$\left.\frac{\partial^2 f(\mathbf{r}')}{\partial x_j' \partial x_i'}\right|_{\mathbf{r}'=0} = \left.\frac{\partial}{\partial x_j'}\frac{x_i - x_i'}{\left[\sqrt{(x_1-x_1')^2+(x_2-x_2')^2+(x_3-x_3')^2}\right]^3}\right|_{\mathbf{r}'=0} = \frac{3x_i x_j - r^2\delta_{ij}}{r^5} \quad . \tag{7}$$

Therefore, the potential can be expanded as

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_o}\int\frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}dV' = \frac{1}{4\pi\varepsilon_o}\left\{\frac{q}{r} + \frac{\mathbf{r}\cdot\mathbf{p}}{r^3} + \frac{1}{6}\sum_{i,j=1}^{3}\frac{\left(3x_i x_j - r^2\delta_{ij}\right)}{r^5}Q_{ij} + \cdots\right\} \quad . \tag{8}$$

where q is the total charge:

$$q = \int \rho(\mathbf{r}')\,dV' \quad , \tag{9}$$

$\mathbf{p}$ is the dipole moment:

$$\mathbf{p} = \int \rho(\mathbf{r}')\,\mathbf{r}'dV' , \tag{10}$$

and Q_{ij} is the quadrupole tensor:

$$Q_{ij} = \int \rho(\mathbf{r}')\left(3x_i'x_j' - r'^2\delta_{ij}\right)dV' \quad . \tag{11}$$

Note that the extra term $\rho(\mathbf{r}')r'^2\delta_{ij}$ in the integral expression of Q_{ij} has no physical contribution due to the fact that

$$\sum_{i,j=1}^{3}\frac{\left(3x_i x_j - r^2\delta_{ij}\right)}{r^5}\left(r'^2\delta_{ij}\right) = \sum_{i=1}^{3}\frac{\left(3x_i^2 - r^2\right)}{r^5}r'^2 = 0 \quad . \tag{12}$$

In terms of Cartesian coordinates, we exhibit the first few multipole moments explicitly:

$$q_{00} = \frac{1}{\sqrt{4\pi}}\int \rho(\mathbf{r}')\,dV' = \frac{1}{\sqrt{4\pi}}q , \tag{13}$$

$$\begin{aligned} q_{11} &= \int \rho(\mathbf{r}')\,r'\,Y_{11}^*(\theta',\phi')\,dV' = -\sqrt{\frac{3}{8\pi}}\int \rho(\mathbf{r}')\,r'\left(\sin\theta'\cos\phi' - i\sin\theta'\sin\phi'\right)dV' \\ &= -\sqrt{\frac{3}{8\pi}}\int \rho(\mathbf{r}')\left(x' - i\,y'\right)dV' = -\sqrt{\frac{3}{8\pi}}\left(p_x - i\,p_y\right) \end{aligned} \quad , \tag{14}$$

$$q_{10} = \int \rho(\mathbf{r}')\,r'\,Y_{10}^*(\theta',\phi')\,dV' = \sqrt{\frac{3}{4\pi}}\int \rho(\mathbf{r}')\,r'\cos\theta'\,dV' = \sqrt{\frac{3}{4\pi}}\,p_z , \tag{15}$$

$$
\begin{aligned}
q_{22} &= \int \rho(\mathbf{r}')\, r'^2\, Y_{22}^*(\theta',\phi')\, dV' = \frac{1}{4}\sqrt{\frac{15}{2\pi}} \int \rho(\mathbf{r}')\, r'^2 \left(\sin\theta'\cos\phi' - i\sin\theta'\sin\phi'\right)^2 dV' \\
&= \frac{1}{4}\sqrt{\frac{15}{2\pi}} \int \rho(\mathbf{r}')\, \left(x' - i\, y'\right)^2 dV' \\
&= \frac{1}{12}\sqrt{\frac{15}{2\pi}} \int \rho(\mathbf{r}')\, \left[\left(3x'^2 - r'^2\right) - 6i\, x'y' - \left(3y'^2 - r'^2\right)\right] dV' \\
&= \frac{1}{12}\sqrt{\frac{15}{2\pi}} \left(Q_{11} - 2iQ_{12} - Q_{22}\right)
\end{aligned} \quad , \qquad (16)
$$

$$
\begin{aligned}
q_{21} &= \int \rho(\mathbf{r}')\, r'^2\, Y_{21}^*(\theta',\phi')\, dV' \\
&= -\frac{1}{2}\sqrt{\frac{15}{2\pi}} \int \rho(\mathbf{r}')\, r'^2 \cos\theta' \left(\sin\theta'\cos\phi' - i\sin\theta'\sin\phi'\right) dV' \quad , \\
&= -\frac{1}{2}\sqrt{\frac{15}{2\pi}} \int \rho(\mathbf{r}')\, z'\left(x' - i\, y'\right) dV' = -\frac{1}{6}\sqrt{\frac{15}{2\pi}} \left(Q_{13} - iQ_{23}\right)
\end{aligned} \qquad (17)
$$

$$
\begin{aligned}
q_{20} &= \int \rho(\mathbf{r}')\, r'^2\, Y_{20}^*(\theta',\phi')\, dV' = \frac{1}{4}\sqrt{\frac{15}{\pi}} \int \rho(\mathbf{r}')\, r'^2 \left(3\cos^2\theta' - 1\right) dV' \\
&= \frac{1}{4}\sqrt{\frac{15}{\pi}} \int \rho(\mathbf{r}')\, \left(3z'^2 - r'^2\right) dV' = \frac{1}{4}\sqrt{\frac{15}{\pi}}\, Q_{33}
\end{aligned} \quad . \qquad (18)
$$

Only the moments with $m \geq 0$ have been given. It can be shown that for a real charge density the moments with $m < 0$ are related through

$$
q_{l,-m}(\theta,\phi) = (-1)^m\, q_{lm}^* \quad . \qquad (19)
$$

Two important remarks need to be made. One remark concerns the relationship between the Cartesian multipole moments and the spherical multipole moments. The former are $(l+1)(l+2)/2$ in number and for $l > 1$ are more numerous than the $(2l+1)$ spherical components. The root of the differences lies in the different rotational transformation properties of the two types of multipole moments; the Cratesian tensors are reducible, the spherical, irreducible. Note that for $l = 2$ we have recognized the difference by defining a traceless Cartensian quadrupole moment in Eq. (11).

The other remark is that the multipole moment coefficients in Eq. (1) generally depend on the choice of origin. The values of q_{lm} for the lowest nonvanishing multipole moment of any charge distribution are independent of the choice of origin of the coordinates, but all higher multiple moments do in general depend on the location of the origin.

The electric field components for a given multipole can be expressed most easily in terms of spherical coordinates. The negative gradient of Eq. (1) is given by

$$\mathbf{E} = \frac{1}{\varepsilon_o}\sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{1}{2l+1}\frac{1}{r^{l+2}} q_{lm} \left[(l+1)Y_{lm}(\theta,\phi)\mathbf{a}_r - \frac{\partial}{\partial\theta}Y_{lm}(\theta,\phi)\mathbf{a}_\theta - \frac{im}{\sin\theta}Y_{lm}(\theta,\phi)\mathbf{a}_\varphi\right] . \quad (20)$$

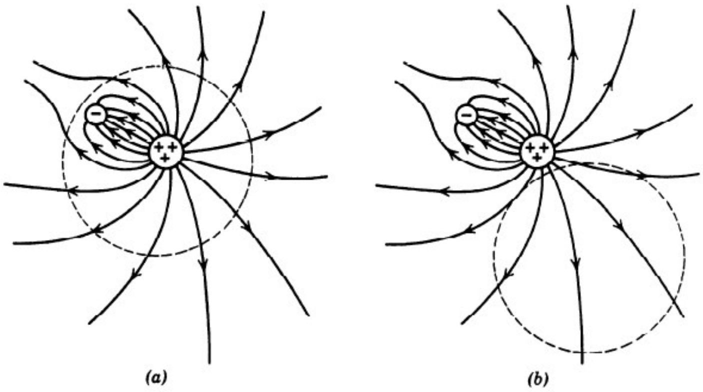
From Eq. (8), the potential for a dipole **p** at the point $\mathbf{r}_o$ is given by

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_o}\frac{\mathbf{p}\cdot(\mathbf{r}-\mathbf{r}_o)}{|\mathbf{r}-\mathbf{r}_o|^3} = \frac{1}{4\pi\varepsilon_o}\frac{\mathbf{p}\cdot\mathbf{n}}{|\mathbf{r}-\mathbf{r}_o|^2} \quad . \quad (21)$$

where $\mathbf{n} = (\mathbf{r}-\mathbf{r}_o)/|\mathbf{r}-\mathbf{r}_o|$ is a unit vector directed from $\mathbf{r}_o$ to $\mathbf{r}$. The electric field resulting from the dipole **p** at the point $\mathbf{r}_o$ is then given by

$$\mathbf{E}(\mathbf{r}) = \frac{-1}{4\pi\varepsilon_o}\nabla\left[\frac{\mathbf{p}\cdot(\mathbf{r}-\mathbf{r}_o)}{|\mathbf{r}-\mathbf{r}_o|^3}\right] = \frac{1}{4\pi\varepsilon_o}\frac{3\mathbf{n}(\mathbf{p}\cdot\mathbf{n})-\mathbf{p}}{|\mathbf{r}-\mathbf{r}_o|^3} \quad . \quad (22)$$

Equation (22) is valid at large enough distances away from the dipole. Equation (22) for the field at a point close to the dipole needs some modification. Consider a localized charge distribution $\rho(\mathbf{r})$ that gives rise to an electric field $\mathbf{E}(\mathbf{r})$ throughout space. We can evaluate the average field by calculating the integral of $\mathbf{E}(\mathbf{r})$ over the volume of a sphere of radius R. We start from a general expression to analyze the two extremes shown in Fig., one in which the sphere contains all of the charge and the other in which the charge lies external to the sphere.



Choosing the origin of coordinates at the center of the sphere, we have the volume integral of the electric field,

$$\int_{r<R} \mathbf{E}(\mathbf{r})\,dV = -\int_{r<R} \nabla\Phi(\mathbf{r})\,dV = -\int_{r=R} R^2\,\Phi(\mathbf{r})\mathbf{n}\,d\Omega \quad . \quad (23)$$

where **n** is the outwardly directed normal. Substitution of

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_o}\int\frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}dV'$$

for the potential leads to

$$\int_{r<R} \mathbf{E}(\mathbf{r})\, dV = -\frac{R^2}{4\pi\varepsilon_o} \int\int_{r=R} \frac{\mathbf{n}}{|\mathbf{r}-\mathbf{r}'|}\, d\Omega\, \rho(\mathbf{r}')\, dV' \ . \tag{24}$$

To perform the angular integration we first note that **n** can be written in terms of the spherical angles (θ, ϕ) as

$$\mathbf{n} = \mathbf{i}\sin\theta\cos\phi + \mathbf{j}\sin\theta\sin\phi + \mathbf{k}\cos\theta \ .$$

Since the different components of **n** are linear combinations of $Y_{lm}(\theta,\phi)$ for l=1 only, the expression of

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{4\pi}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y_{lm}^*(\theta',\phi')\, Y_{lm}(\theta,\phi)$$

and orthogonality of the $Y_{lm}(\theta,\phi)$ can be used to eliminate all but the term l=1 in series. Thus we have

$$\int_{r=R} \frac{\mathbf{n}}{|\mathbf{r}-\mathbf{r}'|}\, d\Omega \ = \frac{r_<}{r_>^2}\int_{r=R} \mathbf{n}\cos\gamma\, d\Omega \ = \frac{4\pi}{3}\mathbf{n}'\frac{r_<}{r_>^2} \ . \tag{25}$$

Thus the integral (24) is

$$\int_{r<R} \mathbf{E}(\mathbf{r})\, dV = -\frac{R^2}{3\varepsilon_o}\int \mathbf{n}'\frac{r_<}{r_>^2}\rho(\mathbf{r}')\, dV' \ , \tag{26}$$

where $r_<(r_>)$ is the smaller (larger) of r' and R. If the sphere of radius R completely encloses the charge density, as indicated in Fig. , then $r_< = r'$ and $r_> = R$ in Eq. (26). The volume integral of the electric field over the sphere then becomes

$$\int_{r<R} \mathbf{E}(\mathbf{r})\, dV = -\frac{\mathbf{p}}{3\varepsilon_o} , \tag{27}$$

where **p** is the electric dipole moment of the charge distribution with respect to the center of the sphere. Note that this volume integral is independent of the size of the spherical region of integration provided all the charge is inside. On the other hand, if the charge is all exterior to the sphere of interest, then $r_> = r'$ and $r_< = R$ in Eq. (26). The volume integral of the electric field over the sphere then becomes

$$\int_{r<R} \mathbf{E}(\mathbf{r})\, dV = -\frac{R^3}{3\varepsilon_o}\int \mathbf{n}'\frac{1}{r'^2}\rho(\mathbf{r}')\, dV' = \frac{4\pi}{3}R^3\mathbf{E}(0) \ . \tag{28}$$

Here we have used the fact that from Coulomb's law the integral can be recognized to be the negative of $4\pi\varepsilon_o$ times the electric field at the center of the sphere. In other words, the average value of the electric field over a spherical volume containing no charge is the value of the field at the center of the sphere. To be consistent with Eq. (27), the electric field for a dipole should be modified as

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_o}\left[\frac{3\mathbf{n}(\mathbf{p}\cdot\mathbf{n})-\mathbf{p}}{|\mathbf{r}-\mathbf{r}_o|^3} - \frac{4\pi}{3}\mathbf{p}\,\delta(\mathbf{r}-\mathbf{r}_o)\right] \quad . \tag{29}$$

2. Interaction of an extended charge distribution with an external field

The multipole expansion of the potential of a charge distribution can be employed to describe the interaction of the charge distribution with an external field. The energy of the charge distribution $\rho(\mathbf{r})$ in an external field $\Phi(\mathbf{r})$ is given by

$$W = \int \rho(\mathbf{r})\Phi(\mathbf{r})\, d^3x \quad . \tag{1}$$

Compared with the formula for the energy derived previously, the factor 1/2 is missed here. It was introduced because the interaction energy of two charges appears twice in the integral. Now the double-counting is excluded, since the charge generating the field $\Phi(\mathbf{r})$ does not belong to the distribution $\rho(\mathbf{r})$. The external field may be expanded in a Taylor series around a suitably chosen origin:

$$\Phi(\mathbf{r}) = \Phi(0) + \mathbf{r}\cdot\nabla\Phi(\mathbf{r})\Big|_{\mathbf{r}=0} + \frac{1}{2}\sum_{i,j=1}^{3} x_i x_j \frac{\partial^2\Phi}{\partial x_i \partial x_j}\bigg|_{\mathbf{r}=0} + \cdots \quad . \tag{2}$$

Utilizing $\mathbf{E}(\mathbf{r}) = -\nabla\Phi(\mathbf{r})$, and therefore also

$$\frac{\partial^2\Phi}{\partial x_i \partial x_j}\bigg|_{\mathbf{r}=0} = -\frac{\partial E_j}{\partial x_i}\bigg|_{\mathbf{r}=0} \quad , \tag{3}$$

the series expansion can be rewritten as

$$\Phi(\mathbf{r}) = \Phi(0) - \mathbf{r}\cdot\mathbf{E}(0) - \frac{1}{2}\sum_{i,j=1}^{3} x_i x_j \frac{\partial E_j}{\partial x_i}\bigg|_{\mathbf{r}=0} + \cdots \quad . \tag{4}$$

To get the quadrupole moments in the last sum, we subtract $\frac{1}{6}r^2\,\nabla\cdot\mathbf{E}$ from each term,

$$\Phi(\mathbf{r}) = \Phi(0) - \mathbf{r}\cdot\mathbf{E}(0) - \frac{1}{6}\sum_{i,j=1}^{3} \left(3x_i x_j - r^2\delta_{ij}\right)\frac{\partial E_j}{\partial x_i}\bigg|_{\mathbf{r}=0} + \cdots \quad . \tag{5}$$

This step does not affect the potential because $\nabla\cdot\mathbf{E} = 0$ for the external field in the considered region due to the fact the field-producing charges are lying outside of it. Performing the integration, we obtain for the energy

$$W = \int \rho(\mathbf{r})\Phi(\mathbf{r})\, d^3x = q\Phi(0) - \mathbf{p}\cdot\mathbf{E}(0) - \frac{1}{6}\sum_{i,j=1}^{3} Q_{ij}\frac{\partial E_j}{\partial x_i}\bigg|_{\mathbf{r}=0} + \cdots \quad . \tag{6}$$

This expansion indicates that different multipoles interact with the outer field in a distinct way: the total charge is connected with the potential, the dipole with the electric field (i.e., with the gradient of the potential), the quadrupole with the derivative of the electric field, etc.

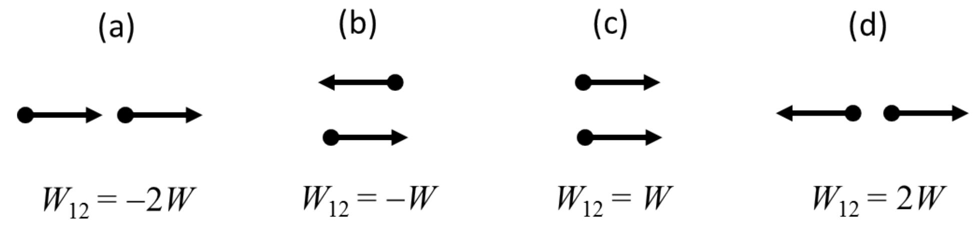
The interaction energy between two dipoles $\mathbf{p}_1$ and $\mathbf{p}_2$ can be obtained directly from Eq. (6) by using the dipole field

$$\mathbf{E}_1(\mathbf{r}) = \frac{-1}{4\pi\varepsilon_o}\nabla\left[\frac{\mathbf{p}_1\cdot(\mathbf{r}-\mathbf{r}_1)}{|\mathbf{r}-\mathbf{r}_1|^3}\right] = \frac{1}{4\pi\varepsilon_o}\frac{3\mathbf{n}(\mathbf{p}_1\cdot\mathbf{n})-\mathbf{p}_1}{|\mathbf{r}-\mathbf{r}_1|^3} \quad . \tag{7}$$

Thus, the mutual potential energy is

$$W_{12} = -\mathbf{p}_2\mathbf{E}_1(\mathbf{r}_2) = \frac{1}{4\pi\varepsilon_o}\frac{\mathbf{p}_2\cdot\mathbf{p}_1 - 3(\mathbf{p}_2\cdot\mathbf{n})(\mathbf{p}_1\cdot\mathbf{n})}{|\mathbf{r}_2-\mathbf{r}_1|^3} \quad . \tag{8}$$

where $\mathbf{n}$ is a unit vector in the direction $\mathbf{r}_2-\mathbf{r}_1$ and it is assumed that $\mathbf{r}_2 \neq \mathbf{r}_1$. The dipole-dipole interaction is attractive or repulsive, depending on the orientation of the dipoles. The energy reaches its minimum if the dipoles are arranged parallel to each other (along a straight line). If $\mathbf{p}_1 \perp \mathbf{n}$ and $\mathbf{p}_2 \perp \mathbf{n}$, then the antiparallel arrangement is favored energetically. In the following figure, various orientations and the corresponding energies are illustrated, where $W_{12} = \frac{1}{4\pi\varepsilon_o}\frac{|\mathbf{p}_1||\mathbf{p}_2|}{|\mathbf{r}_2-\mathbf{r}_1|^3}$.



3. The Electric Field Due to a Polarized Dielectric

Here we use a fairly phenomenological consideration to discuss the electric field arising from charges in matter. We consider a dielectric having charges, electric dipoles, quadrupoles, etc. distributed throughout the material. If the potential is considered to be formed by the charge density $\rho(\vec{r}\,')$ and the dipole moment density $\mathbf{P}(\vec{r}\,')$, then the potential at $\vec{r}$ is given by

$$\Phi(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\left[\int_V \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}d^3r' + \int_V \frac{\mathbf{P}(\vec{r}\,')\cdot(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}d^3r'\right]. \tag{1}$$

With

$$\frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3} = \nabla'\left(\frac{1}{|\vec{r}-\vec{r}\,'|}\right) \quad , \tag{2}$$

Eq. (1) can be rewritten as

$$\Phi(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\left[\int_V \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}d^3r' + \int_V \mathbf{P}(\vec{r}\,')\cdot\nabla'\left(\frac{1}{|\vec{r}-\vec{r}\,'|}\right)d^3r'\right] \quad . \tag{3}$$

Using

$$\mathbf{P}(\vec{r}\,')\cdot\nabla'\left(\frac{1}{|\vec{r}-\vec{r}\,'|}\right) = \nabla'\cdot\left(\frac{\mathbf{P}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\right) - \frac{\nabla'\cdot\mathbf{P}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} \quad , \tag{4}$$

the potential in Eq. (3) can be rewritten as

$$\Phi(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\left[\int_V \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}d^3r' - \int_V \frac{\nabla'\cdot\mathbf{P}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}d^3r' + \int_V \nabla'\cdot\left(\frac{\mathbf{P}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\right)d^3r'\right] \quad . \tag{5}$$

The third volume integral can be converted to a surface integral by Gauss theorem:

$$\int_V \nabla'\cdot\left(\frac{\mathbf{P}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\right)d^3r' = \oint_S \frac{\mathbf{P}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\cdot\mathbf{n}'\,da' \quad . \tag{6}$$

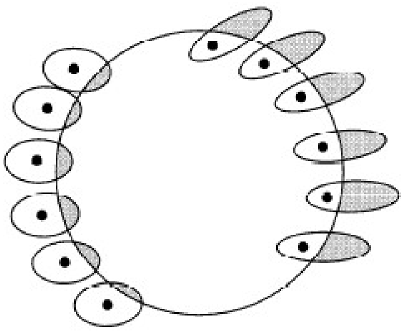
This integral vanishes because approaching to the infinity we have $\mathbf{P}(\vec{r}\,')\cdot\mathbf{n}' = 0$, so that the integrand is zero. Considering $\mathbf{E}(\vec{r}) = -\nabla\Phi(\vec{r})$, we have

$$\mathbf{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\int_V \left[\rho(\vec{r}\,') - \nabla'\cdot\mathbf{P}(\vec{r}\,')\right]\nabla\left(\frac{-1}{|\vec{r}-\vec{r}\,'|}\right)d^3r' \quad . \tag{7}$$

Using $\nabla^2\left(\frac{-1}{|\vec{r}-\vec{r}\,'|}\right) = 4\pi\delta(\vec{r}-\vec{r}\,')$, the divergence of the electric field is given by

$$\nabla\cdot\mathbf{E}(\vec{r}) = \frac{1}{\varepsilon_o}\left[\rho(\vec{r}) - \nabla\cdot\mathbf{P}(\vec{r})\right] \quad . \tag{8}$$

The presence of the divergence of **P** in the effective charge density can be understood qualitatively. If the polarization is nonuniform there can be a net increase or decrease of charge within any small volume, as indicated schematically in Fig. .



The molecules of a dielectric may be classified as either polar or nonpolar. For the case of nonpolar molecules in an electric field, the positive charges will move slightly in the direction of the field while the negative charges move slightly in the opposite direction, creating a polarization of the medium. On the other hand, it the molecules have intrinsic (permanent) dipole moments that in the absence of an electric field are randomly oriented, they will attempt to align with the electric field, and their non-random alignment will lead to a polarization of the medium. In either case, the resulting polarization will be a function of the local electric field. The empirical relation is given by

$$P^i = P_o^i + \chi_j^i \varepsilon_o E^j + \chi_{jk}^{i(2)} \varepsilon_o^2 E^j E^k + \chi_{jkl}^{i(3)} \varepsilon_o^3 E^j E^k E^l + \cdots \quad . \tag{9}$$

Here summation over repeated indices is implied. For isotropic materials, only the diagonal terms of the dielectric susceptibility tensor χ survive, and Eq. (9) becomes merely a power series expansion for the polarization **P**. Materials exhibiting large spontaneous polarization are known as ferroelectrics. In analogy to magnets, ferroelectric objects are known as electrets. The best known example of a ferroelectric crystal is $BaTiO_3$. Mechanical distortions of the crystal may result in large changes of the polarization, giving rise to piezoelectricity. Similarly, changes in temperature give rise to pyroelectricity. A number of crystals have sufficiently large second or third order susceptibility that optical radiation traversing the crystal may excite a polarization with $\cos^2 \omega t$ or $\cos^3 \omega t$ dependence giving rise to the generation of frequency doubled or tripled light. The efficiency of such doubling or tripling would be expected to increase linearly for doubling or quadratically for tripling with incident field strength.

In sufficiently small electric fields, the relationship between E and P for isotropic materials can be simply given by

$$\mathbf{P} = \chi_e \varepsilon_o \mathbf{E} \quad , \tag{10}$$

where the constant χ_e is called the linear dielectric susceptibility of the dielectric. The ratio between the induced molecular dipole and $\varepsilon_o \mathbf{E}$, the polarizing field, is known as the polarizability γ_{mol}:

$$\mathbf{p} = \gamma_{mol} \varepsilon_o \mathbf{E} \, . \tag{11}$$

With the definition of the electric displacement **D**, we have

$$\mathbf{D} = \varepsilon_o \mathbf{E} + \mathbf{P} \quad . \tag{12}$$

Then, Eq. (9) becomes

$$\nabla \cdot \mathbf{D} = \rho \quad . \tag{13}$$

The dipoles of the medium are not a source for **D**, only the so-called free charges act as sources. Using Eq. (10), we have

$$\mathbf{D} = \varepsilon_o \mathbf{E} + \mathbf{P} = \varepsilon_o (1 + \chi_e) \mathbf{E} = \varepsilon \mathbf{E} \quad . \tag{14}$$

The constant e is called the permittivity of the dielectric. The dielectric constant is defined as

$$\kappa = \frac{\varepsilon}{\varepsilon_o} = 1 + \chi_e \quad . \tag{15}$$

In general, since it takes time for dipoles to response to the applied field, all three constants χ_e, ε, and κ are frequency dependent.

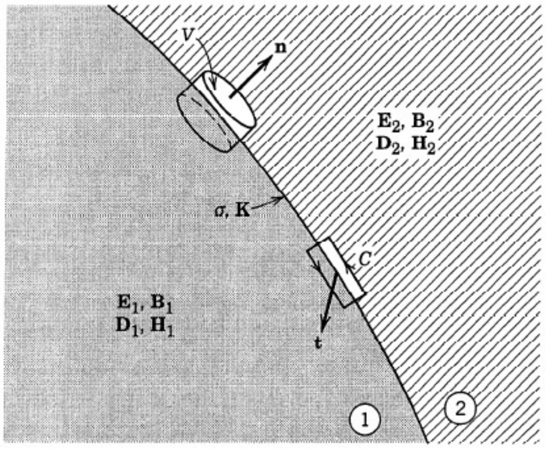
Using the divergence theorem, the differential formulation for the dielectric displacement can be transformed into the integral form:

$$\int_V \nabla \cdot D \, d^3 r = \oint_S \mathbf{D} \cdot \mathbf{n} \, da = Q \quad . \tag{16}$$

On the other hand, the equation $\nabla \times \mathbf{E} = 0$ can be transformed with the Stokes theorem into the integral form:

$$\int_S (\nabla \times \mathbf{E}) \cdot \mathbf{n} \, da = \oint_C \mathbf{E} \cdot d\mathbf{l} = 0 \quad . \tag{17}$$

The integral forms in Eqs. (16) and (17) can be used directly to deduce the relationship of various normal and tangential components of the fields on either side of a surface between different media. Figure shows an appropriate geometrical arrangement. An infinitesimal Gaussian pillbox straddles the boundary surface between two media with different electromagnetic properties. Similarly, the infinitesimal contour C has its long arms on either side of the boundary and is oriented so that the normal to its spanning surface is tangent to the interface.



We first apply the integral statement in Eq. (16) to the volume of the pillbox. In the limit of a very shallow pillbox, the side surface does not contribute to the integrals on the right in Eq. (16). Only the top and bottom surfaces contribute. If the top and the bottom are parallel, tangent to the surface, and of area Δa, then Eq. (16) becomes

$$\oint_S \mathbf{D}\cdot\mathbf{n}\,da = (\mathbf{D}_2 - \mathbf{D}_1)\cdot\mathbf{n}_{21}\,\Delta a = \sigma\,\Delta a \quad . \tag{18}$$

where $\mathbf{n}_{21}$ is a unit vector normal to the surface, directed from region 1 to region 2, and σ is the macroscopic surface-charge density on the boundary surface. Thus the normal components of **D** on either side of the boundary surface are related according to

$$(\mathbf{D}_2 - \mathbf{D}_1)\cdot\mathbf{n}_{21} = \sigma \ . \tag{19}$$

In an analogous manner the infinitesimal Stokesian loop can be used to determine the discontinuities of the tangential components of **E**. If the short arms of the contour C in Fig. are of negligible length and each long arm is parallel to the surface and has length Δl, then the right-hand integral of Eq. (17) is

$$\oint_C \mathbf{E}\cdot d\mathbf{l} = (\mathbf{t}\times\mathbf{n}_{21})\cdot(\mathbf{E}_2 - \mathbf{E}_1)\,\Delta l = 0 \quad . \tag{20}$$

where **t** is a unit vector tangential to the surface. The tangential components of **E** on either side of the boundary are therefore related by

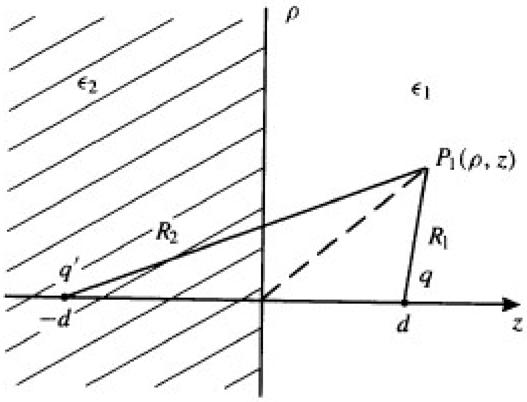
$$(\mathbf{E}_2 - \mathbf{E}_1)\times\mathbf{n}_{21} = 0 \quad . \tag{21}$$

4. Boundary-Value Problems with Dielectrics

The methods of earlier sections for the solution of electrostatic boundary-value problems can readily be extended to handle the presence of dielectrics. Here a few examples of the various techniques applied to dielectric media are treated.

Example

Consider a point charge q embedded in a semi-infinite dielectric ε_1 a distance d away from a plane interface that separates the first medium from another semi-infinite dielectric ε_1. The surface is taken as the plane $z=0$. Find the potential distribution in both half-space by using cylindrical coordinates (ρ,ϕ,z) and derive the polarization surface-charge density at the interface.



<**Solution**>

The potential is composed of partial solutions for the half-spaces $z<0$ and $z>0$ matched by continuity conditions at $z=0$. The conditional equations are

$$\begin{cases} \varepsilon_1 \nabla\cdot\mathbf{E} = q\delta(x)\delta(y)\delta(z-d), & z>0 \\ \varepsilon_2 \nabla\cdot\mathbf{E} = 0, & z<0 \\ \nabla\times\mathbf{E}=0, & \text{everywhere.} \end{cases} \tag{1}$$

The charge lies at (0, 0, d), and the problem is entirely rotationally symmetric about the z-axis; therefore, the angle ϕ is need for the solution. Taking a point P_1 at (ρ, z) in the right-hand half-space, the potential can be calculated by using an image charge q' at (0, 0, $-d$) to consider the effect of dielectric 2. The potential of the two point charges in the half-space 1 is then given by

$$\Phi_1 = \frac{1}{4\pi\varepsilon_1}\left(\frac{q}{\sqrt{\rho^2+(z-d)^2}}+\frac{q'}{\sqrt{\rho^2+(z+d)^2}}\right). \tag{2}$$

Considering the charge-free half-space 2, the potential can be calculated by placing a point charge q'' at (0, 0, *d*) to include the effect of the dielectrics. Then, the potential is given by

$$\Phi_2 = \frac{1}{4\pi\varepsilon_2}\frac{q''}{\sqrt{\rho^2+(z-d)^2}} \quad . \tag{3}$$

The magnitude of the assumed image charges can be evaluated by using the boundary conditions that the tangential component of the electric field intensity and the normal component of the dielectric displacement are continuous at the interface. Consequently, we have

$$\left.\frac{\partial\Phi_1}{\partial\rho}\right|_{z=0} = \left.\frac{\partial\Phi_2}{\partial\rho}\right|_{z=0} \quad , \tag{4}$$

$$\varepsilon_1\left.\frac{\partial\Phi_1}{\partial z}\right|_{z=0} = \varepsilon_2\left.\frac{\partial\Phi_2}{\partial z}\right|_{z=0} \quad . \tag{5}$$

Substituting Eqs. (2) and (3) into Eqs. (4) and (5), we obtain

$$\varepsilon_2(q+q') = \varepsilon_1 q'' \quad , \tag{6}$$

$$q - q' = q'' \quad . \tag{7}$$

With Eqs. (6) and (7), the image charges can be determined as

$$q' = \frac{\varepsilon_1-\varepsilon_2}{\varepsilon_1+\varepsilon_2}q \quad , \tag{8}$$

$$q'' = \frac{2\varepsilon_2}{\varepsilon_1+\varepsilon_2}q \quad . \tag{9}$$

The polarization surface-charge density can be determined by

$$\sigma_p = -(\mathbf{P}_2-\mathbf{P}_1)\cdot\mathbf{n}_{21} \quad . \tag{10}$$

where $\mathbf{n}_{21}$ is a unit vector normal to the surface, directed from region 1 to region 2. When no free charges exist at the interface, Eq. (10) can be conveniently expressed as

$$\sigma_p = \varepsilon_o(\mathbf{E}_2-\mathbf{E}_1)\cdot\mathbf{n}_{21} \quad . \tag{11}$$

This equation manifests that the discontinuity of the electric field intensity at the interface arises from the induced polarization surface-charge density. Explicitly,

$$\begin{aligned}\sigma_p &= \varepsilon_o(\mathbf{E}_2-\mathbf{E}_1)\cdot(-\mathbf{a}_z) = -\varepsilon_o\left.\left(\frac{\partial\Phi_1}{\partial z}-\frac{\partial\Phi_2}{\partial z}\right)\right|_{z=0} \\ &= -\frac{\varepsilon_o}{4\pi}\left[\frac{(q-q')d}{\varepsilon_1(\rho^2+z^2)^{3/2}}-\frac{q''d}{\varepsilon_2(\rho^2+z^2)^{3/2}}\right] = -\frac{q}{2\pi}\frac{\varepsilon_o}{\varepsilon_1}\frac{(\varepsilon_2-\varepsilon_1)}{(\varepsilon_2+\varepsilon_1)}\frac{d}{(\rho^2+z^2)^{3/2}}\end{aligned} \quad . \tag{12}$$

In the limit $\varepsilon_2 >> \varepsilon_1$ the dielectric ε_2 behaves much like a conductor in that the electric field inside it becomes very small and the surface-charge density in Eq. (12) approaches the value appropriate to a conducting surface, apart from a factor of $\varepsilon_o / \varepsilon_1$.

Another method for finding the solution is based on the eigenfucntion expansion. First of all, remember that expanding the free space Green function with the eigenfunctions of the Helmholtz equation, it can be shown that

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \left(\frac{1}{2\pi}\right)^3 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{4\pi}{k^2} e^{ik_x(x-x')} e^{ik_y(y-y')} e^{ik_z(z-z')} dk_x dk_y dk_z \ , \tag{13}$$

where $k_x^2 + k_y^2 + k_z^2 = k^2$. We need to find the eigenfunction of the Helmholtz equation $(\nabla^2 + k^2)\Psi = 0$ for an initial wave in the region $z > 0$ with the boundary condition the same as the present problem, i.e.

$$\left.\frac{\partial \Psi_1}{\partial \rho}\right|_{z=0} = \left.\frac{\partial \Psi_2}{\partial \rho}\right|_{z=0} \ , \tag{14}$$

$$\varepsilon_1 \left.\frac{\partial \Psi_1}{\partial z}\right|_{z=0} = \varepsilon_2 \left.\frac{\partial \Psi_2}{\partial z}\right|_{z=0} \ . \tag{15}$$

Since there are no boundaries in the *x* and *y* directions, the eigenfucntion, the eigenfunction can be expressed as

$$\Psi(x,y,z) = e^{i(k_x x + k_y y)} \psi(z) \ , \tag{16}$$

with

$$\psi(z) = \begin{cases} \psi_1(z) = e^{ik_z z} + Ae^{-ik_z z} \ , & z > 0 \\ \psi_2(z) = Be^{ik_z z} \ , & z < 0 \end{cases} \ . \tag{17}$$

Using the boundary conditions,

$$\psi_1\big|_{z=0} = \psi_2\big|_{z=0} \ , \tag{18}$$

$$\varepsilon_1 \left.\frac{\partial \psi_1}{\partial z}\right|_{z=0} = \varepsilon_2 \left.\frac{\partial \psi_2}{\partial z}\right|_{z=0} \ , \tag{19}$$

it can be found that

$$A = \frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + \varepsilon_2} \ , \tag{20}$$

$$B = \frac{2\varepsilon_1}{\varepsilon_1 + \varepsilon_2} \ . \tag{21}$$

Consequently, the eigenfunction of the present Helmholtz equation is given by

$$\Psi(x,y,z) = e^{i(k_x x + k_y y)}\left[\left(e^{ik_z z} + \frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + \varepsilon_2} e^{-ik_z z}\right)u(z) + \frac{2\varepsilon_1}{\varepsilon_1 + \varepsilon_2} e^{ik_z z} u(-z)\right], \tag{22}$$

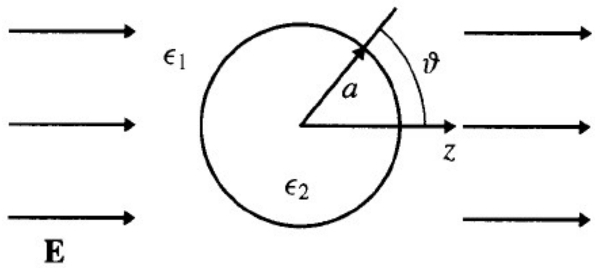
where $u(z)$ is the unit step function. With Eq. (22), the delta source term $\delta(z - z')$ with $z' > 0$ is given by

$$\delta(z - z') = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\left(e^{ik_z z} + \frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + \varepsilon_2} e^{-ik_z z}\right)u(z) + \frac{2\varepsilon_1}{\varepsilon_1 + \varepsilon_2} e^{ik_z z} u(-z)\right] e^{-ik_z z'} dk_z \quad . \tag{23}$$

$$\begin{aligned}\delta(x - x')\delta(y - y')\delta(z - z') = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Bigg\{ e^{i(k_x x + k_y y)}\Bigg[\left(e^{ik_z z} + \frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + \varepsilon_2} e^{-ik_z z}\right)u(z) \\ + \frac{2\varepsilon_1}{\varepsilon_1 + \varepsilon_2} e^{ik_z z} u(-z)\Bigg] dk_x dk_y dk_z \Bigg\}\end{aligned} \quad . \tag{24}$$

Example

A homogeneous sphere of radius *a* and with permittivity ε_2 is embedded in a region with permittivity ε_1. In absence of the sphere, a uniform field $\mathbf{E} = E_o \mathbf{a}_z$ is in this region. Determine the potential and the density of the polarization charge on the spherical surface.



<**Solution**>

Both inside and outside the sphere there are no free charges. Consequently the problem is one of solving the Laplace equation with the proper boundary conditions at $r = a$. From the axial symmetry of the geometry we can take the solution to be of the form:

$$\Phi_{in}(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) \quad , \quad r < a \tag{13}$$

and

$$\Phi_{out}(r,\theta) = \sum_{l=0}^{\infty}\left[B_l r^l + C_l r^{-(l+1)}\right]P_l(\cos\theta) \quad , \quad r > a \tag{14}$$

From the boundary condition at infinity, $\Phi \to -E_o z = -E_o r\cos\theta$, we can find that the only non-vanishing B_l is $B_1 = -E_o$. In addition, $A_l = C_l = 0$ for all $l \neq 1$. The other coefficients are determined from the boundary conditions at $r = a$:

$$E_{1t} = E_{2t} \;\Rightarrow\; -\frac{1}{r}\frac{\partial \Phi_{in}}{\partial \theta}\bigg|_{r=a} = -\frac{1}{r}\frac{\partial \Phi_{out}}{\partial \theta}\bigg|_{r=a} \quad , \tag{15}$$

$$D_{1n} = D_{2n} \;\Rightarrow\; -\varepsilon_2 \frac{\partial \Phi_{in}}{\partial r}\bigg|_{r=a} = -\varepsilon_1 \frac{\partial \Phi_{out}}{\partial r}\bigg|_{r=a} \quad , \tag{16}$$

When the series in Eqs. (13) and (14) are substituted, the boundary conditions in Eqs. (15) and (16) yield

$$A_1 = -E_o + C_1 a^{-3} \quad , \tag{17}$$

$$\varepsilon_2 A_1 = -\varepsilon_1 E_o - 2\varepsilon_1 C_1 a^{-3} \quad . \tag{18}$$

Thus, the coefficients can be found to be

$$A_1 = \frac{-3\varepsilon_1}{2\varepsilon_1 + \varepsilon_2} E_o \quad , \tag{19}$$

$$C_1 = \frac{\varepsilon_2 - \varepsilon_1}{2\varepsilon_1 + \varepsilon_2} E_o\, a^3 \quad . \tag{20}$$

The potential is therefore

$$\Phi_{in} = \frac{-3\varepsilon_1}{2\varepsilon_1 + \varepsilon_2} E_o\, r\cos\theta = -E_o\, r\cos\theta + \left(\frac{\varepsilon_2 - \varepsilon_1}{2\varepsilon_1 + \varepsilon_2}\right) E_o r\cos\theta \quad , \tag{21}$$

$$\Phi_{out} = -E_o\, r\cos\theta + \left(\frac{\varepsilon_2 - \varepsilon_1}{2\varepsilon_1 + \varepsilon_2}\right) E_o \frac{a^3}{r^2}\cos\theta \quad . \tag{22}$$

Outside the sphere the potential is equivalent to the applied field E_o plus the field of an electric dipole at the origin with dipole moment *p* oriented in the direction of the applied field:

$$p = 4\pi\varepsilon_o \left(\frac{\varepsilon_2 - \varepsilon_1}{2\varepsilon_1 + \varepsilon_2}\right) a^3 E_o \quad . \tag{23}$$

The dipole moment can be interpreted as the volume integral of the polarization **P**, which is given by

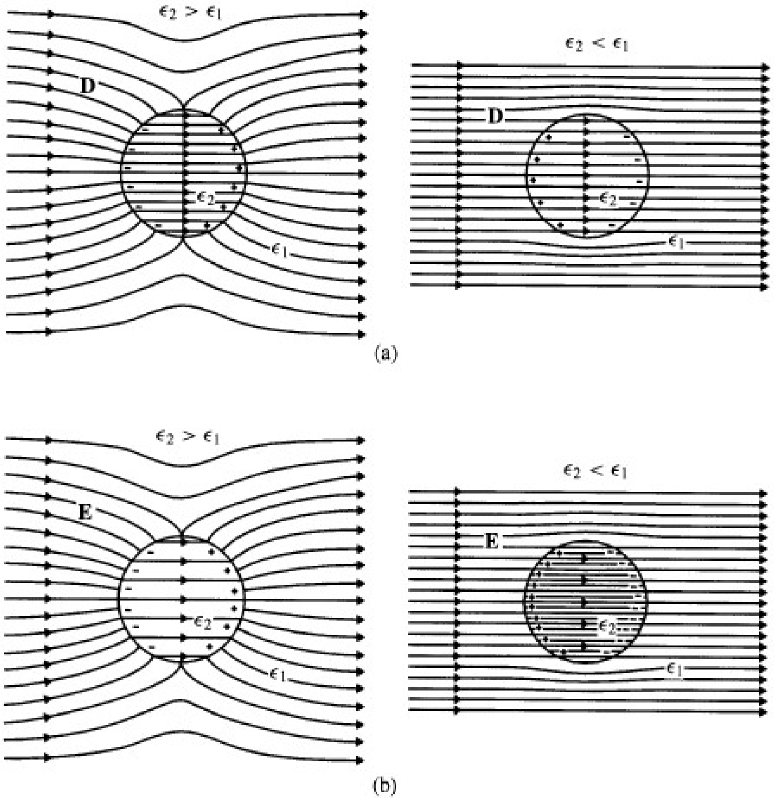
$$\mathbf{P} = 3\varepsilon_o \left(\frac{\varepsilon_2 - \varepsilon_1}{2\varepsilon_1 + \varepsilon_2}\right) \mathbf{E}_o \quad . \tag{24}$$

The polarization-surface-charge density is given by

$$\sigma_{pol} = \mathbf{P}\cdot\mathbf{n} = 3\varepsilon_o \left(\frac{\varepsilon_2 - \varepsilon_1}{2\varepsilon_1 + \varepsilon_2}\right) \mathbf{E}_o \cos\theta \quad . \tag{25}$$

When $\varepsilon_2 > \varepsilon_1$, it can be thought that the polarization-surface-charge density produces an internal field directed oppositely to the applied field, so reducing the field inside the sphere to be

$$\mathbf{E}_{in} = \mathbf{E}_o - \frac{\mathbf{P}}{3\varepsilon_o} \quad . \tag{26}$$



5. Microscopic properties of Matter

Taking a simple classical harmonic oscillator model for an atom or molecule with spring constant $m\omega_o^2$, we can find that the displacement of charge *e* from its equilibrium at frequencies well below the resonant frequency ω_o is given by

$$\Delta\mathbf{x} = \frac{e\mathbf{E}}{m\omega_o^2} \quad , \tag{1}$$

where *m* is the reduced mass of the charge. Consequently the induced dipole moment is

$$\mathbf{p} = e\Delta\mathbf{x} = \frac{e^2\mathbf{E}}{m\omega_o^2} \quad . \tag{2}$$

With the definition of $\mathbf{p} = \gamma_{mol}\varepsilon_o\mathbf{E}$ and using Eq. (2), the polarizability γ_{mol} is then given by

$$\gamma_{mol} = \frac{e^2}{\varepsilon_o m\omega_o^2} \quad . \tag{3}$$

If there are a set of charge e_j with the masses m_j and oscillation frequencies ω_j in each molecule then the molecular polarizability is

$$\gamma_{mol} = \frac{1}{\varepsilon_o}\sum_j \frac{e_j^2}{m_j \omega_j^2} \quad . \tag{4}$$

The susceptibility for gases becomes $\chi = N\gamma_{mol} = Ne^2/\varepsilon_o m\omega_o^2$. Thus for molecular hydrogen with its lowest electronic resonance near $\omega_o \approx 1.8\times 10^{16}$ sec^{-1} ($\lambda \approx 100$ nm) and $N \approx 2.69\times 10^{25}$ m^{-3} at STP (standard temperature and pressure), we obtain $\chi \approx 2.64\times 10^{-4}$, quite close to experimental value. Such good agreement should not be expected for substances other than hydrogen and helium; generally a sum over all resonant frequencies is required to obtain reasonable agreement. It is worth noting that this value should be fairly good up to and above optical frequencies. By contrast, the orientation of polar molecules fails for frequencies approaching rotational frequencies of the molecule, typically a few GHz. Thus water has $\chi \approx 80$ (it has a strong dependence on temperature, varying from 87 at 0^o C to 55 at 100^o C) at low frequencies, decreasing to $\chi \approx 0.8$ at optical frequencies. An exact evaluation for the molecular dipole moment should resort to quantum mechanics to calculate the expectation value of the dipole moment $<\psi_f \,|\sum_j e_j \mathbf{r}_j \,|\psi_i>$.

6. Electrostatic Energy in Dielectric Media

To understand the stored energy in the electric field, we can think of the final configuration of charge as being created by assembling bit by bit the elemental charges, bringing each one in from infinitely far away against the action of the then existing electric field. From the energy of a system of charges in free space, we can obtain

$$W_e = \frac{1}{2}\int \rho(\mathbf{r})\Phi(\mathbf{r})\, d^3x .$$

The above-mentioned equation can be shown to be valid macroscopically only if the behavior is linear. To generalize the formula, we consider a small change in the energy δW_e due to some sort of change $d\rho$ in the macroscopic charge density ρ in all space. The work done for achieving this change is

$$\delta W_e = \int \delta\rho(\mathbf{r})\Phi(\mathbf{r})\, d^3x ,$$

where $\Phi(\mathbf{r})$ is the potential due to the charge density ρ already present. With $\nabla\cdot\mathbf{D} = \rho$, it can be found that

$$\delta W_e = \int (\nabla \cdot \delta \mathbf{D}) \Phi(\mathbf{r}) \, d^3x = \int \mathbf{E} \cdot \delta \mathbf{D} \, d^3x \, ,$$

where the relation $\mathbf{E} = -\nabla \Phi$ has been used. The total energy can be expressed as

$$W_e = \int (\nabla \cdot \delta \mathbf{D}) \Phi(\mathbf{r}) \, d^3x = \int \int_0^D \mathbf{E} \cdot \delta \mathbf{D} \, d^3x \, .$$

If the medium is linear, then $\mathbf{E} \cdot \delta \mathbf{D} = \delta(\mathbf{E} \cdot \mathbf{D})/2$ and the total energy is given by

$$W_e = \frac{1}{2} \int \mathbf{D} \cdot \mathbf{E} \, d^3x = \frac{1}{2} \int \mathbf{D} \cdot (-\nabla \Phi(\mathbf{r})) \, d^3x = \frac{1}{2} \int (\nabla \cdot \mathbf{D}) \Phi(\mathbf{r}) \, d^3x = \frac{1}{2} \int \rho(\mathbf{r}) \Phi(\mathbf{r}) \, d^3x$$

Consequently, for a linear medium, the rate of change of the stored energy of the electric field is given by

$$\frac{dW_e}{dt} = \int \frac{\partial \mathbf{D}}{\partial t} \cdot \mathbf{E} \, d^3x$$

Chapter Six: Magnetostatics and Faraday's Law

1. Biot and Savart Law

In 1819 Oersted observed the phenomenon that wires carrying electric currents can produce deflections of permanent magnetic dipoles placed in their neighborhood. This phenomenon indicates that the currents are sources of magnetic-flux density. Biot and Savart in 1820 first established the basic experimental laws relating the magnetic induction B to the currents. On the other hand, Ampère during 1820-1825 performed much more elaborate and thorough to establish the law of force between one current and another. The conclusion is that if $d\mathbf{l}$ is an element of length of a wire with a current I at $\boldsymbol{r}'$ and $\boldsymbol{r}$ is the coordinate of an observation point P, as shown in Fig. , then the elemental flux density $d\mathbf{B}$ at the point P is given by

$$d\mathbf{B}(\mathbf{r})=\frac{\mu_o}{4\pi}I\frac{d\mathbf{l}\times(\mathbf{r}-\mathbf{r}')}{|\,\mathbf{r}-\mathbf{r}'\,|^3}\quad .$$

Ampère did not directly determine the relation between currents and magnetic induction, but considered rather the force that one current-carrying wire experiences in the presence of another. Experimental results revealed that the force experienced by a current element $I_1 d\mathbf{l}_1$ in the presence of a magnetic induction $\mathbf{B}$ is given by

$$d\mathbf{F}=I_1\left(d\mathbf{l}_1\times\mathbf{B}\right)\quad .$$

When the magnetic induction $\mathbf{B}$ is due to a closed current loop #2 with current I_2, the total force experienced by a closed current loop #1 is

$$\mathbf{F}_{12}=\frac{\mu_o}{4\pi}I_1 I_2\oint\oint\frac{d\mathbf{l}_1\times\left[d\mathbf{l}_2\times(\mathbf{r}_1-\mathbf{r}_2)\right]}{|\,\mathbf{r}_1-\mathbf{r}_2\,|^3}\quad .$$

The line integrals are taken around the two loops, as shown in Fig.. With the formula

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C})=\mathbf{B}(\mathbf{A}\cdot\mathbf{C})-\mathbf{C}(\mathbf{A}\cdot\mathbf{B})\quad ,$$

the integrand can be put in a form that

$$\frac{d\mathbf{l}_1\times\left[d\mathbf{l}_2\times(\mathbf{r}_1-\mathbf{r}_2)\right]}{|\,\mathbf{r}_1-\mathbf{r}_2\,|^3}=d\mathbf{l}_2\left[\frac{d\mathbf{l}_1\cdot(\mathbf{r}_1-\mathbf{r}_2)}{|\,\mathbf{r}_1-\mathbf{r}_2\,|^3}\right]-\frac{(\mathbf{r}_1-\mathbf{r}_2)}{|\,\mathbf{r}_1-\mathbf{r}_2\,|^3}\left(d\mathbf{l}_1\cdot d\mathbf{l}_2\right)\quad .$$

The first term involves a perfect differential in the integral over $d\mathbf{l}_1$. As a consequence, it gives no contribution to the integral in Eq. (), when the paths are closed or extend to infinity. Then Ampère's force law is given by

$$\mathbf{F}_{12}=-\frac{\mu_o}{4\pi}I_1 I_2\oint\oint\frac{(\mathbf{r}_1-\mathbf{r}_2)}{|\,\mathbf{r}_1-\mathbf{r}_2\,|^3}\left(d\mathbf{l}_1\cdot d\mathbf{l}_2\right)\quad .$$

This equation explicitly displays symmetry and satisfies Newton's third law.

Each of two long, parallel, straight wires a distance d apart, carrying currents I_1 and I_2, experiences a force per unit length directed perpendicularly toward the other wire and of magnitude,

$$\frac{dF}{dl} = \frac{\mu_o}{2\pi}\frac{I_1 I_2}{d} \quad .$$

The force is attractive (repulsive) if the currents flow in the same (opposite) directions. The forces that exist between current-carrying wires can be used to define magnetic-flux density in a way that is independent of permanent magnetic dipoles. For a current density $\mathbf{J}(\mathbf{r})$ in an external magnetic-flux density $\mathbf{B}(\mathbf{r})$, the elementary force law implies that the total force on the current distribution is

$$\mathbf{F} = \int \mathbf{J}(\mathbf{r}) \times \mathbf{B}(\mathbf{r})\, d^3 r \quad .$$

The total torque is given by

$$\mathbf{N} = \int \mathbf{r} \times \left[\mathbf{J}(\mathbf{r}) \times \mathbf{B}(\mathbf{r})\right] d^3 r \quad .$$

Example 1: A circular loop of radius a carrying current I lies in the x-y plane with its center at the origin. Find the magnetic induction field at a point on the z-axis.

Solution: In cylindrical coordinates, $Id\mathbf{l} = (Ia)d\phi'\, \mathbf{a}_{\phi'}$ and $\mathbf{r} - \mathbf{r}' = z\, \mathbf{a}_\mathbf{z} - a\,\mathbf{a}_{\rho'}$. Thus,

$$\mathbf{B}(0,0,z) = \frac{\mu_o}{4\pi} I \oint \frac{d\mathbf{l} \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} = \frac{\mu_o}{4\pi} I \cdot \int_0^{2\pi} \frac{az\,\mathbf{a}_{\rho'} + a^2\,\mathbf{a}_\mathbf{z}}{(z^2 + a^2)^{3/2}} d\phi' = \frac{\mu_o}{2} I \frac{a^2}{(z^2 + a^2)^{3/2}} \mathbf{a}_\mathbf{z}$$

where we have used the fact that $\int_0^{2\pi} \mathbf{a}_{\rho'}\, d\phi' = 0$. In terms of the magnetic moment, $\mathbf{m} = I\pi a^2\, \mathbf{a}_\mathbf{z}$, of the loop, the result at large distance can be approximated as

$$\mathbf{B}(0,0,z) = 2\frac{\mu_o}{4\pi}\frac{\mathbf{m}}{R^3}$$

2. Differential Equations of Magnetostatics and Ampère's Law

The basic law

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_o}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} d^3 r' \quad .$$

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_o} \int \frac{\rho(\mathbf{r}')(\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} d^3 r' \quad .$$

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_o}{4\pi}\nabla\times\int\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^3r' \quad .$$

$$\nabla\cdot\mathbf{B}(\mathbf{r}) = 0 \quad .$$

$$\nabla\times\mathbf{B}(\mathbf{r}) = \frac{\mu_o}{4\pi}\nabla\times\nabla\times\int\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^3r' \quad .$$

$$\nabla\times\nabla\times\mathbf{A} = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A} \,,$$

$$\nabla\times\mathbf{B}(\mathbf{r}) = \frac{\mu_o}{4\pi}\nabla\int\mathbf{J}(\mathbf{r}')\cdot\nabla\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)d^3r' - \frac{\mu_o}{4\pi}\int\mathbf{J}(\mathbf{r}')\cdot\nabla^2\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)d^3r' \quad .$$

$$\nabla\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) = \nabla'\left(\frac{-1}{|\mathbf{r}-\mathbf{r}'|}\right)$$

$$\nabla^2\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) = -4\pi\delta(\mathbf{r}-\mathbf{r}')$$

$$\nabla\times\mathbf{B}(\mathbf{r}) = -\frac{\mu_o}{4\pi}\nabla\int\mathbf{J}(\mathbf{r}')\cdot\nabla'\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)d^3r' + \mu_o\mathbf{J}(\mathbf{r})$$

$$\nabla\times\mathbf{B}(\mathbf{r}) = \mu_o\mathbf{J}(\mathbf{r}) + \frac{\mu_o}{4\pi}\nabla\int\frac{\nabla'\cdot\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^3r' \quad .$$

$$\nabla\cdot\mathbf{J} = 0$$

$$\nabla\times\mathbf{B}(\mathbf{r}) = \mu_o\mathbf{J}(\mathbf{r}), \quad \int_S\nabla\times\mathbf{B}(\mathbf{r})\cdot\mathbf{n}\,da = \mu_o\int_S\mathbf{J}(\mathbf{r})\cdot\mathbf{n}\,da, \qquad \oint_C\mathbf{B}(\mathbf{r})\cdot d\mathbf{l} = \mu_o\int_S\mathbf{J}(\mathbf{r})\cdot\mathbf{n}\,da$$

$$\oint_C\mathbf{B}(\mathbf{r})\cdot d\mathbf{l} = \mu_o I$$

3. Vector Potential

The basic equations for magnetostatics are

$$\nabla\cdot\mathbf{B}(\mathbf{r}) = 0$$

$$\nabla\times\mathbf{B}(\mathbf{r}) = \mu_o\mathbf{J}(\mathbf{r})$$

A general method of solving them is to exploit the equation $\nabla\cdot\mathbf{B}(\mathbf{r}) = 0$. If $\nabla\cdot\mathbf{B}(\mathbf{r}) = 0$ everywhere, **B** must be the curl of some vector field **A(r)**, called the vector potential

$$\mathbf{B}(\mathbf{r}) = \nabla\times\mathbf{A}(\mathbf{r})$$

From $\mathbf{B}(\mathbf{r}) = \frac{\mu_o}{4\pi}\nabla\times\int\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^3r'$, the general form of **A** is

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_o}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} d^3r' + \nabla\psi(\mathbf{r}) \quad .$$

The added gradient of an arbitrary scalar function ψ shows that for a given magnetic induction **B**, the vector potential can be freely transformed according to

$$\mathbf{A} \rightarrow \mathbf{A} + \nabla\psi \quad .$$

This transformation is called a gauge transformation. The freedom of gauge transformations makes $\nabla\cdot\mathbf{A}$ have any convenient functional form.

Substituting $\mathbf{B}(\mathbf{r}) = \nabla\times\mathbf{A}(\mathbf{r})$ into $\nabla\times\mathbf{B}(\mathbf{r}) = \mu_o\mathbf{J}(\mathbf{r})$, we find

$$\nabla\times\nabla\times\mathbf{A}(\mathbf{r}) = \mu_o\mathbf{J}(\mathbf{r})$$

$$\nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A} = \mu_o\mathbf{J}(\mathbf{r})$$

With the Coulomb gauge $\nabla\cdot\mathbf{A} = 0$, each rectangular component of the vector potential satisfies the Poisson equation

$$\nabla^2\mathbf{A} = -\mu_o\mathbf{J}(\mathbf{r})$$

It is clear that the solution for A in unbounded space is $\mathbf{A}(\mathbf{r}) = \frac{\mu_o}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} d^3r' + \nabla\psi(\mathbf{r})$

with ψ=constant, i.e.,

$$\mathbf{A}(\vec{r}) = \frac{\mu_o}{4\pi}\int_V \frac{\mathbf{J}(\vec{r}')}{|\vec{r}-\vec{r}'|} d^3r'$$

The condition ψ=constant can be understood as follows. The Coulomb gauge $\nabla\cdot\mathbf{A} = 0$ indicates that $\nabla^2\psi(\mathbf{r}) = 0$. If $\nabla^2\psi(\mathbf{r}) = 0$ holds in all space, ψ must be at most a constant provided there are no sources at infinity.

Example 2: A circular loop of radius *a* carrying current *I* lies in the *x-y* plane with its center at the origin. Find the vector potential and the magnetic induction field at a point on the *z*-axis.

Solution: To begin with, considering a current loop with a radius *a* in the plane perpendicular to *z*-axis with an expansion angle of α, the current density can be expressed as

$$\mathbf{J} = \frac{1}{r'^2} Ia\delta(\cos\theta' - \cos\alpha)\delta(r' - a/\sin\alpha)\mathbf{a}_{\phi'} \quad .$$

Note that the unit vector $\mathbf{a}_{\phi'}$ can be in terms of the coordinates of the observation point:

$$\mathbf{a}_{\phi'} = \cos(\phi-\phi')\,\mathbf{a}_\phi + \sin(\phi-\phi')\left(\sin\theta\,\mathbf{a}_r + \cos\theta\,\mathbf{a}_\theta\right)$$

Using $\cos\alpha = 0$ for the current *I* lying in the *x-y* plane, the current density can be written as

$$\mathbf{J} = \frac{1}{a} I\delta(\cos\theta')\delta(r'-a)\mathbf{a}_{\phi'}$$
$$= \frac{1}{a} I\delta(\cos\theta')\delta(r'-a)\left[\cos(\phi-\phi')\,\mathbf{a}_\phi + \sin(\phi-\phi')\left(\sin\theta\,\mathbf{a}_r + \cos\theta\,\mathbf{a}_\theta\right)\right].$$

Substituting the current density into the formula of the vector potential, we have

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_o}{4\pi}\int_V \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} d^3r'$$
$$= \frac{\mu_o Ia}{4\pi}\int_0^{2\pi} \frac{\cos(\phi-\phi')\,\mathbf{a}_\phi + \sin(\phi-\phi')\left(\sin\theta\,\mathbf{a}_r + \cos\theta\,\mathbf{a}_\theta\right)}{\sqrt{r^2+a^2-2ra\sin\theta\cos(\phi-\phi')}} d\phi' .$$

The denominator term in the integration can be expanded as the form of the power series of the term $\cos(\phi-\phi')$. Therefore, it can be confirmed that only the component along $\mathbf{a}_\phi$ survives because the orthogonality between $\cos(\phi-\phi')$ and $\sin(\phi-\phi')$. Since the azimuthal integration is symmetry, the vector potential for the $\mathbf{a}_\phi$ component can be conveniently evaluated with $\phi = 0$:

$$A_\phi(\mathbf{r}) = \frac{\mu_o Ia}{4\pi}\int_0^{2\pi} \frac{\cos\phi'}{\sqrt{r^2+a^2-2ra\sin\theta\cos\phi'}} d\phi' .$$

Restricting the discussion to large distances from the current loop, $r >> a$, the integration can be approximated as

$$A_\phi(\mathbf{r}) = \frac{\mu_o Ia}{4\pi\, r}\int_0^{2\pi} \cos\phi'\left[1 - \frac{1}{2}\left(\frac{a^2}{r^2} - 2\frac{a}{r}\sin\theta\cos\phi'\right)\right] d\phi' = \frac{\mu_o Ia^2}{4\, r^2}\sin\theta .$$

From $\mathbf{B}(\mathbf{r}) = \nabla\times\mathbf{A}(\mathbf{r})$ and

$$\nabla\times\mathbf{A}(\mathbf{r}) = \frac{1}{r^2\sin\theta}\begin{vmatrix} \mathbf{a}_r & r\,\mathbf{a}_\theta & r\sin\theta\,\mathbf{a}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ A_r & rA_\theta & r\sin\theta\, A_\phi \end{vmatrix} .$$

Thus,

$$B_r = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\, A_\phi\right)$$

$$B_\theta = -\frac{1}{r}\frac{\partial}{\partial r}\left(rA_\phi\right) .$$

Far from the loop, the fields are given by

$$B_r = \frac{\mu_o I\pi a^2}{2\pi\, r^3}\cos\theta$$

$$B_\theta = \frac{\mu_o I\pi a^2}{4\pi\, r^3}\sin\theta \quad .$$

Comparison with the electrostatic dipole fields shows that the magnetic fields far away from a current loop are dipole in character. By analogy with electrostatics, the magnetic dipole moment can be defined to be $m = I\pi a^2$. Next a spherical harmonic expansion is used to point out similarities and differences between the magnetostatic and electrostatic problems. Before discussing this problem, we consider the following example for convenience.

Example 3: A localized cylindrically symmetric current distribution is such that the current flows only in the azimuthal direction; the current density is a function only of r and θ (or ρ and z): $\mathbf{J} = J(r,\theta)\mathbf{a}_{\phi'}$. The distribution is "hollow" in the sense that there is a current-free region near the origin, as well as outside. Find the azimuthal component of the vector potential.

Solution: The expansion for $1/|\mathbf{r}-\mathbf{r}'|$ is given by

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}^*(\theta',\phi')\,Y_{lm}(\theta,\phi) \quad .$$

This can be written entirely in terms of real functions as

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}\left\{P_l(\cos\theta')P_l(\cos\theta) + 2\sum_{m=1}^{l}\frac{(l-m)!}{(l+m)!}\cos[m(\phi-\phi')]\,P_l^m(\cos\theta')\,P_l^m(\cos\theta)\right\}$$

the current density can be expressed as

$$\mathbf{J} = J(r',\theta')\mathbf{a}_{\phi'} = J(r',\theta')\big[\cos(\phi-\phi')\,\mathbf{a}_\phi + \sin(\phi-\phi')\big(\sin\theta\,\mathbf{a}_r + \cos\theta\,\mathbf{a}_\theta\big)\big] \ ,$$

where the unit vector $\mathbf{a}_{\phi'}$ is in terms of the coordinates of the observation point. Using the orthogonal property for the terms $\cos[m(\phi-\phi')]$, only the terms with $m=1$ can contribute to the integration. So, the vector potential can be expressed as

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_o}{4\pi}\int_V \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^3r'$$
$$= \mathbf{a}_\phi\frac{\mu_o}{4\pi}\int J(r',\theta')\sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}\frac{1}{l(l+1)}\,P_l^1(\cos\theta')\,P_l^1(\cos\theta)\,d^3r' \quad .$$

To evaluate the radial component of magnetic induction from

$$B_r = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\, A_\phi\right) = -\frac{1}{r}\frac{d}{dx}\left[\sqrt{1-x^2}\,A_\phi(\mathbf{r})\right] .$$

Form $\left[(1-x^2)P_l'(x)\right]' + l(l+1)P_l(x) = 0$, we have

$$-\frac{d}{dx}[\sqrt{(1-x^2)}P_l^1(x)] = \frac{d}{dx}\left[(1-x^2)\frac{dP_l(x)}{dx}\right] = -l(l+1)P_l(x) \quad .$$

Therefore,

$$B_r(\mathbf{r}) = -\frac{\mu_o}{4\pi r}\int J(r',\theta')\sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}\,P_l^1(\cos\theta')\,P_l(\cos\theta)\,d^3r' .$$

Similarly, with

$$B_\theta = -\frac{1}{r^2}\frac{\partial}{\partial r}\left(rA_\phi\right) \quad ,$$

we can find

$$B_\theta(\mathbf{r}) = -\frac{\mu_o}{4\pi r}\int J(r',\theta')\sum_{l=0}^{\infty}\frac{\partial}{\partial r}\left(r\frac{r_<^l}{r_>^{l+1}}\right)\frac{1}{l(l+1)}\,P_l^1(\cos\theta')\,P_l^1(\cos\theta)\,d^3r'$$

Remember that

$$Y_{l\,-m}(\theta,\phi) = (-1)^m\,Y_{l\,m}^*(\theta,\phi)$$

$$Y_{lm}(\theta,\phi) = (-1)^m\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}e^{im\phi}P_l^m(\cos\theta)$$

$$P_l^m(x) = (1-x^2)^{m/2}\frac{d^mP_l(x)}{dx^m}$$

$$P_l(x) = \sum_{k=0}^{l}\frac{(-1)^k}{2^l}\frac{1}{k!(l-k)!}\frac{d^l}{dx^l}x^{2l-2k} = \frac{1}{2^l\,l!}\frac{d^l}{dx^l}(x^2-1)^l$$

$$P_l(x) = \sum_{k=0}^{[l/2]}\frac{(-1)^k}{2^l}\frac{(2l-2k)!}{k!(l-k)!(l-2k)!}x^{l-2k}$$

$$(1-2xt+t^2)^{-1/2} = \sum_{l=0}^{\infty}\frac{(2l)!}{2^{2l}(l!)^2}(2xt-t^2)^l = \sum_{l=0}^{\infty}\frac{(2l)!}{2^{2l}(l!)^2}t^l\sum_{k=0}^{l}\frac{(-1)^k\,l!}{k!(l-k)!}(2x)^{l-k}t^k$$

$$= \sum_{l=0}^{\infty}\sum_{k=0}^{l}\frac{(-1)^k}{2^{2l}}\frac{(2l)!}{l!k!(l-k)!}(2x)^{l-k}t^{l+k}$$

$$(1-x^2)P_l'(x) = x(l+1)P_l(x) - (l+1)P_{l+1}(x)$$

$$\left[(1-x^2)P_l'(x)\right]' + l(l+1)P_l(x) = 0$$

$$P_{2n}(0) = (-1)^n \frac{(2n)!}{2^{2n}(n!)^2} = (-1)^n \frac{(2n-1)!!}{(2n)!!}$$

Example 4: Using the form of the differential equation, find the vector potential and the magnetic induction field in spherical coordinates for a circular loop of radius a carrying current I lies in the planes of $z = d\cdot\cos\alpha$ and $d = a/\sin\alpha$ with its center at the z-axis.

Solution: In the Coulomb gauge, $\nabla^2 \mathbf{A} = -\mu_o \mathbf{J}(\mathbf{r})$. Considering that the current flows only in the azimuthal direction: $\mathbf{J} = J(r,\theta)\mathbf{a}_\phi$. In spherical coordinates r, θ, and ϕ, the Poisson equation for $A_\phi(r,\theta)\mathbf{a}_\phi$ can be written in the form

$$\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)\right]A_\phi(r,\theta)\,\mathbf{a}_\phi = -\mu_o J(r,\theta)\,\mathbf{a}_\phi$$

Note that

$$\frac{\partial}{\partial\phi^2}\mathbf{a}_\phi = -\mathbf{a}_\phi .$$

The differential equation can be rewritten as

$$\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} - \frac{1}{\sin^2\theta}\right)\right]A_\phi(r,\theta) = -\mu_o J(r,\theta) .$$

First considering the homogeneous equation

$$\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} - \frac{1}{\sin^2\theta}\right)\right]A_\phi(r,\theta) = 0$$

The solution $A_\phi(r,\theta)$ can be expressed as the product of a radial part and an angular part,

$$A_\phi(r,\theta) = \frac{U(r)}{r}P(\theta) .$$

Substituting the form into the equation yields

$$P\frac{d^2U}{dr^2} + \frac{U}{r^2}\frac{1}{\sin\theta}\frac{d}{d\theta}\sin\theta\frac{dP}{d\theta} - \frac{UP}{r^2\sin^2\theta} = 0 .$$

Multiplying Eq. () by r^2/UP can result in

$$\frac{r^2}{U}\frac{d^2U}{dr^2} + \frac{1}{P}\left[\frac{1}{\sin\theta}\frac{d}{d\theta}\sin\theta\frac{dP}{d\theta} - \frac{1}{\sin^2\theta}\right] = 0 .$$

If Eq. () is to hold for arbitrary values of the independent coordinates, each of the terms must be separately constant:

$$\frac{r^2}{U}\frac{d^2U}{dr^2} = l(l+1)$$

$$\frac{1}{P}\frac{1}{\sin\theta}\frac{d}{d\theta}\sin\theta\frac{dP}{d\theta} - \frac{1}{\sin^2\theta} = -l(l+1).$$

The solution of the radial differential equation is

$$R(r) = \frac{U}{r} = Ar^l + Br^{-l-1} \qquad .$$

In terms of $x = \cos\theta$, the θ equation for $P(\theta)$ is usually expressed as

$$\frac{d}{dx}\left[(1-x^2)\frac{dP_l^m(x)}{dx}\right] + \left[l(l+1) - \frac{m^2}{1-x^2}\right]P_l^m(x) = 0$$

with m = 1. It has been shown that

$$\int_{-1}^{1} P_l^m(x)\, P_{l'}^m(x)dx = \frac{2}{(2l+1)}\frac{(l+m)!}{(l-m)!}\delta_{l,l'}\,.$$

In terms of the $P_l^m(x)$, we can the function $\delta(\cos\theta - \cos\theta')$ expand as

$$\delta(\cos\theta - \cos\theta') = \sum_{l=0}^{\infty}\frac{(2l+1)}{2}\frac{(l-m)!}{(l+m)!}P_l^m(\cos\theta')\, P_l^m(\cos\theta) \quad .$$

For m = 1, we have

$$\delta(\cos\theta - \cos\theta') = \sum_{l=0}^{\infty}\frac{(2l+1)}{2}\frac{1}{l(l+1)}P_l^1(\cos\theta')\, P_l^1(\cos\theta)$$

The current density can be expressed as

$$J(r,\theta) = \int J(r',\theta')\frac{1}{2\pi r'^2}\delta(\cos\theta - \cos\theta')\delta(r-r')\, d^3r' \quad .$$

Based on the superposition principle, it is useful to consider the vector potential due to the current density $J(r',\theta')\frac{1}{2\pi r'^2}\delta(\cos\theta - \cos\theta')\delta(r-r')$:

$$\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} - \frac{1}{\sin^2\theta}\right)\right]A_\phi(r,\theta) = -\frac{\mu_o J(r',\theta')}{2\pi r'^2}\delta(\cos\theta - \cos\theta')\delta(r-r')\,.$$

Similar to the expansion for the $\delta(\cos\theta - \cos\theta')$, the vector potential for this equation can be expanded with the basis $P_l^1(\cos\theta)$ as

$$A_\phi(r,\theta) = \sum_{l=0}^{\infty} g_l(r,r')\frac{(2l+1)}{2}\frac{1}{l(l+1)}P_l^1(\cos\theta')\, P_l^1(\cos\theta) \quad .$$

Then we can obtain an 1D differential equation:

$$\left[\frac{d}{dr}\left(r^2\frac{d}{dr}\right) - l(l+1)\right]g_l(r,r') = -\frac{\mu_o J(r',\theta')}{2\pi}\delta(r-r')\,.$$

The function $g_l(r,r')$ can be solved to be

$$g_l(r,r') = \frac{1}{2l+1}\frac{\mu_o J(r',\theta')}{2\pi}\frac{r_<^l}{r_>^{l+1}} \quad ,$$

where $r_<(r_>)$ is the smaller (larger) of r and r'. Therefore, the vector potential due to the current density $J(r',\theta')\frac{1}{2\pi r'^2}\delta(\cos\theta - \cos\theta')\delta(r-r')$ is given by

$$A_\phi(r,\theta) = \frac{\mu_o}{4\pi}\sum_{l=0}^{\infty} J(r',\theta')\frac{r_<^l}{r_>^{l+1}}\frac{1}{l(l+1)} P_l^1(\cos\theta')\, P_l^1(\cos\theta) \quad .$$

In general, the vector potential due to the current density $J(r,\theta)$ is then given by

$$A_\phi(r,\theta) = \frac{\mu_o}{4\pi}\int J(r',\theta')\sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}\frac{1}{l(l+1)} P_l^1(\cos\theta')\, P_l^1(\cos\theta)\, d^3r' \,.$$

Example 5: A circular loop of radius a carrying current I lies in the planes of $z = d\cdot\cos\alpha$ and $d = a/\sin\alpha$ with its center at the z-axis. Find the vector potential and the magnetic induction field in spherical coordinates.

Solution: Using the result of Example 3,

$$A_\phi(\mathbf{r}) = \frac{\mu_o}{4\pi}\int J(r',\theta')\sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}\frac{1}{l(l+1)} P_l^1(\cos\theta')\, P_l^1(\cos\theta)\, d^3r' \quad .$$

$$B_r(\mathbf{r}) = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\, A_\phi\right) = -\frac{\mu_o}{4\pi\, r}\int J(r',\theta')\sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}} P_l^1(\cos\theta')\, P_l\ (\cos\theta)\, d^3r'$$

$$B_\theta(\mathbf{r}) = -\frac{\mu_o}{4\pi}\int J(r',\theta')\sum_{l=0}^{\infty}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{r_<^l}{r_>^{l+1}}\right)\frac{1}{l(l+1)} P_l^1(\cos\theta')\, P_l^1(\cos\theta)\, d^3r' \quad .$$

The current density in spherical coordinates is given by

$$J(r',\theta') = \frac{I\sin\alpha}{r'}\delta(\cos\theta' - \cos\alpha)\delta(r'-d)$$

Substituting this current density into the equations, the vector potential and magnetic induction fields are given by

$$\begin{aligned}\mathbf{A}(\mathbf{r}) &= \mathbf{a}_\phi\frac{\mu_o I}{4\pi}\int\frac{\sin\alpha}{r'}\delta(\cos\theta'-\cos\alpha)\delta(r'-d)\sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}\frac{1}{l(l+1)} P_l^1(\cos\theta')\, P_l^1(\cos\theta)\, d^3r' \\ &= \mathbf{a}_\phi\frac{\mu_o I d}{2}\sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}\frac{\sin\alpha}{l(l+1)} P_l^1(\cos\alpha) P_l^1(\cos\theta)\end{aligned} \quad ,$$

$$B_r(\mathbf{r}) = -\frac{\mu_o I d}{2r}\sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}\sin\alpha P_l^1(\cos\alpha) P_l\ (\cos\theta) \quad ,$$

$$B_\theta(\mathbf{r}) = -\frac{\mu_o Id}{2}\sum_{l=0}^{\infty}\frac{1}{r}\sin\alpha\frac{\partial}{\partial r}\left(r\frac{r_<^l}{r_>^{l+1}}\right)\frac{1}{l(l+1)}P_l^1(\cos\alpha)\,P_l^1(\cos\theta) \quad .$$

where $r_<(r_>)$ is the smaller (larger) of r and d. For the special case $\alpha = \pi/2$, $d = a$, the fields are then given by

$$A_\phi(\mathbf{r}) = \frac{\mu_o Ia}{2}\sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}\frac{1}{l(l+1)}P_l^1(0)P_l^1(\cos\theta)\,,$$

$$B_r(\mathbf{r}) = -\frac{\mu_o Ia}{2r}\sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}P_l^1(0)P_l\,(\cos\theta) \quad ,$$

$$B_\theta(\mathbf{r}) = -\frac{\mu_o Ia}{2}\sum_{l=0}^{\infty}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{r_<^l}{r_>^{l+1}}\right)\frac{1}{l(l+1)}P_l^1(0)\,P_l^1(\cos\theta) \quad .$$

Using $P_l^m(x) = (-1)^m(1-x^2)^{m/2}\frac{d^m P_l\,(x)}{dx^m}$ and $(1-x^2)P_l'(x) = x(l+1)P_l(x) - (l+1)P_{l+1}(x)$, it can be found that

$$P_l^1(0) = -P_l'(0) = (l+1)P_{l+1}(0) \quad .$$

Substituting $l = 2n+1$ and using $P_{2n+2}(0) = (-1)^{n+1}\frac{(2n+1)!!}{2^{n+1}(n+1)!}$, we obtain

$$P_{2n+1}^1(0) = (-1)^{n+1}\frac{(2n+1)!!}{2^n n!} \quad .$$

Therefore, the vector field can be expressed as

$$\begin{aligned} A_\phi(\mathbf{r}) &= \frac{\mu_o Ia}{2}\sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}\frac{1}{l(l+1)}P_l^1(0)P_l^1(\cos\theta) \\ &= -\frac{\mu_o Ia}{4}\sum_{n=0}^{\infty}\frac{r_<^{2n+1}}{r_>^{2n+2}}(-1)^n\frac{(2n-1)!!}{2^n(n+1)!}P_{2n+1}^1(\cos\theta) \end{aligned} .$$

The radial component of magnetic induction is then given by

$$\begin{aligned} B_r(\mathbf{r}) &= -\frac{\mu_o Ia}{2\,r}\sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}P_l^1(0)P_l\,(\cos\theta) \\ &= \frac{\mu_o Ia}{2r}\sum_{n=0}^{\infty}\frac{r_<^{2n+1}}{r_>^{2n+2}}(-1)^n\frac{(2n+1)!!}{2^n n!}P_{2n+1}(\cos\theta) \end{aligned} .$$

The θ component of B is similarly

$$B_\theta(\mathbf{r}) = -\frac{\mu_o Ia}{4}\sum_{n=0}^{\infty}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{r_<^{2n+1}}{r_>^{2n+2}}\right)(-1)^{n+1}\frac{(2n-1)!!}{2^n(n+1)!}P_{2n+1}^1(\cos\theta)$$

$$= -\frac{\mu_o Ia^2}{4}\sum_{n=0}^{\infty}\begin{Bmatrix} -\left(\frac{2n+2}{2n+1}\right)\frac{1}{a^3}\left(\frac{r}{a}\right)^{2n} \\ \frac{1}{r^3}\left(\frac{a}{r}\right)^{2n}\end{Bmatrix}(-1)^n\frac{(2n+1)!!}{2^n(n+1)!}P_{2n+1}^1(\cos\theta) .$$

The upper line holds for $r < a$, and the lower line for $r > a$. For $r >> a$, only the $n = 0$ term in the series is important. Since $P_1^1(\cos\theta) = \sin\theta$, the expressions can be reduced to

$B_r = \frac{\mu_o I\pi a^2}{2\pi\, r^3}\cos\theta$ and $B_\theta = \frac{\mu_o I\pi a^2}{4\pi\, r^3}\sin\theta$.

Another mode of attack on the problem of the planar loop is to employ an expansion in cylindrical waves. The expansion for $1/|\mathbf{r}-\mathbf{r}'|$ in cylindrical coordinates is given by

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{2}{\pi}\sum_{m=-\infty}^{\infty}\int_0^{\infty} I_m(k\rho_<)K_m(k\rho_>)\, e^{im(\phi-\phi')}\cos[k(z-z')]dk$$
$$= \frac{4}{\pi}\int_0^{\infty}\left\{\frac{1}{2}I_0(k\rho_<)K_0(k\rho_>) + \sum_{m=1}^{\infty}\cos[m(\phi-\phi')]I_m(k\rho_<)K_m(k\rho_>)\right\}\cos[k(z-z')]dk ,$$

where $\rho_<(\rho_>)$ is the smaller (larger) of ρ and ρ'. Alternatively, it can be expressed as

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \sum_{m=-\infty}^{\infty}\int_0^{\infty} e^{-k(z_>-z_<)}\, e^{im(\phi-\phi')}J_m(k\rho)J_m(k\rho')\, dk$$
$$= 2\int_0^{\infty}\left\{\frac{1}{2}e^{-k(z_>-z_<)}\, J_0(k\rho)J_0(k\rho') + \sum_{m=1}^{\infty}e^{-k(z_>-z_<)}\cos[m(\phi-\phi')]J_m(k\rho)J_m(k\rho')\right\}dk ,$$

where $z_<(z_>)$ is the smaller (larger) of z and z'. For a current loop lying in the x-y plane, the current density in cylindrical coordinates can be written as

$$\mathbf{J} = aI\delta(z')\delta(\rho'-a)\mathbf{a}_{\phi'} = aI\delta(z')\delta(\rho'-a)\left[\cos(\phi-\phi')\,\mathbf{a}_\phi + \sin(\phi-\phi')\mathbf{a}_\rho\right] .$$

Therefore, the vector potential can be expressed as

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_o}{4\pi}\int_V \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^3r'$$
$$= \mathbf{a}_\phi\frac{\mu_o Ia}{\pi}\int_0^{\infty} I_1(k\rho_<)K_1(k\rho_>)\cos(kz)dk ,$$

where $\rho_<(\rho_>)$ is the smaller (larger) of ρ and a. Alternatively, it can be expressed as

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_o}{4\pi}\int_V \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} d^3r' \\ = \mathbf{a}_\phi \frac{\mu_o Ia}{2}\int_0^\infty e^{-k|z|} J_1(k\rho) J_1(ka) dk$$

From $\mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A}(\mathbf{r})$ and

$$\nabla \times \mathbf{A}(\mathbf{r}) = \frac{1}{\rho}\begin{vmatrix} \mathbf{a}_\rho & \mathbf{a}_\phi & \mathbf{a}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_\rho & \rho A_\phi & A_z \end{vmatrix} .$$

Thus,

$$B_z = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho A_\phi\right) ,$$

$$B_\rho = -\frac{\partial}{\partial z}\left(A_\phi\right) .$$

Remember that

$$J_{m-1}(\rho) + J_{m+1}(\rho) = \frac{2m}{\rho} J_m(\rho)$$

$$J_{m-1}(\rho) - J_{m+1}(\rho) = 2J'_m(\rho)$$

With the derived results to evaluate the fields in the interior region of the loop, $\rho_< = \rho$ and $\rho_> = a$, we have

$$B_z(\mathbf{r}) = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho A_\phi\right) = \frac{\mu_o Ia}{\pi}\int_0^\infty \left[\frac{I_1(k\rho)}{\rho} + kI'_1(k\rho)\right] K_1(ka)\cos(kz) dk ,$$

$$B_\rho(\mathbf{r}) = -\frac{\partial}{\partial z}\left(A_\phi\right) = \frac{\mu_o Ia}{\pi}\int_0^\infty kI_1(k\rho_<) K_1(k\rho_>)\sin(kz) dk .$$

For $\rho \to 0$, we can find $B_\rho(0,0,z) = 0$ and

$$B_z(0,0,z) = \frac{\mu_o Ia}{\pi}\int_0^\infty k K_1(ka)\cos(kz) dk = \frac{\mu_o Ia}{\pi}\frac{\partial}{\partial z}\int_0^\infty K_1(ka)\sin(kz) dk .$$

Using $-K'_0(\rho) = K_1(\rho)$ and $\int_0^\infty K_0(ka)\cos(kz) dk = \pi/(2\sqrt{a^2+z^2})$, we have

$$B_z(0,0,z) = \frac{\mu_o Ia}{\pi}\frac{\partial}{\partial z}\int_0^\infty \frac{z}{a} K_0(ka)\cos(kz) dk = \frac{\mu_o I}{2}\frac{\partial}{\partial z}\frac{z}{\sqrt{a^2+z^2}} = \frac{\mu_o I}{2}\frac{a^2}{\left(a^2+z^2\right)^{3/2}} .$$

On the other hand, we can use another form to obtain

$$B_z(\mathbf{r}) = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\, A_\phi\right) = \frac{\mu_o I a}{2}\int_0^\infty e^{-k|z|}\left[\frac{J_1(k\rho)}{\rho} + kJ_1'(k\rho)\right]J_1(ka)dk \quad ,$$

$$B_\rho(\mathbf{r}) = -\frac{\partial}{\partial z}\left(A_\phi\right) = \frac{\mu_o I a}{2}\int_0^\infty k\frac{|z|}{z}e^{-k|z|}\; J_1(k\rho)J_1(ka)dk \quad .$$

For $\rho \to 0$, we can find $B_\rho(0,0,z) = 0$

$$B_z(0,0,z) = \frac{\mu_o I a}{2}\int_0^\infty ke^{-k|z|}\; J_1(ka)dk = \frac{\mu_o I a}{2}\frac{\partial}{\partial z}\int_0^\infty e^{-k|z|}\; J_1(ka)dk \quad .$$

Using $-J_0'(\rho) = J_1(\rho)$ and $\int_0^\infty e^{-k|z|}\; J_0(ka)dk = 1/\sqrt{a^2+z^2}$, we have

$$B_z(0,0,z) = \frac{\mu_o I a}{2}\frac{\partial}{\partial z}\int_0^\infty e^{-k|z|}\; J_1(ka)dk = \frac{\mu_o I a}{2}\frac{\partial}{\partial z}\frac{z}{\sqrt{a^2+z^2}} = \frac{\mu_o I}{2}\frac{a^2}{\left(a^2+z^2\right)^{3/2}}$$

Example 6: Helmholtz coils: two coaxial, parallel, circular loops of radius *a* carrying current *I* lie in the planes of *z* = *b*/2 and *z* = –*b*/2 with their centers at the origin. Find the magnetic induction field at a point on the *z*-axis.

Solution: Using the result of Example 3,

$$A_\phi(\mathbf{r}) = \frac{\mu_o}{4\pi}\int J(r',\theta')\sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}\frac{1}{l(l+1)}\,P_l^1(\cos\theta')\,P_l^1(\cos\theta)\,d^3r' \quad .$$

$$B_r(\mathbf{r}) = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\, A_\phi\right) = \frac{\mu_o}{4\pi\, r}\int J(r',\theta')\sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}\,P_l^1(\cos\theta')\,P_l\,(\cos\theta)\,d^3r'$$

$$B_\theta(\mathbf{r}) = -\frac{\mu_o}{4\pi}\int J(r',\theta')\sum_{l=0}^{\infty}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{r_<^l}{r_>^{l+1}}\right)\frac{1}{l(l+1)}\,P_l^1(\cos\theta')\,P_l^1(\cos\theta)\,d^3r' \quad .$$

The current density for Helmholtz coils can be expressed as

$$J(r',\theta') = \frac{I\sin\alpha}{r'}\delta(\cos\theta' - \cos\alpha)\delta(r'-d) + \frac{I\sin\alpha}{r'}\delta(\cos\theta' - \cos(\pi-\alpha))\delta(r'-d)\,.$$

Here $\sin\alpha = a/d$ and $d^2 = \sqrt{(b/2)^2 + a^2}$. Therefore,

$$A_\phi(\mathbf{r}) = \frac{\mu_o}{4\pi}\int J(r',\theta')\sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}\frac{1}{l(l+1)}P_l^1(\cos\theta')\,P_l^1(\cos\theta)\,d^3r'$$
$$= \frac{\mu_o Id}{2}\sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}\frac{\sin\alpha}{l(l+1)}\left[P_l^1(\cos\alpha)+P_l^1(-\cos\alpha)\,\right]P_l^1(\cos\theta)$$
$$= \frac{\mu_o Id}{2}2\sum_{n=0}^{\infty}\frac{r_<^{2n+1}}{r_>^{2n+2}}\frac{\sin\alpha}{(2n+1)(2n+2)}P_{2n+1}^1(\cos\alpha)\,P_{2n+1}^1(\cos\theta)$$

$$B_r(\mathbf{r}) = \frac{\mu_o}{4\pi\, r}\int J(r',\theta')\sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}P_l^1(\cos\theta')\,P_l\,(\cos\theta)\,d^3r'$$
$$= \frac{\mu_o Id}{2\,r}\sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}\sin\alpha\left[P_l^1(\cos\alpha)+P_l^1(-\cos\alpha)\,\right]P_l\,(\cos\theta)\quad .$$
$$= \frac{\mu_o Id}{r}\sum_{n=0}^{\infty}\frac{r_<^{2n+1}}{r_>^{2n+2}}\sin\alpha\,P_{2n+1}^1(\cos\alpha)\,P_{2n+1}(\cos\theta)$$

For the observation on the z-axis and near the origin, $\theta = 0$, $r_< = z$, and $r_> = d$. Therefore, the magnetic induction can be expressed as

$$B_r(0,0,z) = \frac{\mu_o Id}{z}\sum_{n=0}^{\infty}\frac{z^{2n+1}}{d^{2n+2}}\sin\alpha\,P_{2n+1}^1(\cos\alpha)$$
$$= \frac{\mu_o I}{d}\sin^2\alpha\left[1\frac{dP_1\,(\cos\alpha)}{d\cos\alpha}+\left(\frac{z}{d}\right)^2\frac{dP_3\,(\cos\alpha)}{d\cos\alpha}+\left(\frac{z}{d}\right)^4\frac{dP_5\,(\cos\alpha)}{d\cos\alpha}+\cdots\right]$$
$$= \frac{\mu_o I}{d}\sin^2\alpha\left[1+\left(\frac{z}{d}\right)^2\frac{d}{d\cos\alpha}\left(\frac{5}{2}\cos^3\alpha-\frac{3}{2}\cos\alpha\right)\right.$$
$$\left.+\left(\frac{z}{d}\right)^4\frac{d}{d\cos\alpha}\frac{1}{8}\left(63\cos^5\alpha-70\cos^3\alpha+15\cos\alpha\right)+\cdots\right]$$
$$= \frac{\mu_o I}{d}\sin^2\alpha\times\left[1+\left(\frac{z}{d}\right)^2\frac{3}{2}\left(5\cos^1\alpha-1\right)+\left(\frac{z}{d}\right)^4\frac{15}{8}\left(21\cos^4\alpha-14\cos^2\alpha+1\right)+\cdots\right]$$
$$= \frac{\mu_o Ia^2}{d^3}\left[1+\left(\frac{z}{d}\right)^2\frac{3}{2}\frac{(b^2-a^2)}{d^2}+\left(\frac{z}{d}\right)^4\frac{15}{8}\frac{\left(2a^4-6a^2b^2+b^4\right)}{2d^4}+\cdots\right]$$

Therefore, $B_z(0,0,z)$ is nearly a constant when $b = a$. In other words, the pair of coils yields an approximately uniform field along the axis if it is positioned at a distance equal to the radius of the coils. Coils of this kind are called Helmholtz coils.

Example 7: A sphere of radius R carries a uniform surface-charge distribution σ. The sphere is rotated about a diameter with constant angular velocity ω. Find the vector potential and magnetic-flux density both inside and outside the sphere. This serves as a simple model for the magnetic field of the Earth.

Solution: The current density is given by

$$\mathbf{J} = \rho\,\mathbf{v}\,,$$

where $\rho = \sigma\,\delta(r' - R)$ and $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}'$.

$$\mathbf{J} = \sigma\,\omega\, r'\delta(r' - R)\left(\cos\theta'\,\mathbf{a}_{r'} - \sin\theta'\,\mathbf{a}_{\theta'}\right)\times \mathbf{a}_{r'}$$
$$= \sigma\,\omega\, r'\delta(r' - R)\sin\theta'\,\mathbf{a}_{\phi'}$$

In terms of the spherical coordinates of the observation point,

$$\mathbf{a}_{\phi'} = \cos(\phi - \phi')\,\mathbf{a}_{\phi} + \sin(\phi - \phi')\cos(\theta - \theta')\mathbf{a}_{r} + \sin(\phi - \phi')\sin(\theta - \theta')\mathbf{a}_{\theta}\,.$$

The expansion for $1/|\mathbf{r} - \mathbf{r}'|$ is given by

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}}\,Y_{lm}^{*}(\theta',\phi')\,Y_{lm}(\theta,\phi)$$

Using the orthogonal property for the terms $\cos[m(\phi - \phi')]$, the vector potential can be expressed as

$$A_{\phi}(\mathbf{r}) = \frac{\mu_o}{4\pi}\int_V \frac{\sigma\,\omega\, r'\delta(r' - R)\sin\theta'\cos(\phi - \phi')}{\left|\mathbf{r} - \mathbf{r}'\right|}\,d^3r'$$
$$= \frac{\mu_o\sigma\,\omega R^3}{3}\frac{r_<}{r_>^2}\sin\theta$$

Example 8: Two coaxial, parallel, circular loops of radius *a* carrying current *I* lie in the planes of $z = b/2$ and $z = -b/2$ with their centers at the origin. Find the magnetic induction field at a point on the *z*-axis.

Solution: In cylindrical coordinates, $Id\mathbf{l}_1 = Id\mathbf{l}_2 = (Ia)d\phi'\,\mathbf{a}_{\phi'}$. In terms of the coordinates of the observation point,

$$\mathbf{a}_{\phi'} = \cos(\phi - \phi')\,\mathbf{a}_{\phi} + \sin(\phi - \phi')\mathbf{a}_{\rho}\quad .$$

The position vectors for the two coils are $\mathbf{r}_1' = (b/2)\,\mathbf{a}_\mathbf{z} + a\,\mathbf{a}_{\rho'}$ and $\mathbf{r}_2' = (-b/2)\,\mathbf{a}_\mathbf{z} + a\,\mathbf{a}_{\rho'}$.

Note that

$$\mathbf{a}_{\rho'} = \cos(\phi - \phi')\,\mathbf{a}_{\rho} - \sin(\phi - \phi')\mathbf{a}_{\phi}\quad .$$

Therefore, we have

$$\mathbf{r} - \mathbf{r}_1' = (z - b/2)\,\mathbf{a}_\mathbf{z} + [\rho - a\cos(\phi - \phi')]\,\mathbf{a}_{\rho} + a\sin(\phi - \phi')\mathbf{a}_{\phi}\quad ,$$

$$\mathbf{r}-\mathbf{r}_2' = (z+b/2)\,\mathbf{a_z} + [\rho - a\cos(\phi-\phi')]\,\mathbf{a}_\rho + a\sin(\phi-\phi')\mathbf{a}_\phi \quad .$$

The distances are given by

$$|\,\mathbf{r}-\mathbf{r}_1'| = \sqrt{(z-b/2)^2 + a^2 + \rho^2 - 2a\rho\cos(\phi-\phi')} \quad ,$$

$$|\,\mathbf{r}-\mathbf{r}_1'| = \sqrt{(z+b/2)^2 + a^2 + \rho^2 - 2a\rho\cos(\phi-\phi')} \quad .$$

The terms $d\mathbf{l}\times(\mathbf{r}-\mathbf{r}')$ are given by

$$\begin{aligned} d\mathbf{l}_1\times(\mathbf{r}-\mathbf{r}_1') &= (Ia)d\phi'\,[\cos(\phi-\phi')\,\mathbf{a}_\phi + \sin(\phi-\phi')\mathbf{a}_\rho] \\ &\quad \times\left[(z-b/2)\,\mathbf{a_z} + (\rho-a)\cos(\phi-\phi')\,\mathbf{a}_\rho + a\sin(\phi-\phi')\mathbf{a}_\phi\right] \quad , \\ &= (Ia)d\phi'\,[(a-\rho\cos^2(\phi-\phi'))\mathbf{a_z} + (z-b/2)\cos(\phi-\phi')\,\mathbf{a}_\rho - (z-b/2)\sin(\phi-\phi')\mathbf{a}_\phi] \end{aligned}$$

$$\begin{aligned} & d\mathbf{l}_2\times(\mathbf{r}-\mathbf{r}_2') \\ &= (Ia)d\phi'\,[(a-\rho\cos^2(\phi-\phi'))\mathbf{a_z} + (z+b/2)\cos(\phi-\phi')\,\mathbf{a}_\rho - (z+b/2)\sin(\phi-\phi')\mathbf{a}_\phi] \quad . \end{aligned}$$

For the observation on the z-axis, $\rho = 0$, we have

$$\mathbf{r}-\mathbf{r}_1' = (z-b/2)\,\mathbf{a_z} - a\cos(\phi-\phi')\,\mathbf{a}_\rho + a\sin(\phi-\phi')\mathbf{a}_\phi \quad ,$$

$$\mathbf{r}-\mathbf{r}_2' = (z+b/2)\,\mathbf{a_z} - a\cos(\phi-\phi')\,\mathbf{a}_\rho + a\sin(\phi-\phi')\mathbf{a}_\phi \quad .$$

Therefore, the term $d\mathbf{l}\times(\mathbf{r}-\mathbf{r}')$ is given by

$$\begin{aligned} d\mathbf{l}_1\times(\mathbf{r}-\mathbf{r}_1') &= (Ia)d\phi'\,[\cos(\phi-\phi')\,\mathbf{a}_\phi + \sin(\phi-\phi')\mathbf{a}_\rho] \\ &\quad \times\left[(z-b/2)\,\mathbf{a_z} - a\cos(\phi-\phi')\,\mathbf{a}_\rho + a\sin(\phi-\phi')\mathbf{a}_\phi\right] \quad , \\ &= (Ia)d\phi'\,[a\mathbf{a_z} + (z-b/2)\cos(\phi-\phi')\,\mathbf{a}_\rho - (z-b/2)\sin(\phi-\phi')\mathbf{a}_\phi] \end{aligned}$$

$$d\mathbf{l}_2\times(\mathbf{r}-\mathbf{r}_2') = (Ia)d\phi'\,[a\mathbf{a_z} + (z+b/2)\cos(\phi-\phi')\,\mathbf{a}_\rho - (z+b/2)\sin(\phi-\phi')\mathbf{a}_\phi] \quad .$$

The distances are given by $|\,\mathbf{r}-\mathbf{r}_1'| = \sqrt{(z-b/2)^2 + a^2}$ and $|\,\mathbf{r}-\mathbf{r}_2'| = \sqrt{(z+b/2)^2 + a^2}$. With all results, we have

$$\begin{aligned} \mathbf{B}(0,0,z) &= \frac{\mu_o}{4\pi} I\left[\oint \frac{d\mathbf{l}_1\times(\mathbf{r}-\mathbf{r}_1')}{|\,\mathbf{r}-\mathbf{r}_1'|^3} + \oint \frac{d\mathbf{l}_2\times(\mathbf{r}-\mathbf{r}_2')}{|\,\mathbf{r}-\mathbf{r}_2'|^3}\right] \\ &= \frac{\mu_o}{2} Ia^2\left[\frac{1}{[(z-b/2)^2+a^2]^{3/2}} + \frac{1}{[(z+b/2)^2+a^2]^{3/2}}\right]\mathbf{a_z} \end{aligned} \quad .$$

An expansion of B_z about $z = 0$ yields

$$B_z(0,0,z) = \frac{\mu_o I a^2}{d^3}\left[1 + \frac{3(b^2-a^2)z^2}{2d^4} + \cdots\right] \quad .$$

where $d^2 = \sqrt{(b/2)^2 + a^2}$. Therefore, $B_z(0,0,z)$ is nearly a constant when $b = a$. In other words, the pair of coils yields an approximately uniform field along the axis if it is positioned at a distance equal to the radius of the coils. Coils of this kind are called Helmholtz coils.

4. Magnetic fields of a localized current distribution

$$\mathbf{A}(\vec{r}) = \frac{\mu_o}{4\pi}\int_V \frac{\mathbf{J}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} d^3r'$$

$$\frac{1}{|\vec{r}-\vec{r}\,'|} = \frac{1}{|\vec{r}\,|} + \frac{\vec{r}\,'\cdot\vec{r}}{|\vec{r}\,|^3} + \cdots$$

$$A_i(\vec{r}) = \frac{\mu_o}{4\pi}\left[\frac{1}{|\vec{r}\,|}\int_V J_i(\vec{r}\,')d^3r' + \frac{\vec{r}}{|\vec{r}\,|^3}\cdot\int_V J_i(\vec{r}\,')\vec{r}\,'d^3r' + \cdots\right]$$

$$\nabla\cdot(fg\mathbf{J}) = g\nabla f\cdot\mathbf{J} + f\nabla g\cdot\mathbf{J} + fg\nabla\cdot\mathbf{J}$$

Using $g = x_i'$, $f = 1$, and $\nabla'\cdot\mathbf{J} = 0$

It can be shown that $\int_V J_i(\vec{r}\,')d^3r' = 0$

Using $g = x_i'$, $f = x_j'$, and $\nabla'\cdot\mathbf{J} = 0$

It can be shown that $\int_V x_j' J_i(\vec{r}\,')d^3r' = -\int_V x_i' J_j(\vec{r}\,')d^3r'$

$$\begin{aligned}\vec{r}\cdot\int_V J_i(\vec{r}\,')\vec{r}\,'d^3r' &= \sum_j x_j \int_V J_i(\vec{r}\,')\,x_j'\,d^3r' \\ &= \frac{1}{2}\sum_j x_j \int_V \left(J_i(\vec{r}\,')x_j' - J_j(\vec{r}\,')x_i'\right)d^3r' \\ &= -\frac{1}{2}\sum_j \varepsilon_{ijk}\,x_j \int_V \left[\vec{r}\,'\times\mathbf{J}(\vec{r}\,')\right]_k d^3r' \\ &= -\frac{1}{2}\left[\vec{r}\times\int_V \vec{r}\,'\times\mathbf{J}(\vec{r}\,')\,d^3r'\right]_i\end{aligned}$$

Define $\mathrm{M}(\vec{r}) = \frac{1}{2}\left[\vec{r}\times\mathbf{J}(\vec{r})\right]$ and $\mathbf{m} = \frac{1}{2}\int_V \vec{r}\,'\times\mathbf{J}(\vec{r}\,')\,d^3r'$

$$\mathbf{A}(\vec{r}) = \frac{\mu_o}{4\pi}\frac{\mathbf{m}\times\vec{r}}{|\vec{r}\,|^3}$$

$$\mathbf{B}(\mathbf{r}) = \nabla\times\mathbf{A}(\mathbf{r}) = \frac{\mu_o}{4\pi}\nabla\times\left(\frac{\mathbf{m}\times\vec{r}}{|\vec{r}|^3}\right)$$

$$\nabla\times\left(\frac{\mathbf{m}\times\vec{r}}{|\vec{r}|^3}\right) = \nabla\left(\frac{1}{r^3}\right)\times\mathbf{m}\times\vec{r} + \frac{1}{r^3}\nabla\times(\mathbf{m}\times\vec{r})$$

$$\nabla\left(\frac{1}{r^3}\right)\times(\mathbf{m}\times\mathbf{r}) = \frac{-3}{r^5}\mathbf{r}\times(\mathbf{m}\times\mathbf{r}) = \frac{3\mathbf{n}(\mathbf{n}\cdot\mathbf{m})-3\mathbf{m}}{r^3}$$

Note $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$

Using $\nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} + (\nabla\cdot\mathbf{B})\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} - (\nabla\cdot\mathbf{A})\mathbf{B}$

$$\frac{1}{r^3}\nabla\times(\mathbf{m}\times\mathbf{r}) = \frac{1}{r^3}\left[(\mathbf{r}\cdot\nabla)\mathbf{m} + (\nabla\cdot\mathbf{r})\mathbf{m} - (\mathbf{m}\cdot\nabla)\mathbf{r} - (\nabla\cdot\mathbf{m})\mathbf{r}\right] = \frac{1}{r^3}\{3\mathbf{m}-\mathbf{m}\}$$

$$\mathbf{B}(\mathbf{r}) = \nabla\times\mathbf{A}(\mathbf{r}) = \frac{\mu_o}{4\pi}\left[\frac{3\mathbf{n}(\mathbf{n}\cdot\mathbf{m})-\mathbf{m}}{r^3}\right]$$

With $\mathbf{m} = \frac{1}{2}\int_V \vec{r}\times\mathbf{J}(\vec{r})\,d^3r = \frac{I}{2}\oint \vec{r}\times d\vec{l}$, we can obtain $|\mathbf{m}| = I\times(\text{Area})$

For the current density $\mathbf{J} = \sum_i q_i\,\vec{v}_i\delta(\vec{r}-\vec{r}_i)$, we obtain $\mathbf{m} = \frac{1}{2}\sum_i q_i\,(\vec{r}_i\times\vec{v}_i) = \sum_i \frac{q_i}{2M_i}\mathbf{L}_i$

If all the particles in motion have the same charge-to-mass ratio $q_i/M_i = e/M$, then

$$\mathbf{m} = \frac{e}{2M}\sum_i \mathbf{L}_i = \frac{e}{2M}\mathbf{L}$$

$$\mathbf{A}(\vec{r}) = \frac{\mu_o}{4\pi}\left[\int_V \frac{\mathbf{J}(\vec{r}')}{|\vec{r}-\vec{r}'|}d^3r' + \int_V \frac{\mathbf{M}(\vec{r}')\times(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}d^3r'\right]$$

$$\mathbf{A}(\vec{r}) = \frac{\mu_o}{4\pi}\left[\int_V \frac{\mathbf{J}(\vec{r}')}{|\vec{r}-\vec{r}'|}d^3r' + \int_V \mathbf{M}(\vec{r}')\times\nabla'\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)d^3r'\right]$$

$$\mathbf{A}(\vec{r}) = \frac{\mu_o}{4\pi}\left[\int_V \frac{\mathbf{J}(\vec{r}')}{|\vec{r}-\vec{r}'|}d^3r' + \int_V \frac{\nabla'\times\mathbf{M}(\vec{r}')}{|\vec{r}-\vec{r}'|}d^3r'\right]$$

$\mathbf{B}(\vec{r}) = \nabla\times\mathbf{A}(\vec{r})$ and $\nabla^2\left(\frac{-1}{|\vec{r}-\vec{r}'|}\right) = -4\pi\delta(\vec{r}-\vec{r}')$

$$\nabla\times\mathbf{B}(\vec{r}) = \mu_o\left[\mathbf{J}(\vec{r}) + \nabla\times\mathbf{M}(\vec{r})\right]$$

$\nabla \times \mathbf{H}(\vec{r}) = \mathbf{J}(\vec{r})$ with $\mathbf{H}(\vec{r}) = \dfrac{\mathbf{B}(\vec{r})}{\mu_o} - \mathbf{M}(\vec{r})$

$$\int \frac{\cos\theta'}{|\mathbf{r}-\mathbf{r}'|} d\Omega'$$

5. The Magnetic Scalar Potential

Although not the same as theoretical importance as the magnetic vector potential, the magnetic scalar potential is extremely useful for solving problems involving magnetic fields. Considering the magnetic induction produced by a closed loop carrying a current *I*, in the region of space where $\mathbf{J}(\vec{r}) = 0$, we have $\nabla \times \mathbf{B}(\vec{r}) = 0$ to express the magnetic induction as the gradient of a scalar potential:

$$\mathbf{B} = -\nabla \Phi_M \quad .$$

The change in any scalar function Φ_M due to an infinitesimal change $d\mathbf{r}$ is given by

$$d\Phi_M = \nabla \Phi_M \cdot d\mathbf{r} = -\mathbf{B} \cdot d\mathbf{r} \quad .$$

Using the Biot-Svart law express the magnetic induction, we have

$$d\Phi_M = -\frac{\mu_o}{4\pi} I \oint \frac{d\mathbf{l} \times (\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} \cdot d\mathbf{r} \quad .$$

It is interesting to relate this expression to the solid angle Ω, subtended by the current loop. As shown in Fig.,

$$\Omega = \int_S \frac{d\mathbf{A} \cdot (-\mathbf{R})}{R^3}$$

with $\mathbf{R} = \mathbf{r} - \mathbf{r}'$, where –**R** is a vector pointing from the observer at the field point to a point on the surface enclosed by the loop. If the observer moves by an amount $d\mathbf{r}$**,** the solid angle subtended by the loop will change. Consequently,

$$\oint \frac{d\mathbf{l} \times (\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} \cdot d\mathbf{r} = \oint \frac{(d\mathbf{r} \times d\mathbf{l}) \cdot (\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} = d\Omega \quad .$$

Comparing $d\Omega$ with $d\Phi_M$, we find that

$$\Phi_M = -\frac{\mu_o}{4\pi} I\Omega \quad .$$

Example 1: A circular loop of radius *a* carrying current *I* lies in the *x*-*y* plane with its center at the origin. Find the magnetic scalar potential at a point below the center of a circular current loop. Use the scalar potential to find the magnetic induction field at a point on the *z*-axis.

Solution: In cylindrical coordinates, $\mathbf{R} = \mathbf{r} - \mathbf{r}' = z\,\mathbf{a_z} - r'\mathbf{a}_r$. Thus, for $z<0$, we find

$$\Omega = \int_S \frac{d\mathbf{A}\cdot(-\mathbf{R})}{R^3} = \int_0^a\int_0^{2\pi} \frac{-zr'}{(z^2+r'^2)^{3/2}}d\phi dr' = 2\pi\frac{z}{(z^2+r'^2)^{1/2}}\Bigg|_0^a = 2\pi\left[\frac{z}{(z^2+a^2)^{1/2}}+1\right].$$

From this result, we have

$$\mathbf{B}(0,0,z) = \frac{\mu_o I}{4\pi}\nabla\Omega = \frac{\mu_o I}{2}\left[\frac{1}{(z^2+a^2)^{1/2}} - \frac{z^2}{(z^2+a^2)^{3/2}}\right]\mathbf{a}_z = \frac{\mu_o I}{2}\frac{a^2}{(z^2+a^2)^{3/2}}\mathbf{a}_z \quad .$$

Note that for z>0, the expression for Ω becomes

$$\Omega = \int_S \frac{d\mathbf{A}\cdot(-\mathbf{R})}{R^3} = \int_0^a\int_0^{2\pi} \frac{-zr'}{(z^2+r'^2)^{3/2}}d\phi dr' = 2\pi\frac{z}{(z^2+r'^2)^{1/2}}\Bigg|_0^a = 2\pi\left[\frac{z}{(z^2+a^2)^{1/2}}-1\right]$$

Hard Ferromagnets

A practical case concerns hard ferromagnetic, having a magnetization that is essentially independent of applied fields for moderate field strengths. Such materials can be treated as if they had a fixed, specified magnetization $\mathbf{M}(\mathbf{r})$. From $\nabla\cdot\mathbf{B}(\mathbf{r}) = 0$ and $\mathbf{B}(\mathbf{r}) = \mu_o[\mathbf{H}(\mathbf{r})+\mathbf{M}(\mathbf{r})]$, we have

$$\nabla\cdot\mathbf{H}(\mathbf{r}) = -\nabla\cdot\mathbf{M}(\mathbf{r}) \quad .$$

Furthermore, in the region of space where $\mathbf{J}(\vec{r}) = 0$, we have $\nabla\times\mathbf{H}(\vec{r}) = 0$ to make

$$\mathbf{H} = -\nabla\Phi_M \quad .$$

Therefore, combining the two equations yields

$$\nabla\cdot\nabla\Phi_M = \nabla\cdot\mathbf{M}(\mathbf{r}) \quad .$$

It becomes a magnetostatic Poisson equation,

$$\nabla^2\Phi_M = -\rho_M$$

with the effective magnetic-charge density

$$\rho_M = -\nabla\cdot\mathbf{M}(\mathbf{r}) \quad .$$

When there are no boundary surfaces, the solution for the potential Φ_M is given by

$$\Phi_M(\mathbf{r}) = \frac{-1}{4\pi}\int_{\text{all space}} \frac{\nabla'\cdot\mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^3r' \quad .$$

where the volume of integration *V* contains all the magnetization. The above equation can be transformed into a more useful form. Using the identity

$$\frac{\nabla'\cdot\mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} = \nabla'\cdot\frac{\mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} - \mathbf{M}(\mathbf{r}')\cdot\nabla'\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) \quad ,$$

we rewrite Φ_M as

$$\Phi_M(\mathbf{r}) = \frac{-1}{4\pi}\int_{\text{all space}} \nabla' \cdot \frac{\mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} d^3r' + \frac{1}{4\pi}\int_{\text{all space}} \mathbf{M}(\mathbf{r}')\cdot\nabla'\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) d^3r' \quad .$$

Since **M** vanishes on the boundary of the volume of integration, the first integral can be shown to be a vanishing surface integral with the divergence theorem. Thus the potential Φ_M becomes

$$\begin{aligned}\Phi_M(\mathbf{r}) &= \frac{1}{4\pi}\int_{\text{all space}} \mathbf{M}(\mathbf{r}')\cdot\nabla'\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) d^3r' \\ &= \frac{1}{4\pi}\int_{\text{all space}} \mathbf{M}(\mathbf{r}')\cdot\nabla\left(\frac{-1}{|\mathbf{r}-\mathbf{r}'|}\right) d^3r' \quad . \\ &= -\frac{1}{4\pi}\nabla\cdot\int_{\text{all space}} \frac{\mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} d^3r'\end{aligned}$$

It is sometimes convention to treat the problem that the magnetization **M** has a finite volume V and surface S and falls suddenly to zero at the surface S. To deal with the discontinuity of **M** at the boundary of the material, the integration is divided into two region: one is the volume V and the other is the volume outside V. Thus

$$\Phi_M(\mathbf{r}) = \frac{-1}{4\pi}\int_V \frac{\nabla'\cdot\mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} d^3r' + \frac{-1}{4\pi}\int_{\text{outside}} \frac{\nabla'\cdot\mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} d^3r'$$

Using

$$\frac{\nabla'\cdot\mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} = \nabla'\cdot\frac{\mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} - \mathbf{M}(\mathbf{r}')\cdot\nabla'\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) \quad ,$$

the second integral can be expressed as

$$\frac{1}{4\pi}\int_{\text{outside}} \frac{\nabla'\cdot\mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} d^3r' = \frac{1}{4\pi}\int_{\text{outside}} \nabla'\cdot\frac{\mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} d^3r' - \frac{1}{4\pi}\int_{\text{outside}} \mathbf{M}(\mathbf{r}')\cdot\nabla'\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) d^3r' .$$

The second integral over outside the volume V vanishes as M is zero. The first integral can be rewritten as a surface integral with the divergence theorem,

$$\frac{1}{4\pi}\int_{\text{outside}} \frac{\nabla'\cdot\mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} d^3r' = \frac{-1}{4\pi}\oint_S \frac{\mathbf{M}(\mathbf{r}')\cdot\mathbf{n}'}{|\mathbf{r}-\mathbf{r}'|} da' .$$

The negative sign is due to the fact that $\mathbf{n}'$ is the outwardly directed normal of the volume V. Then we obtain

$$\Phi_M(\mathbf{r}) = \frac{-1}{4\pi}\int_V \frac{\nabla'\cdot\mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^3r' + \frac{1}{4\pi}\oint_S \frac{\mathbf{M}(\mathbf{r}')\cdot\mathbf{n}'}{|\mathbf{r}-\mathbf{r}'|}da' \quad .$$

An important special case is that of uniform magnetization throughout the volume *V*. Then the first term vanishes; only the surface integral over *S* contributes.

$$Y_{lm}(\theta,\phi) = (-1)^m\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}e^{im\phi}P_l^m(\cos\theta)$$

$$= \frac{(-1)^{l+m}}{2^l l!}\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}e^{im\phi}\sin^m\theta\frac{d^{l+m}}{d(\cos\theta)^{l+m}}(1-\cos^2\theta)^l$$

$$Y_{l\,-m}(\theta,\phi) = (-1)^m\; Y^*_{l\,m}(\theta,\phi)$$

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}}\,Y^*_{l\,m}(\theta',\phi')\,Y_{lm}(\theta,\phi)$$

$$\mathbf{n} = \mathbf{i}\sin\theta\,\cos\phi + \mathbf{j}\sin\theta\,\sin\phi + \mathbf{k}\cos\theta$$

$$Y_{00}(\theta,\phi) = \sqrt{\frac{1}{4\pi}}$$

$$Y_{10}(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_{11}(\theta,\phi) = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_{22}(\theta,\phi) = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{i2\phi}$$

$$Y_{21}(\theta,\phi) = -\frac{1}{2}\sqrt{\frac{15}{2\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_{22}(\theta,\phi) = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

Example: Find the magnetic scalar potential of a magnetized sphere of radius *a* having magnetization $\mathbf{M}(\mathbf{r}) = M_o\mathbf{a}_z$.

Solution: Using the form

$$\Phi_M(\mathbf{r}) = -\frac{1}{4\pi}\nabla\cdot\int_{\text{all space}}\frac{\mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^3r' ,$$

we have

$$\Phi_M(\mathbf{r}) = -\frac{M_o}{4\pi}\frac{\partial}{\partial z}\int_{\text{all space}}\frac{1}{|\mathbf{r}-\mathbf{r}'|}d^3r' \quad .$$

Using the fact

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|}=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}^*(\theta',\phi')\,Y_{lm}(\theta,\phi),$$

it can be shown that only the $l=0$ term survives the angular integration and the integral is written as

$$\Phi_M(\mathbf{r})=-M_o\frac{\partial}{\partial z}\int_0^a\frac{1}{r_>}r'^2\,dr' \quad .$$

Outside the sphere, $r>a$, the integral is explicitly given by

$$\Phi_M(\mathbf{r})=-M_o\frac{\partial}{\partial z}\int_0^a\frac{1}{r}r'^2\,dr'=-\frac{M_oa^3}{3}\frac{\partial}{\partial z}\frac{1}{r}=\frac{M_oa^3}{3}\frac{z}{r^3}=\frac{M_oa^3}{3}\frac{\cos\theta}{r^2} \quad .$$

Inside the sphere, $r<a$, the integral is explicitly given by

$$\Phi_M(\mathbf{r})=-M_o\frac{\partial}{\partial z}\left(\int_0^r\frac{1}{r}r'^2\,dr'+\int_r^a r'\,dr'\right)=-M_o\frac{\partial}{\partial z}\left(\frac{1}{3}r^2-\frac{1}{2}(a^2-r^2)\right)=\frac{M_o}{3}r\cos\theta .$$

As a result, we have

$$\mathbf{H}_{out}=-\nabla\Phi_M=\frac{M_oa^3}{3r^3}(2\cos\theta\,\mathbf{a}_r+\sin\theta\,\mathbf{a}_\theta)$$

$$\mathbf{B}_{out}=\mu_o\mathbf{H}_{out}=\mu_o\frac{M_oa^3}{3r^3}(2\cos\theta\,\mathbf{a}_r+\sin\theta\,\mathbf{a}_\theta)$$

and

$$\mathbf{H}_{in}=-\nabla\Phi_M=-\frac{M_o}{3}(\cos\theta\,\mathbf{a}_r-\sin\theta\,\mathbf{a}_\theta)$$

$$\mathbf{B}_{in}=\mu_o(\mathbf{H}_{in}+\mathbf{M})=\mu_o\frac{2M_o}{3}(\cos\theta\,\mathbf{a}_r-\sin\theta\,\mathbf{a}_\theta)$$

It can be checked that the boundary conditions on **B** and **H** are satisfied, i.e. the tangential **H** fields and the normal **B** field are continuous.

Magnetic shielding

A permeable body is placed in the region of empty space where a certain magnetic induction $\mathbf{B}_o$ exists. The lines of magnetic induction are modified. Concerning media of very high permeability, the field lines are expected to tend to be normal to the surface of the body. Similar to conductors, if the body is hollow, it is expected that the field in the cavity is smaller than the external field, vanishing in the limit $\mu\to\infty$. Such a reduction in field is regarded as the magnetic shielding provided by the permeable material. It is practically important, since essentially field-free regions are often necessary or desirable for experimental purposes or for the reliable working of electronic devices..

Example: A spherical shell of permeable material having inner radius *a* and outer radius *b* is placed in an initially uniform magnetic induction field $\mathbf{B}_o$. Find the magnetic induction field inside the spherical shell.

Solution: In the absence of any free currents, $\nabla\times\mathbf{H}=0$, implying that **H** may be written as the gradient of a potential, $\mathbf{H}=-\nabla\Phi_M$. Consequently,

$$\nabla\cdot\mathbf{B}=\nabla\cdot(\mu\mathbf{H})=-\nabla\cdot(\mu\nabla\Phi_M)=0 \quad .$$

For piecewise constant μ, we have

$$\mu\nabla^2\Phi_M=0 \quad .$$

Taking the z axis along the initial field $\mathbf{B}_o$, we have $\mathbf{B}=-\mu\nabla\Phi_M$ and $\mathbf{B}_o=\mu_o\mathbf{H}_o$. Thus at sufficiently large distance from the shell, $\Phi_M(r\to\infty)=-H_o z=-H_o r\cos\theta$. Expanding $\Phi_M(\mathbf{r})$ in spherical polar coordinates in each of the three regions, explicitly putting in the asymptotic form of $\Phi_M(\mathbf{r})$ for large *r*, we get

$$r>b\,,\qquad \Phi_M(\mathbf{r})=-H_o r\cos\theta+\sum_\ell \frac{A_\ell}{r^{\ell+1}}P_\ell(\cos\theta)$$

$$a<r<b\,,\qquad \Phi_M(\mathbf{r})=\sum_\ell\left(B_\ell r^\ell+\frac{C_\ell}{r^{\ell+1}}\right)P_\ell(\cos\theta)$$

$$r<a\,,\qquad \Phi_M(\mathbf{r})=\sum_\ell D_\ell r^\ell P_\ell(\cos\theta)$$

The boundary conditions at *r*=*a* and *r*=*b* are

$$H_\theta(a_+)=H_\theta(a_-)\qquad H_\theta(b_+)=H_\theta(b_-)$$

$$B_r(a_+)=B_r(a_-)\qquad B_r(b_+)=B_r(b_-) \quad .$$

In terms of the magnetic scalar potential, the boundary conditions are given by

$$\left.\frac{\partial\Phi_M}{\partial\theta}\right|_{a_+}=\left.\frac{\partial\Phi_M}{\partial\theta}\right|_{a_-}\qquad \left.\frac{\partial\Phi_M}{\partial\theta}\right|_{b_+}=\left.\frac{\partial\Phi_M}{\partial\theta}\right|_{b-}$$

$$\mu\left.\frac{\partial\Phi_M}{\partial r}\right|_{a_+}=\mu_o\left.\frac{\partial\Phi_M}{\partial r}\right|_{a_-}\qquad \mu\left.\frac{\partial\Phi_M}{\partial r}\right|_{b_+}=\mu_o\left.\frac{\partial\Phi_M}{\partial r}\right|_{b_-} \quad .$$

Using arguments akin to those used in the electrostatic examples, but more laboriously, we can show that all the coefficients with $\ell\neq 1$ vanish. The equations for $\ell=1$ are given by

$$B_1 a+\frac{C_1}{a^2}=D_1 a\qquad B_1 b+\frac{C_1}{b^2}=-H_o b+\frac{A_1}{b^2}$$

$$\mu\left(B_1-\frac{2C_1}{a^3}\right)=\mu_o D_1\qquad \mu\left(B_1-\frac{2C_1}{b^3}\right)=\mu_o\left(-H_o-\frac{2A_1}{b^3}\right)$$

After some algebra, we have

$$C_1=\frac{a^3}{3}\left(\frac{\mu-\mu_o}{\mu}\right)D_1$$

$$B_1=\left(\frac{2\mu+\mu_o}{3\mu}\right)D_1$$

$$\left(\frac{2\mu+\mu_o}{3\mu}\right)D_1+\frac{a^3}{3b^3}\left(\frac{\mu-\mu_o}{\mu}\right)D_1=-H_o+\frac{A_1}{b^3}$$

$$\left(\frac{2\mu+\mu_o}{3\mu_o}\right)D_1-\frac{2a^3}{3b^3}\left(\frac{\mu-\mu_o}{\mu_o}\right)D_1=-H_o-\frac{2A_1}{b^3}$$

$$D_1=\frac{-3H_o}{\left[2\left(\frac{2\mu+\mu_o}{3\mu}\right)+\left(\frac{2\mu+\mu_o}{3\mu_o}\right)\right]+\frac{2a^3}{3b^3}\left[\left(\frac{\mu-\mu_o}{\mu}\right)-\left(\frac{\mu-\mu_o}{\mu_o}\right)\right]}$$

$$A_1=\frac{D_1}{9\mu_o\mu}\left(a^3-b^3\right)(2\mu+\mu_o)(\mu-\mu_o)$$

$$D_1=\frac{-9H_o\mu_o\mu}{[(2\mu+\mu_o)(2\mu_o+\mu)]-\frac{2a^3}{b^3}(\mu-\mu_o)^2}$$

$$A_1=\frac{\left(b^3-a^3\right)(2\mu+\mu_o)(\mu-\mu_o)}{[(2\mu+\mu_o)(2\mu_o+\mu)]-\frac{2a^3}{b^3}(\mu-\mu_o)^2}H_o$$

The exterior potential

$$\Phi_M(\mathbf{r})=-H_o r\cos\theta+\frac{A_1}{r^2}\cos\theta$$

consists of that for a uniform field *Ho* plus the field of a dipole with magnetic moment $4\pi A_1$, oriented parallel to $\mathbf{B}_o$. Inside the cavity, there is a uniform magnetic induction $\mathbf{B}=-\mu_o\mathbf{H}_o D_1$. When the permeability of the shell μ is much greater than that of vacuum, the coefficients

$$A_1\approx b^3H_o$$

and

$$D_1=\frac{-9H_o\mu_o}{2\left(1-\frac{a^3}{b^3}\right)\mu}\quad .$$

For shields of high permeability, m ranges from $10^3\ \mu_o$ to $10^9\ \mu_o$; even relatively thin shells cause a great reduction of **B** in the interior shell.

Faraday's Law of induction

Here the effect of a slow variation in the electromagnetic fields is considered. The meaning of the slowly varying fields is that the sources do not change considerably during the time it takes for their fields to propagate to any point in the region of interest. It is clear that charged particles experience a force when they move through a magnetic field. It is expected and experimentally verified that the same force is experienced by charged particles whether they move while the magnetic field is stationary or they are stationary while the magnetic field is moved in the same relative manner. At the position of the particle, a moving source of magnetic field is perceived as a temporally varying magnetic field. Any local field interpretation would therefore require that the force on the particle depend on $\partial \mathbf{B}/\partial t$. It will evolve that the force felt by such a stationary particle must be reinterpreted as resulting from an electric field.

When a charge is forced to move through a magnet induction field, it is subjected to a force due to motion through the field $\mathbf{F} = q\mathbf{v}\times\mathbf{B}$. In this case, it is important to note that the magnetic field does not do any work on the charge. The agent that produces or maintains **v** does the work. Let us consider the electromotive force (EMF) for a mobile loop placed in a static electric and magnetic field. The force on a charge attached to the moving loop is then $q(\mathbf{E}+\mathbf{v}\times\mathbf{B})$, leading to an electromotive force (EMF)

$$\mathcal{E} = \oint_{C(t)} (\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot d\mathbf{l} \quad .$$

For a static field, $\mathcal{E} = \oint_{C(t)} \mathbf{E}\cdot d\mathbf{l} = 0$. To begin with, let us consider a loop whose shape does not change to move through a magnetic induction field whose strength varies with position. The sides of the moving loop will evidently experience a time-dependent field. For simplifying discussion, consider a small rectangular loop of dimensions δ*x* and δ*y* in the *x*-*y* plane, moving in the *x*-direction through a magnetic induction field whose *z* component varies linearly with *x*. The electromotive force (EMF) generated around the moving loop is generally given by

$$\mathcal{E} = \oint_{C} (\mathbf{v}\times\mathbf{B})\cdot d\mathbf{l} \quad .$$

When the field is spatially dependent, the integral can be expanded as

$$\mathcal{E} = \int_{\delta y} (-vB_z(x,y)dy + vB_z(x+dx,y)dy) = -\iint_{\delta x\delta y} v\frac{\partial B_z}{\partial x}dxdy \quad .$$

It is known that physical phenomena are the same when viewed by two observes moving with a constant velocity **v** relative to one another. It is expected and experimentally verified that the same EMF is induced in the loop whether it is moved while the magnetic induction field is stationary or it is held fixed while the magnetic induction field in moved in the same relative manner. Mathematically, the term $v(\partial B_z/\partial x)$ in the integrand can be replaced with $(\partial B_z/\partial t)$ to manifest the different moving coordinate systems. Therefore, if instead of moving the loop we move the magnet responsible for the field above, the same EMF is required. However, since now the velocity **v** = 0, implying there can be no contribution from $\mathbf{v}\times\mathbf{B}$. In other words, the line integral of the electric field cannot vanish when we have temporally varying fields. Equivalently, we must have

$$\mathcal{E}=\oint_C \mathbf{E}\cdot d\mathbf{l}=-\iint_S \frac{\partial \mathbf{B}}{\partial t}\cdot \mathbf{n}da \quad .$$

Next, the loop is assumed to be stretched and deformed with velocity **v**. For a static field, $\mathcal{E}=\oint_{C(t)} \mathbf{E}\cdot d\mathbf{l}=0$. Therefore the EMF can be rearranged as

$$\mathcal{E}=\oint_{C(t)} (d\mathbf{l}\times\mathbf{v})\cdot\mathbf{B}\,.$$

During a time *dt*, a segment of the loop of length *dl* moves to change the area within the loop by $\mathbf{n}da=-(d\mathbf{l}\times\mathbf{v}dt)$. Thus, the EMF can be expressed as

$$\mathcal{E}=-\int_{S(t)}(da/dt)\mathbf{n}\cdot\mathbf{B}$$

6. Energy in the magnetic field

Considering a single circuit with a constant current *I* flowing in it, when the flux through the circuit changes, an electromotive force $\mathcal{E}$ is induced around it. To keep the current constant, the sources do work to maintain the current at the rate

$$\frac{dW_m}{dt}=-I\mathcal{E}=I\frac{d\Phi_B}{dt}$$

the negative sign following from Lenz's law. As a result, if the flux change through a circuit carrying a current I is $\delta\Phi_B$, the work done by the source is

$$\delta W_m = I\,\delta\Phi_B$$

To derive the energy in the loop, the current density distribution is broken up into element current loops. For each element loop, the increment of work done against the induced emf is

$$\Delta(\delta W_m) = (J\Delta\sigma)\int_S \delta\mathbf{B}\cdot\mathbf{n}\,da \ .$$

With $\mathbf{B} = \nabla\times\mathbf{A}$, we can obtain

$$\Delta(\delta W_m) = (J\Delta\sigma)\int_S (\nabla\times\delta\mathbf{A})\cdot\mathbf{n}\,da$$

Using the Stokes's theorem, this equation can be written as

$$\Delta(\delta W_m) = (J\Delta\sigma)\oint_C \delta\mathbf{A}\cdot d\mathbf{l} \ .$$

With $J\,\Delta\sigma\, d\mathbf{l} = \mathbf{J}\, d^3r$, the sum over all such element loops will be the volume integral. Hence the total increment of work done by the external sources due to a change $\delta\mathbf{A}(\mathbf{r})$ in the vector potential is

$$\delta W_m = \int \delta\mathbf{A}\cdot\mathbf{J}\, d^3x$$

Using Ampere's law $\nabla\times\mathbf{H} = \mathbf{J}$, this equation can be rewritten as

$$\delta W_m = \int \delta\mathbf{A}\cdot\nabla\times\mathbf{H}\, d^3x = \int \mathbf{H}\cdot\nabla\times\delta\mathbf{A}\, d^3x = \int \mathbf{H}\cdot\delta\mathbf{B}\, d^3x$$

This relation is the magnetic equivalent of the electrostatic equation

$$\delta W_e = \int(\nabla\cdot\delta\mathbf{D})\Phi(\mathbf{r})\, d^3x = \int\mathbf{E}\cdot\delta\mathbf{D}\, d^3x \, .$$

In its present form it is applicable to all magnetic media, including ferromagnetic substances. When the medium is para- or diamagnetic, so that a linear relation exists between **H** and **B**, then $\mathbf{H}\cdot\delta\mathbf{B} = \delta(\mathbf{H}\cdot\mathbf{B})/2$. When the fields are increased from zero to their final values, the total magnetic energy is

$$W_m = \frac{1}{2}\int \mathbf{H}\cdot\mathbf{B}\, d^3x$$

Consequently, the total energy density for the linear media can be denoted by

$$u = \frac{1}{2}(\mathbf{E}\cdot\mathbf{D} + \mathbf{H}\cdot\mathbf{B})\,.$$

Chapter Seven: Maxwell Equations and gauge transformations

The almost independent nature of electric and magnetic phenomena disappears when time-dependent problems are considered. Time-varying magnetic fields give rise to electric fields and vice versa. The full import of the interconnection between electric and magnetic fields and their essential sameness becomes clear only within the framework of special relativity. Here we content ourselves with examining the basic phenomena and deducing the set of equations known as the Maxwell equations, which describe the feature of electromagnetic fields. We next discuss vector and scalar potentials, gauge transformations, and Green functions for the wave equation.

Modification of Ampere's law

If we consider Ampere's law in the form

$$\nabla \times \mathbf{H} = \mathbf{J}$$

and take the divergence of both sides, it is found that

$$\nabla \cdot \mathbf{J} = 0$$

Note that the divergence of a curl vanishes. Thus, if the equation of continuity

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0$$

needs to be satisfied, then Ampere's law must be modified. This modification was first carried out by J. C. Maxwell. Maxwell started with Gauss's law $\nabla \cdot \mathbf{D} = \rho$ to get

$$\frac{\partial \rho}{\partial t} = \nabla \cdot \frac{\partial \mathbf{D}}{\partial t}.$$

Replacing this in the continuity equation yields

$$\nabla \cdot \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) = 0.$$

Thus, Maxwell proposed rewriting Ampere's law in the form

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}.$$

This form clearly reduces to the original Ampere's law when the charge density ρ is independent of time. The quantity

$$\mathbf{J}_D = \frac{\partial \mathbf{D}}{\partial t}$$

is referred to as the displacement current. In fact, Maxwell has another more ohysical reason for introducing the displacement current. Maxwell proposed that the displacement current through the electric between the plates of the capacitor to keep the current density to be continuous. Thus, we have the complete set of Maxwell equations to be given by

$$\begin{aligned}
&\nabla \cdot \mathbf{D} = \rho \\
&\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \\
&\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \\
&\nabla \cdot \mathbf{B} = 0
\end{aligned} .$$

These four equations need to be supplemented by constitutive relations relating **D** and **E**, **B** and **H**, as well as **J** and **E**. The simplest constitutive relations hold for non-ferromagnetic, linear, isotropic materials. They are an empirical statement about the material:

$$\begin{aligned}
&\mathbf{D} = \varepsilon \mathbf{E} \\
&\mathbf{B} = \mu \mathbf{H} \\
&\mathbf{J} = \sigma \mathbf{E}
\end{aligned} .$$

Vector and scalar potentials

The Maxwell equations consists of a set of coupled first-order partial differential equations relating the various components of electric and magnetic fields. It is often convenient to introduce potentials, obtaining a smaller number of second-order equations, while satisfying some of the Maxell equations identically. Here we use the scalar potential Φ and the vector potential A that have been already introduced in electrostatics and magnetostatics.

Since $\nabla \cdot \mathbf{B} = 0$ still holds, **B** can be defined in terms of a vector potential:

$$\mathbf{B} = \nabla \times \mathbf{A} .$$

Then the other homogeneous equation $\nabla \times \mathbf{E} + \partial \mathbf{B} / \partial t = 0$, Faraday's law, can be written

$$\nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) = 0 .$$

This indicates that the quantity with vanishing curl in Eq. () can be written as the gradient of some scalar function, namely, a scalar potential Φ:

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla \Phi \quad \Rightarrow \quad \mathbf{E} = -\nabla \Phi - \frac{\partial \mathbf{A}}{\partial t}$$

The definition of **B** and **E** in terms of the potentials **A** and Φ satisfies identically the two homogeneous Maxwell equations. Putting these two equations into two inhomogeneous Maxwell equations for the vacuum case, we can obtain

$$\nabla \cdot \left(-\nabla\Phi - \frac{\partial \mathbf{A}}{\partial t} \right) = \frac{\rho}{\varepsilon_o}$$

$$\nabla \times \nabla \times \mathbf{A} = \mu_o \mathbf{J} + \frac{1}{c^2}\frac{\partial}{\partial t}\left(-\nabla\Phi - \frac{\partial \mathbf{A}}{\partial t} \right)$$

where we have used $\mu_o \varepsilon_o = 1/c^2$. Using $\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$, the last equation can be written

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_o \mathbf{J} + \nabla\left(\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t} \right) \ .$$

The most convenient gauge for radiation problems is the so-called Lorenz gauge,

$$\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t} = 0 \ .$$

With the Lorenz gauge condition, the equations for Φ and $\mathbf{A}$ can be written

$$\nabla^2 \Phi - \frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} = -\frac{\rho}{\varepsilon_o}$$

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_o \mathbf{J} \ .$$

Thus, with the help of the gauge condition, we now have the decoupled equations for the potentials.

Gauge transformations, Lorenz gauge, Coulomb gauge

Since $\mathbf{B}$ is defined through $\mathbf{B} = \nabla \times \mathbf{A}$, the vector potential is arbitrary to the extent that the gradient of some scalar function Λ can be added. Thus $\mathbf{B}$ is left unchanged by the transformation,

$$\mathbf{A} \to \mathbf{A}' = \mathbf{A} + \nabla\Lambda \ .$$

For the electric field $\mathbf{E} = -\nabla\Phi - \partial \mathbf{A}/\partial t$ to be unchanged as well, the scalar potential must be simultaneously transformed,

$$\Phi \to \Phi' = \Phi - \frac{\partial \Lambda}{\partial t} .$$

The freedom implied by the last two Equations () and () means that we can choose a set of potentials ($\mathbf{A}$, Φ) to satisfy the Lorenz condition. To see this transformation, suppose that the original potentials $\mathbf{A}$, Φ do not satisfy the Lorenz condition. Then let us make a gauge transformation to potentials $\mathbf{A}'$, Φ' and demand that $\mathbf{A}'$, Φ' satisfy the Lorenz condition:

$$\nabla \cdot \mathbf{A}' + \frac{1}{c^2}\frac{\partial \Phi'}{\partial t} = 0 = \nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t} + \nabla^2 \Lambda - \frac{1}{c^2}\frac{\partial^2 \Lambda}{\partial t^2}$$

Thus, provided a gauge function Λ can be found to satisfy

$$\nabla^2 \Lambda - \frac{1}{c^2}\frac{\partial^2 \Lambda}{\partial t^2} = -\left(\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t} \right)$$

the new potentials $\mathbf{A}'$, Φ' will satisfy the Lorenz condition and the two inhomogeneous wave equations in ( ).

Even for potentials that satisfy the Lorenz condition there is arbitrariness. Evidently, the restricted gauge transformation,

$$\mathbf{A} \rightarrow \mathbf{A} + \nabla \Lambda$$
$$\Phi \rightarrow \Phi - \frac{\partial \Lambda}{\partial t}$$

where

$$\nabla^2 \Lambda - \frac{1}{c^2}\frac{\partial^2 \Lambda}{\partial t^2} = 0$$

preserves the Lorenz condition, provided $\mathbf{A}$, Φ satisfy it initially. All potentials in this restricted class are said to belong to the Lorenz gauge. The Lorenz gauge is commonly used, first because it leads to the wave equations which treat Φ and $\mathbf{A}$ on equivalent footings, and second because it is a concept independent of the coordinate system chosen and so fits naturally into the considerations of special relativity.

Another useful gauge for the potentials is the so-called Coulomb, radiation, or transverse gauge:

$$\nabla \cdot \mathbf{A} = 0 .$$

From

$$\nabla \cdot \mathbf{E} = \nabla \cdot \left(-\nabla \Phi - \frac{\partial \mathbf{A}}{\partial t} \right) = \frac{\rho}{\varepsilon_o} ,$$

The scalar potential can be found to satisfy the Poisson equation,

$$\nabla^2 \Phi = -\rho / \varepsilon_o$$

with solution,

$$\Phi(\mathbf{r}, t) = \frac{1}{4\pi\varepsilon_o} \int \frac{\rho(\mathbf{r}', t)}{|\mathbf{r} - \mathbf{r}'|} d^3 r' .$$

The scalar potential is just the instantaneous Coulomb potential due to the charge density ρ(**r**,t). This is the origin of the name “Coulomb gauge”.

The vector potential A satisfies the inhomogeneous wave equation,

$$\nabla^2\mathbf{A}-\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2}=-\mu_o\mathbf{J}+\frac{1}{c^2}\nabla\frac{\partial\Phi}{\partial t}.$$

The current density can be written as the sum of two terms

$$\mathbf{J}=\mathbf{J}_l+\mathbf{J}_t,$$

where $\mathbf{J}_l$ is called the longitudinal or irrotational current and has $\nabla\times\mathbf{J}_l=0$, while $\mathbf{J}_t$ is called the transverse or solenoidal current and has $\nabla\cdot\mathbf{J}_t=0$. The following derivation shows that

$$\frac{1}{c^2}\nabla\frac{\partial\Phi}{\partial t}=\mu_o\mathbf{J}_l \quad .$$

Using

$$\nabla^2\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)=-4\pi\delta(\mathbf{r}-\mathbf{r}')$$

and

$$\nabla\times\nabla\times\mathbf{A}=\nabla(\nabla\cdot\mathbf{A})-\nabla^2\mathbf{A},$$

we have

$$\begin{aligned}\mathbf{J}(\mathbf{r},t)&=\int\mathbf{J}(\mathbf{r}',t)\,\delta(\mathbf{r}-\mathbf{r}')\,d^3r'\\&=\frac{-1}{4\pi}\int\mathbf{J}(\mathbf{r}',t)\nabla^2\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)d^3r'\\&=\frac{1}{4\pi}\int\left[\nabla\times\nabla\times\frac{\mathbf{J}(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|}-\nabla\left(\nabla\cdot\frac{\mathbf{J}(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|}\right)\right]d^3r'\end{aligned}$$

Based on $\nabla\times\mathbf{J}_l=0$ and $\nabla\cdot\mathbf{J}_t=0$, we can identify that

$$\mathbf{J}_t(\mathbf{r},t)=\frac{1}{4\pi}\int\nabla\times\nabla\times\frac{\mathbf{J}(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|}\,d^3r'=\frac{1}{4\pi}\nabla\times\nabla\times\int\frac{\mathbf{J}(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|}\,d^3r'$$

and

$$\mathbf{J}_l(\mathbf{r},t)=\frac{-1}{4\pi}\nabla\int\left(\nabla\cdot\frac{\mathbf{J}(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|}\right)d^3r'.$$

The last equation can be further simplified as

$$\mathbf{J}_l(\mathbf{r},t) = \frac{-1}{4\pi}\nabla\int \mathbf{J}(\mathbf{r}',t)\cdot\nabla'\left(\frac{-1}{|\mathbf{r}-\mathbf{r}'|}\right)d^3r'$$
$$= \frac{-1}{4\pi}\nabla\int \frac{\nabla'\cdot\mathbf{J}(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|}d^3r'$$
$$= \frac{1}{4\pi}\nabla\frac{\partial}{\partial t}\int \frac{\rho(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|}d^3r'$$
$$= \varepsilon_o\nabla\frac{\partial}{\partial t}\Phi(\mathbf{r},t) = \frac{1}{c^2\mu_o}\nabla\frac{\partial}{\partial t}\Phi(\mathbf{r},t)$$

Therefore the source for the wave equation for **A** can be expressed entirely in terms of the transverse current:

$$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = -\mu_o\mathbf{J}_t .$$

This is the origin of the name transverse gauge. The name "radiation gauge" stems from the fact that transverse radiation fields are given by the vector potential alone, the instantaneous Coulomb potential contributing only to the near fields. This gauge is particularly useful in quantum electrodynamics. A quantum-mechanical description of photons necessitates quantization of only the vector potential. The Coulomb or transverse gauge is often used when no sources are present. Then Φ = 0, and **A** satisfies the homogeneous wave equation. The fields are given by

$$\mathbf{E} = \frac{\partial\mathbf{A}}{\partial t}$$
$$\mathbf{B} = \nabla\times\mathbf{A} .$$

The inhomogeneous wave equation

With the Lorenz gauge condition, the equations for Φ and **A** can be written

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = -\frac{\rho}{\varepsilon_o}$$
$$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = -\mu_o\mathbf{J} .$$

In order to relate the radiation fields to the sources, it is usually needed to solve the inhomogeneous wave equations for the potentials. The two equations are not independent because ρ and **J** are related by the continuity equation. The electric and magnetic fields may be obtained from the solutions of the equations as

$$\mathbf{E} = -\nabla\Phi - \frac{\partial\mathbf{A}}{\partial t} ,$$

$$\mathbf{B} = \nabla \times \mathbf{A} \ .$$

In practice it is sufficient to evaluate **A** since **B** may be found directly as its curl, and outside the source, **E** may be obtained from **B**.

Green Functions for the wave equation

Both wave equations have the same form as

$$\nabla^2 \psi - \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} = -f(\mathbf{r},t) \quad .$$

One commonly useful method of solving the wave equation is based on the Fourier analysis to deal with only one frequency component. After the single-frequency solution has been found, the time dependent solution can be found by summing the frequency components. With the Fourier transform, the source function $f(\mathbf{r},t)$ and the frequency spectrum $F(\mathbf{r},\omega)$ can be related by

$$f(\mathbf{r},t) = \frac{1}{\sqrt{2\pi}}\int_0^\infty F(\mathbf{r},\omega)e^{-i\omega t}d\omega \quad ,$$

$$F(\mathbf{r},\omega) = \frac{1}{\sqrt{2\pi}}\int_0^\infty f(\mathbf{r},t)e^{i\omega t}dt \quad .$$

In the same way, the solution ψ can be expressed as

$$\psi(\mathbf{r},t) = \frac{1}{\sqrt{2\pi}}\int_0^\infty \Psi(\mathbf{r},\omega)e^{-i\omega t}d\omega \quad ,$$

$$\Psi(\mathbf{r},\omega) = \frac{1}{\sqrt{2\pi}}\int_0^\infty \psi(\mathbf{r},t)e^{i\omega t}dt \quad .$$

Substituting the Fourier components into the wave equation, we obtain

$$(\nabla^2 + k^2)\Psi(\mathbf{r},\omega) = -F(\mathbf{r},\omega) \quad ,$$

where $k = \omega/c$. The solution of this equation can be synthesized with the Green's function that satisfies the equation

$$(\nabla^2 + k^2)G_k(\mathbf{r},\mathbf{r}') = -4\pi\delta(\mathbf{r}-\mathbf{r}') \quad .$$

It is clear that the Green's function that is the solution of the unit point source. The frequency component $\Psi(\mathbf{r},\omega)$ of the total solution of the source function $f(\mathbf{r},t)$ can be found by integrating all point source solutions with the appropriate weight $F(\mathbf{r},\omega)$:

$$\Psi(\mathbf{r},\omega) = \frac{1}{4\pi}\int F(\mathbf{r}',\omega)G_k(\mathbf{r},\mathbf{r}')d^3r' \quad .$$

The Green's function $G_k(\mathbf{r},\mathbf{r}')$ can be conveniently found by using the property of the spherical symmetry about $\mathbf{r}'$. Denoting the distance from the source by $R = |\mathbf{r} - \mathbf{r}'|$, the Green's function at points other than $R = 0$ must satisfy

$$\frac{1}{R}\frac{d^2}{dR^2}(RG_k) + k^2 G_k = 0 \quad .$$

It is well known that the solution of this equation is given by

$$RG_k^{(\pm)} = Ce^{\pm ikR} \quad .$$

Thus the general solution for the Green function can be expressed as

$$G_k(R) = AG_k^{(+)}(R) + BG_k^{(-)} \quad .$$

where

$$G_k^{(\pm)} = \frac{e^{\pm ikR}}{R} \quad .$$

In the limit of $R \to 0$, the wave equation reduces to the Poisson equation, since $kR << 1$. With the help of $\nabla^2(1/R) = -4\pi\delta(R)$, the coefficients of A and B need to satisfy $A + B = 1$. We further evaluate $\Psi(\mathbf{r},\omega)$

$$\Psi^{(\pm)}(\mathbf{r},\omega) = \frac{1}{4\pi}\int F(\mathbf{r}',\omega)\frac{e^{\pm ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}d^3r' .$$

The time-dependent solution $\psi(\mathbf{r},t)$ can be obtained by taking the inverse Fourier transform

$$\psi^{(\pm)}(\mathbf{r},t) = \frac{1}{\sqrt{2\pi}}\int_0^\infty \left[\frac{1}{4\pi}\int F(\mathbf{r}',\omega)\frac{e^{\pm ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}d^3r'\right]e^{-i\omega t}d\omega \quad .$$

With the definition of $t' = t \mp \dfrac{|\mathbf{r}-\mathbf{r}'|}{c}$, the solution is rewritten as

$$\begin{aligned}\psi^{(\pm)}(\mathbf{r},t) &= \frac{1}{4\pi}\int_0^\infty \frac{1}{|\mathbf{r}-\mathbf{r}'|}\left[\frac{1}{\sqrt{2\pi}}\int_0^\infty F(\mathbf{r}',\omega)e^{-i\omega t'}d\omega\right]d^3r' \\ &= \frac{1}{4\pi}\int_0^\infty \frac{f(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|}d^3r'\end{aligned} .$$

Physically, the term with the + sign (the retarded solution) states that the present potential at **r** was caused by the source a travel time *R*/c earlier. The term with the – sign (the advanced solution) means that the current potential depends on the behavior of the source in the future at $t' = t + R/c$. For the retarded solution, it is sometimes written explicitly as

$$\psi^{(\pm)}(\mathbf{r},t) = \frac{1}{4\pi}\int_0^\infty \frac{[f(\mathbf{r}',t')]_{ret}}{|\mathbf{r}-\mathbf{r}'|}d^3r' \quad .$$

The square bracket $[\ ]_{ret}$ means that the time t' is to be evaluated at the retarded time, $t' = t - |\mathbf{r}-\mathbf{r}'|/c$. Use of the retarded solution for wave equations

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = -\frac{\rho}{\varepsilon_o}$$

$$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = -\mu_o\mathbf{J} \ .$$

yields

$$\Phi(\mathbf{r},t) = \frac{1}{4\pi\varepsilon_o}\int_0^\infty \frac{[\rho(\mathbf{r}',t')]_{ret}}{R} d^3r' \quad ,$$

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_o}{4\pi}\int_0^\infty \frac{[\mathbf{J}(\mathbf{r}',t')]_{ret}}{R} d^3r'$$

where it is convenient to use $\mathbf{R} = \mathbf{r} - \mathbf{r}'$, $R = |\mathbf{r} - \mathbf{r}'|$, and $\hat{\mathbf{R}} = \mathbf{R}/R$ (in the following).

These solutions were first given by Lorenz. In principle, from these two solutions the electric and magnetic fields can be computed, but it is often useful to have retarded integration solutions for the fields in terms of the sources. In the following, these solutions will be derived.

Retarded solutions for the Fields: Jefimenko's generalizations of the Coulomb and Biot-Savart Laws

The electric and magnetic fields may be obtained from the solutions of the equations as

$$\mathbf{E} = -\nabla\Phi - \frac{\partial\mathbf{A}}{\partial t} \quad ,$$

$$\mathbf{B} = \nabla\times\mathbf{A} \ .$$

The retarded solutions for the fields can be immediately written in the preliminary forms

$$\mathbf{E}(\mathbf{r},t) = \frac{-1}{4\pi\varepsilon_o}\int_0^\infty \nabla\left(\frac{[\rho(\mathbf{r}',t')]_{ret}}{R}\right)d^3r' - \frac{\mu_o}{4\pi}\int_0^\infty \frac{\partial}{\partial t}\left(\frac{[\mathbf{J}(\mathbf{r}',t')]_{ret}}{R}\right)d^3r' \quad ,$$

$$\mathbf{B}(\mathbf{r},t) = \frac{\mu_o}{4\pi}\int_0^\infty \nabla\times\left(\frac{[\mathbf{J}(\mathbf{r}',t')]_{ret}}{R}\right)d^3r' \ .$$

The time derivative in the integrand has the property

$$\frac{\partial}{\partial t}[\mathbf{J}(\mathbf{r}',t')]_{ret} = \left[\frac{\partial\mathbf{J}(\mathbf{r}',t')}{\partial t'}\right]_{ret} \quad .$$

Moreover, we have

$$\nabla[\rho(\mathbf{r}',t')]_{ret} = \nabla(t')\left[\frac{\partial\rho(\mathbf{r}',t')}{\partial t'}\right]_{ret} = -\frac{\hat{\mathbf{R}}}{cR}\left[\frac{\partial\rho(\mathbf{r}',t')}{\partial t'}\right]_{ret}$$

and

$$\nabla\times[\mathbf{J}(\mathbf{r}',t')]_{ret} = \nabla(t')\times\left[\frac{\partial\mathbf{J}(\mathbf{r}',t')}{\partial t'}\right]_{ret} = -\frac{\hat{\mathbf{R}}}{cR}\times\left[\frac{\partial\mathbf{J}(\mathbf{r}',t')}{\partial t'}\right]_{ret} \ .$$

After some algebra, we can obtain

$$\begin{aligned}\mathbf{E}(\mathbf{r},t) &= \frac{-1}{4\pi\varepsilon_o}\int_0^\infty \left\{[\rho(\mathbf{r}',t')]_{ret}\nabla\left(\frac{1}{R}\right) + \frac{\nabla[\rho(\mathbf{r}',t')]_{ret}}{R}\right\}d^3r' - \frac{1}{4\pi\varepsilon_o}\int_0^\infty \frac{1}{c^2R}\left[\frac{\partial \mathbf{J}(\mathbf{r}',t')}{\partial t'}\right]_{ret} d^3r' \\ &= \frac{1}{4\pi\varepsilon_o}\int_0^\infty \left\{[\rho(\mathbf{r}',t')]_{ret}\left(\frac{\hat{\mathbf{R}}}{R^2}\right) + \left[\frac{\partial\rho(\mathbf{r}',t')}{\partial t'}\right]_{ret}\frac{\hat{\mathbf{R}}}{cR} - \frac{1}{c^2R}\left[\frac{\partial \mathbf{J}(\mathbf{r}',t')}{\partial t'}\right]_{ret}\right\}d^3r'\end{aligned},$$

$$\begin{aligned}\mathbf{B}(\mathbf{r},t) &= \frac{\mu_o}{4\pi}\int_0^\infty \left\{\nabla\left(\frac{1}{R}\right)\times[\mathbf{J}(\mathbf{r}',t')]_{ret} + \frac{\nabla\times[\mathbf{J}(\mathbf{r}',t')]_{ret}}{R}\right\}d^3r' \\ &= \frac{\mu_o}{4\pi}\int_0^\infty \left\{[\mathbf{J}(\mathbf{r}',t')]_{ret}\times\left(\frac{\hat{\mathbf{R}}}{R^2}\right) + \left[\frac{\partial \mathbf{J}(\mathbf{r}',t')}{\partial t'}\right]_{ret}\times\left(\frac{\hat{\mathbf{R}}}{cR}\right)\right\}d^3r'\end{aligned}.$$

If the charge and current densities are time independent, the expressions reduce to the familiar static expressions. The terms involving the time derivatives and the retardation provide the generalizations to time-dependent sources. These two results, sometimes known as Jefimenko's generalizations of the Coulomb and Biot-Savart laws, were popularized in Jefimenko's text. Alternatively, the fields can be expressed as

$$\mathbf{E}(\mathbf{r},t) = \frac{1}{4\pi\varepsilon_o}\int_0^\infty \left\{[\rho(\mathbf{r}',t')]_{ret}\left(\frac{\hat{\mathbf{R}}}{R^2}\right) + \frac{\partial}{c\partial t}\left[\frac{\rho(\mathbf{r}',t')\hat{\mathbf{R}}}{R}\right]_{ret} - \frac{\partial}{c^2\partial t}\left[\frac{\mathbf{J}(\mathbf{r}',t')}{R}\right]_{ret}\right\}d^3r',$$

$$\mathbf{B}(\mathbf{r},t) = \frac{\mu_o}{4\pi}\int_0^\infty \left\{[\mathbf{J}(\mathbf{r}',t')]_{ret}\times\left(\frac{\hat{\mathbf{R}}}{R^2}\right) + \frac{\partial}{c\partial t}\left[\mathbf{J}(\mathbf{r}',t')\times\left(\frac{\hat{\mathbf{R}}}{R}\right)\right]_{ret}\right\}d^3r'$$

Considering a point charge q moving with a velocity $\mathbf{v} = \dot{\mathbf{r}}_q(t')$, the charge and current densities are

$$\rho(\mathbf{r}',t') = q\delta(\mathbf{r}' - \mathbf{r}_q(t'))$$

$$\mathbf{J}(\mathbf{r}',t') = q\dot{\mathbf{r}}_q(t')\delta(\mathbf{r}' - \mathbf{r}_q(t'))$$

Note that $\mathbf{r}_q(t')$ is the position of charge at t'. The retarded integrations for the charge and current densities are given by

$$[\rho(\mathbf{r}',t')]_{ret} = \int q\delta(\mathbf{r}' - \mathbf{r}_q(t'))\,\delta\left(t' - \left(t - \frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)\right)dt'$$

$$[\mathbf{J}(\mathbf{r}',t')]_{ret} = \int q\dot{\mathbf{r}}_q(t')\delta(\mathbf{r}' - \mathbf{r}_q(t'))\,\delta\left(t' - \left(t - \frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)\right)dt'$$

The further evaluation can be done with the property of delta function

$$\int g(x)\delta(f(x)-y)dx = \left.\frac{g(x)}{|df/dx|}\right|_{y=f(x)}$$

Here $y = t$, $f(t') = t' + \left|\mathbf{r}-\mathbf{r}_q(t')\right|/c$, and

$$\frac{df(t')}{dt'} = 1 + \frac{1}{c}\frac{d\left|\mathbf{r}-\mathbf{r}_q(t')\right|}{dt'}$$

$$\left|\mathbf{r}-\mathbf{r}_q(t')\right| = \sqrt{(x-x_q(t'))^2+(y-y_q(t'))^2+(z-z_q(t'))^2}$$

$$\frac{d\left|\mathbf{r}-\mathbf{r}_q(t')\right|}{dt'} = -\frac{(x-x_q(t'))\frac{dx_q(t')}{dt'}+(y-y_q(t'))\frac{dy_q(t')}{dt'}+(z-z_q(t'))\frac{dz_q(t')}{dt'}}{\sqrt{(x-x_q(t'))^2+(y-y_q(t'))^2+(z-z_q(t'))^2}}$$

With $R(t') = \left|\,\mathbf{r}-\mathbf{r}_q(t')\,\right|$, we obtain

$$\frac{dR(t')}{dt'} = -\mathbf{n}\cdot\mathbf{v}(t'),$$

where $\mathbf{n} = \left(\mathbf{r}-\mathbf{r}_q(t')\right)/\left|\mathbf{r}-\mathbf{r}_q(t')\right|$ is an instantaneous unit vector. We can have

$$\frac{df(t')}{dt'} = 1 + \frac{1}{c}\frac{dR(t')}{dt'} = 1 - \boldsymbol{\beta}\cdot\mathbf{n}.$$

Setting $\xi = 1-\boldsymbol{\beta}\cdot\mathbf{n}$, the retarded integrations for the charge and current densities are given by

$$\left[\rho(\mathbf{r}',t')\right]_{ret} = \frac{1}{\xi}q\left[\delta\left(\mathbf{r}'-\mathbf{r}_q(t')\right)\right]_{ret} \quad ,$$

$$\left[\mathbf{J}(\mathbf{r}',t')\right]_{ret} = \frac{1}{\xi}q\left[\mathbf{v}(t')\delta\left(\mathbf{r}'-\mathbf{r}_q(t')\right)\right]_{ret} \quad .$$

Substituting these results into the expressions for the fields, we have

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{4\pi\varepsilon_o}\left\{\left(\frac{\hat{\mathbf{R}}}{\xi R^2}\right)_{ret} + \frac{\partial}{c\partial t}\left(\frac{\hat{\mathbf{R}}}{\xi R}\right)_{ret} - \frac{1}{c^2}\frac{\partial}{\partial t}\left[\frac{\mathbf{v}}{\xi R}\right]_{ret}\right\} \quad ,$$

$$\mathbf{B}(\mathbf{r},t) = \frac{\mu_o q}{4\pi}\left\{\left(\frac{\mathbf{v}\times\hat{\mathbf{R}}}{\xi R^2}\right)_{ret} + \frac{\partial}{c\partial t}\left(\frac{\mathbf{v}\times\hat{\mathbf{R}}}{\xi R}\right)_{ret}\right\} \quad .$$

Spherical wave solutions of the scalar wave equation

Spherical harmonic expansions for the solutions of the Laplace or Poisson equations were employed in potential problems with spherical boundaries or to develop multipole expansions of charge densities and their fields. We extend spherical harmonic expansions to the development of spherical wave solutions of the scalar wave equation for radiating sources.

A scalar field $\psi(\mathbf{r},t)$ satisfying the source-free wave equation is given by

$$\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = 0 \quad .$$

With the Fourier transform, the solution ψ can be expressed as

$$\psi(\mathbf{r},t) = \frac{1}{\sqrt{2\pi}}\int_0^\infty \Psi(\mathbf{r},\omega)e^{-i\omega t}d\omega \quad ,$$

$$\Psi(\mathbf{r},\omega) = \frac{1}{\sqrt{2\pi}}\int_0^\infty \psi(\mathbf{r},t)e^{i\omega t}dt \quad .$$

Substituting the Fourier components into the wave equation, we obtain

$$(\nabla^2 + k^2)\Psi(\mathbf{r},k) = 0 \quad ,$$

where $k = \omega/c$. For problems possessing symmetry properties about some origin, it is convenient to have fundamental solutions appropriate to spherical coordinates. The separation of the angular and radial variables follows the expansion

$$\Psi(\mathbf{r},k) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} A_{l,m}f_l(r)Y_{l,m}(\theta,\phi) \quad .$$

The radial functions $f_l(r)$ satisfy the radial equation, independent of m,

$$\left[\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr} - \frac{l(l+1)}{r^2} + k^2\right]f_l(r) = 0 \quad .$$

With the substitution,

$$f_l(r) = r^{-1/2}u_l(r) \quad ,$$

Eq. () is transformed into

$$\left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{(l+1/2)^2}{r^2} + k^2\right]u_l(r) = 0 \quad .$$

This equation is just the Bessel equation with $m = l + 1/2$. Thus the solution for $f_l(r)$ are

$$f_l(r) = A_l r^{-1/2}J_{l+1/2}(kr) + B_l r^{-1/2}N_{l+1/2}(kr) \quad .$$

The solutions are customarily expressed as spherical Bessel and Hankel functions, denoted by $j_l(x)$, $n_l(x)$, $h_l^{(1,2)}(x)$, as follows:

$$j_l(x) = \left(\frac{\pi}{2x}\right)^{1/2} J_{l+1/2}(x) \quad ,$$

$$n_l(x) = \left(\frac{\pi}{2x}\right)^{1/2} N_{l+1/2}(x) \quad ,$$

$$h_l^{(1,2)}(x) = \left(\frac{\pi}{2x}\right)^{1/2}\left[J_{l+1/2}(x) \pm iN_{l+1/2}(x)\right] \ .$$

It can be further shown that

$$j_l(x) = (-x)^l\left(\frac{1}{x}\frac{d}{dx}\right)^l\left(\frac{\sin x}{x}\right) \ ,$$

$$n_l(x) = -(-x)^l\left(\frac{1}{x}\frac{d}{dx}\right)^l\left(\frac{\cos x}{x}\right) \ .$$

For the first few values of l the explicit forms are given by

$$\begin{cases} j_0(x) = \dfrac{\sin x}{x} \\ n_0(x) = -\dfrac{\cos x}{x} \end{cases} \Rightarrow \quad h_0^{(1)}(x) = -i\frac{e^{ix}}{x} \ ,$$

$$\begin{cases} j_1(x) = \dfrac{\sin x}{x^2} - \dfrac{\cos x}{x} \\ n_1(x) = -\dfrac{\cos x}{x^2} - \dfrac{\sin x}{x} \end{cases} \Rightarrow \quad h_1^{(1)}(x) = -\frac{e^{ix}}{x}\left(1 + \frac{i}{x}\right) \ ,$$

$$\begin{cases} j_2(x) = \left(\dfrac{3}{x^3} - \dfrac{1}{x}\right)\sin x - \dfrac{3}{x^2}\cos x \\ n_2(x) = -\left(\dfrac{3}{x^3} - \dfrac{1}{x}\right)\cos x - \dfrac{3}{x^2}\sin x \end{cases} \Rightarrow \quad h_2^{(1)}(x) = i\frac{e^{ix}}{x}\left(1 + \frac{3i}{x} - \frac{3}{x^2}\right) \ .$$

From the series

$$J_m(x) = \sum_{j=0}^{\infty} \frac{(-1)^j}{\Gamma(j+m+1)j!}\left(\frac{x}{2}\right)^{2j+m} \tag{7}$$

$$J_{-m}(x) = \sum_{j=0}^{\infty} \frac{(-1)^j}{\Gamma(j-m+1)j!}\left(\frac{x}{2}\right)^{2j-m} , \tag{8}$$

the small argument limits are given by

$$j_l(x) \to \frac{x^l}{(2l+1)!!}\left(1 - \frac{x^2}{2(2l+3)} + \cdots\right) \ ,$$

$$n_l(x) \to -\frac{(2l-1)!!}{x^{l+1}}\left(1 - \frac{x^2}{2(1-2l)} + \cdots\right) \ .$$

Similarly the large argument limits are

$$j_l(x) \to \frac{1}{x}\sin\left(x - \frac{l\pi}{2}\right) \ ,$$

$$n_l(x) \to -\frac{1}{x}\cos\left(x - \frac{l\pi}{2}\right)$$

$$h_l^{(1)}(x) \to (-i)^{l+1}\frac{e^{ix}}{x} \quad .$$

The spherical Bessel functions satisfy the recursion formuals,

$$z_{l-1}(x) + z_{l+1}(x) = \frac{2l+1}{x} z_l(x) \quad ,$$

$$l z_{l-1}(x) - (l+1) z_{l+1}(x) = (2l+1) z_l'(x) \, .$$

The Wronskians of the various pairs are

$$W(j_l, n_l) = \frac{1}{i} W(j_l, h_l^{(1)}) = -W(n_l, h_l^{(1)}) = \frac{1}{x^2} \quad .$$

The general solution for the Helmholtz equation

$$(\nabla^2 + k^2)\Psi(\mathbf{r}, k) = 0$$

in spherical coordinates can be written

$$\Psi(\mathbf{r}, k) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left[A_{l,m} h_l^{(1)}(r) + B_{l,m} h_l^{(2)}(r) \right] Y_{l,m}(\theta, \phi) \quad .$$

where the coefficients $A_{l,m}$ and $B_{l,m}$ are determined by the boundary conditions.

The outgoing wave Green function $G_k(\mathbf{r}, \mathbf{r}')$, which is appropriate to the equation,

$$(\nabla^2 + k^2) G_k(\mathbf{r}, \mathbf{r}') = -4\pi\delta(\mathbf{r} - \mathbf{r}') \quad ,$$

is given by

$$G_k(\mathbf{r}, \mathbf{r}') = \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r} - \mathbf{r}'|} \quad .$$

The spherical wave expansion for $G_k(\mathbf{r}, \mathbf{r}')$ can be obtained in exactly the same way as was done for the Poisson equation. Thus the expansion of the Green function is

$$G_k(\mathbf{r}, \mathbf{r}') = \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r} - \mathbf{r}'|} = 4\pi i k \sum_{l=0}^{\infty} j_l(kr_<)\, h_l^{(1)}(kr_>) \sum_{m=-l}^{l} Y_{l\,m}^*(\theta', \phi')\, Y_{lm}(\theta, \phi)$$

The energy conservation law of electrodynamics: The Poynting vector

Considering the work dE_{mech} done by the electromagnetic fields on the charge dq contained in a small volume d^3x, moving through the field with velocity $\mathbf{v}$ when it is displaced through distance $d\boldsymbol{l}$, the work performed by the Lorentz force is given by

$$dE_{mech} = dq(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot d\vec{l} = dq(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \mathbf{v}dt$$

Replacing dq with ρd^3x, the total rate of doing work by the fields is a small volume V is

$$\frac{dE_{mech}}{dt} = \int_V \rho(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \mathbf{v}\, d^3x = \int_V \mathbf{J} \cdot \mathbf{E}\, d^3x$$

Using the Maxwell-Ampere law, we can obtain

$$\int_V \mathbf{J} \cdot \mathbf{E}\, d^3x = \int_V \left(\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} \right) \cdot \mathbf{E}\, d^3r .$$

From the Faraday's law, the above-mentioned equation can be modified as

$$\int_V \mathbf{J} \cdot \mathbf{E}\, d^3x = \int_V \left(\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} \right) \cdot \mathbf{E} - \left(\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} \right) \cdot \mathbf{H}\, d^3r$$

Note that since $\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$, the added term does not lead to any influence.

Using $\nabla \cdot (\mathbf{E} \times \mathbf{H}) = (\nabla \times \mathbf{E}) \cdot \mathbf{H} - (\nabla \times \mathbf{H}) \cdot \mathbf{E}$, we obtain

$$\int_V \mathbf{J} \cdot \mathbf{E}\, d^3x = -\int_V \left[\nabla \cdot (\mathbf{E} \times \mathbf{H}) + \mathbf{E} \cdot \left(\frac{\partial \mathbf{D}}{\partial t} \right) + \mathbf{H} \cdot \left(\frac{\partial \mathbf{B}}{\partial t} \right) \right] d^3r$$

Before proceeding to the further derivation, we review section 4.7 to understand the stored energy in the electric field. From the energy of a system of charges in free space, we can obtain

$$W_e = \frac{1}{2} \int \rho(\mathbf{r}) \Phi(\mathbf{r})\, d^3x .$$

The above-mentioned equation can be shown to be valid macroscopically only if the behavior is linear. To generalize the formula, we consider a small change in the energy δW_e due to some sort of change $d\rho$ in the macroscopic charge density ρ in all space. The work done for achieving this change is

$$\delta W_e = \int \delta\rho(\mathbf{r}) \Phi(\mathbf{r})\, d^3x ,$$

where $\Phi(\mathbf{r})$ is the potential due to the charge density ρ already present. With $\nabla \cdot \mathbf{D} = \rho$, it can be found that

$$\delta W_e = \int (\nabla \cdot \delta \mathbf{D}) \Phi(\mathbf{r})\, d^3x = \int \mathbf{E} \cdot \delta \mathbf{D}\, d^3x ,$$

where the relation $\mathbf{E} = -\nabla \Phi$ has been used. The total energy can be expressed as

$$W_e = \int (\nabla \cdot \delta \mathbf{D}) \Phi(\mathbf{r})\, d^3x = \int \int_0^D \mathbf{E} \cdot \delta \mathbf{D}\, d^3x .$$

If the medium is linear, then $\mathbf{E} \cdot \delta \mathbf{D} = \delta(\mathbf{E} \cdot \mathbf{D})/2$ and the total energy is given by

$$W_e = \frac{1}{2} \int \mathbf{D} \cdot \mathbf{E}\, d^3x = \frac{1}{2} \int \mathbf{D} \cdot (-\nabla \Phi(\mathbf{r}))\, d^3x = \frac{1}{2} \int (\nabla \cdot \mathbf{D}) \Phi(\mathbf{r})\, d^3x = \frac{1}{2} \int \rho(\mathbf{r}) \Phi(\mathbf{r})\, d^3x$$

Consequently, for a linear medium, the rate of change of the stored energy of the electric field

is given by

$$\frac{dW_e}{dt} = \int \frac{\partial \mathbf{D}}{\partial t} \cdot \mathbf{E}\, d^3x$$

Now, we review section 5.16 to understand the energy in the magnetic field. Considering a single circuit with a constant current I flowing in it, when the flux through the circuit changes, an electromotive force $\mathcal{E}$ is induced around it. To keep the current constant, the sources do work to maintain the current at the rate

$$\frac{dW_m}{dt} = -I\mathcal{E} = I\frac{d\Phi_B}{dt}$$

the negative sign following from Lenz's law. As a result, if the flux change through a circuit carrying a current I is $\delta\Phi_B$, the work done by the source is

$$\delta W_m = I\,\delta\Phi_B$$

To derive the energy in the loop, the current density distribution is broken up into element current loops. For each element loop, the increment of work done against the induced emf is

$$\Delta(\delta W_m) = (J\Delta\sigma)\int_S \delta\mathbf{B}\cdot\mathbf{n}\,da \ .$$

With $\mathbf{B} = \nabla\times\mathbf{A}$, we can obtain

$$\Delta(\delta W_m) = (J\Delta\sigma)\int_S (\nabla\times\delta\mathbf{A})\cdot\mathbf{n}\,da$$

Using the Stokes's theorem, this equation can be written as

$$\Delta(\delta W_m) = (J\Delta\sigma)\oint_C \delta\mathbf{A}\cdot d\mathbf{l} \ .$$

With $J\,\Delta\sigma\, d\mathbf{l} = \mathbf{J}\,d^3r$, the sum over all such element loops will be the volume integral. Hence the total increment of work done by the external sources due to a change $\delta\mathbf{A}(\mathbf{r})$ in the vector potential is

$$\delta W_m = \int \delta\mathbf{A}\cdot\mathbf{J}\,d^3x$$

Using Ampere's law $\nabla\times\mathbf{H} = \mathbf{J}$, this equation can be rewritten as

$$\delta W_m = \int \delta\mathbf{A}\cdot\nabla\times\mathbf{H}\,d^3x = \int \mathbf{H}\cdot\nabla\times\delta\mathbf{A}\,d^3x = \int \mathbf{H}\cdot\delta\mathbf{B}\,d^3x$$

This relation is the magnetic equivalent of the electrostatic equation

$$\delta W_e = \int(\nabla\cdot\delta\mathbf{D})\Phi(\mathbf{r})\,d^3x = \int \mathbf{E}\cdot\delta\mathbf{D}\,d^3x \,.$$

In its present form it is applicable to all magnetic media, including ferromagnetic substances. When the medium is para- or diamagnetic, so that a linear relation exists between **H** and **B**, then $\mathbf{H}\cdot\delta\mathbf{B}=\delta(\mathbf{H}\cdot\mathbf{B})/2$. When the fields are increased from zero to their final values, the total magnetic energy is

$$W_m=\frac{1}{2}\int \mathbf{H}\cdot\mathbf{B}\,d^3x$$

Consequently, the total energy density for the linear media can be denoted by

$$u=\frac{1}{2}(\mathbf{E}\cdot\mathbf{D}+\mathbf{H}\cdot\mathbf{B}).$$

In terms of u, the following equation

$$\int_V \mathbf{J}\cdot\mathbf{E}\,d^3x=-\int_V\left[\nabla\cdot(\mathbf{E}\times\mathbf{H})+\mathbf{E}\cdot\left(\frac{\partial\mathbf{D}}{\partial t}\right)+\mathbf{H}\cdot\left(\frac{\partial\mathbf{B}}{\partial t}\right)\right]d^3r$$

can be expressed as

$$\int_V \mathbf{J}\cdot\mathbf{E}\,d^3x=-\int_V\left[\nabla\cdot(\mathbf{E}\times\mathbf{H})+\frac{\partial u}{\partial t}\right]d^3r$$

Since the volume V is arbitrary, this can be cast into the form of a differential continuity equation or conservation law,

$$-\mathbf{J}\cdot\mathbf{E}=\nabla\cdot\mathbf{S}+\frac{\partial u}{\partial t}$$

where the Poynting vector **S**, representing energy flow, is given by

$$\mathbf{S}=\mathbf{E}\times\mathbf{H}.$$

The physical meaning of Poynting's theorem is that the time rate of change of electromagnetic energy within a certain volume, plus the energy flowing out through the boundary surfaces of the volume per unit time, is equal to the negative of the total work done by the fields on the sources within the volume. Then Poynting's theorem expresses the conservation of energy for the combined system as

$$\frac{dE}{dt}=\frac{dE_{mech}}{dt}+\frac{dE_{field}}{dt}=-\int_V\nabla\cdot(\mathbf{E}\times\mathbf{H})\,d^3r=-\oint_S \mathbf{S}\cdot\mathbf{n}\,da,$$

where the total field energy within V is

$$E_{field}=\int_V u\,d^3r=\frac{\varepsilon_o}{2}\int_V(\mathbf{E}^2+c^2\mathbf{B}^2)\,d^3r$$

The momentum conservation law: The Maxwell stress tensor

For the conservation of linear momentum, we consider the sum of all the momenta of all the particles in the volume $\mathbf{P}_{mech}$. With Newton's second law and $\mathbf{F}=q(\mathbf{E}+\mathbf{v}\times\mathbf{B})$, we have

$$\frac{d\mathbf{P}_{mech}}{dt}=\int_V(\rho\mathbf{E}+\mathbf{J}\times\mathbf{B})\,d^3r$$

Using the Maxwell equations, the integrand can be expressed as

$$\rho \mathbf{E} + \mathbf{J} \times \mathbf{B} = \varepsilon_o \mathbf{E} \nabla \cdot \mathbf{E} + \left(\nabla \times \frac{\mathbf{B}}{\mu_o} - \varepsilon_o \frac{\partial \mathbf{E}}{\partial t} \right) \times \mathbf{B}$$

$$\rho \mathbf{E} + \mathbf{J} \times \mathbf{B} = \varepsilon_o \mathbf{E} \nabla \cdot \mathbf{E} + \frac{\mathbf{B}}{\mu_o} \nabla \cdot \mathbf{B} + \left(\nabla \times \frac{\mathbf{B}}{\mu_o} - \varepsilon_o \frac{\partial \mathbf{E}}{\partial t} \right) \times \mathbf{B} + \varepsilon_o \left(\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} \right) \times \mathbf{E}$$

$$\rho \mathbf{E} + \mathbf{J} \times \mathbf{B} = \varepsilon_o \left(\mathbf{E} \nabla \cdot \mathbf{E} - \mathbf{E} \times \nabla \times \mathbf{E} + c^2 \mathbf{B} \nabla \cdot \mathbf{B} - c^2 \mathbf{B} \times \nabla \times \mathbf{B} \right) - \frac{\partial}{\partial t} \frac{1}{c^2} \left(\mathbf{E} \times \mathbf{H} \right)$$

The rate of change of mechanical momentum can now be expressed as

$$\frac{d\mathbf{P}_{mech}}{dt} + \frac{d\mathbf{P}_{field}}{dt} = \int_V \varepsilon_o \left(\mathbf{E} \nabla \cdot \mathbf{E} - \mathbf{E} \times \nabla \times \mathbf{E} + c^2 \mathbf{B} \nabla \cdot \mathbf{B} - c^2 \mathbf{B} \times \nabla \times \mathbf{B} \right) d^3 r$$

where

$$\mathbf{P}_{field} = \int_V \frac{1}{c^2} \left(\mathbf{E} \times \mathbf{H} \right) d^3 r$$

The integrand can be interpreted as a density of electromagnetic momentum, i.e.,

$$\mathbf{g} = \frac{1}{c^2} \left(\mathbf{E} \times \mathbf{H} \right)$$

Note that the momentum density **g** is proportional to the energy-flux density **S**, with proportionality constant c^{-2}. For discussing the momentum flow, we consider the expression

$$\begin{aligned} \left[\mathbf{E} \nabla \cdot \mathbf{E} - \mathbf{E} \times \nabla \times \mathbf{E} \right]_x &= E_x \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \right) - E_y \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right) + E_z \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} \right) \\ &= \frac{\partial}{\partial x}(E_x E_x) + \frac{\partial}{\partial y}(E_x E_y) + \frac{\partial}{\partial z}(E_x E_z) - \frac{1}{2} \frac{\partial}{\partial x} \left(\mathbf{E} \cdot \mathbf{E} \right) \end{aligned}$$

Consequently, we can write the αth component as

$$\left[\mathbf{E} \nabla \cdot \mathbf{E} - \mathbf{E} \times \nabla \times \mathbf{E} \right]_\alpha = \sum_\beta \frac{\partial}{\partial x_\beta} \left(E_\alpha E_\beta - \frac{1}{2} \left(\mathbf{E} \cdot \mathbf{E} \right) \delta_{\alpha\beta} \right)$$

A completely analogous result for the magnetic field is given by

$$\left[\mathbf{B} \nabla \cdot \mathbf{B} - \mathbf{B} \times \nabla \times \mathbf{B} \right]_\alpha = \sum_\beta \frac{\partial}{\partial x_\beta} \left(B_\alpha B_\beta - \frac{1}{2} \left(\mathbf{B} \cdot \mathbf{B} \right) \delta_{\alpha\beta} \right)$$

With these relations, the integrand in the equation of the rate of change of mechanical momentum can be expressed as the divergence of a tensor

$$\left(\frac{d\mathbf{P}_{mech}}{dt} + \frac{d\mathbf{P}_{field}}{dt} \right)_\alpha = \int_V \varepsilon_o \left[\sum_\beta \frac{\partial}{\partial x_\beta} \left(E_\alpha E_\beta + c^2 B_\alpha B_\beta - \frac{1}{2} \left(\mathbf{E} \cdot \mathbf{E} + c^2 \mathbf{B} \cdot \mathbf{B} \right) \delta_{\alpha\beta} \right) \right] d^3 r$$

With the definition of the Maxwell stress tensor $T_{\alpha\beta}$ as

$$T_{\alpha\beta} = \varepsilon_o \left(E_\alpha E_\beta + c^2 B_\alpha B_\beta - \frac{1}{2} \left(\mathbf{E} \cdot \mathbf{E} + c^2 \mathbf{B} \cdot \mathbf{B} \right) \delta_{\alpha\beta} \right),$$

we can have

$$\left(\frac{d\mathbf{P}_{mech}}{dt}+\frac{d\mathbf{P}_{field}}{dt}\right)_{\alpha}=\int_V \sum_{\beta}\frac{\partial}{\partial x_{\beta}}T_{\alpha\beta}\; d^3r=\oint_S \sum_{\beta}T_{\alpha\beta}\, n_{\beta}\; da$$

where n_β means the βth component of the unit normal vector (direction cosine) perpendicular to the area da. Evidently, the term $\sum_{\beta}T_{\alpha\beta}\, n_{\beta}$ is the αth component of the flow per unit area of momentum across the surface S into the volume V. In other words, it is the force per unit area transmitted across the surface S and acting on the combined system of particles and fields inside V. Therefore, it can be used to calculate the forces acting on material objects in electromagnetic fields by enclosing the objects with a boundary surface S. Explicitly, the Maxwell stress tensor can be expressed in matrix form as

$$\begin{bmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{bmatrix}=\varepsilon_o\begin{bmatrix} E_x^2+c^2B_x^2-u & E_xE_y+c^2B_xB_y & E_xE_z+c^2B_xB_z \\ E_yE_x+c^2B_yB_x & E_y^2+c^2B_y^2-u & E_yE_z+c^2B_yB_z \\ E_zE_x+c^2B_zB_x & E_zE_y+c^2B_zB_y & E_z^2+c^2B_z^2-u \end{bmatrix}$$

The trace of the Maxwell stress tensor Tr(T) is equal to the negative of the electromagnetic energy density, -u.

A similar result can be obtained for the angular momentum of the fields. Starting from the definition of the mechanical torque

$$\boldsymbol{\tau}=\frac{d\,\mathbf{L}_{mech}}{dt}=\int_V \mathbf{r}\times\left(\rho\mathbf{E}+\mathbf{J}\times\mathbf{B}\right)d^3r$$

With the previous result of

$$\mathbf{f}=\rho\mathbf{E}+\mathbf{J}\times\mathbf{B}=-\frac{\partial\mathbf{g}}{\partial t}+\nabla\cdot\vec{\mathbf{T}},$$

we can obtain

$$\boldsymbol{\tau}=\frac{d\,\mathbf{L}_{mech}}{dt}=\int_V \mathbf{r}\times\left(-\frac{\partial\mathbf{g}}{\partial t}+\nabla\cdot\vec{\mathbf{T}}\right)d^3r$$

Defining

$$\mathbf{L}_{field}=\int_V \mathscr{L}_{field}\; d^3r=\int_V \mathbf{r}\times\mathbf{g}\; d^3r$$

we can get

$$\frac{d\,\mathbf{L}_{mech}}{dt}+\frac{d\,\mathbf{L}_{field}}{dt}=\int_V \mathbf{r}\times\left(\nabla\cdot\vec{\mathbf{T}}\right)d^3r$$

Since $\mathbf{r}\times(\nabla\cdot\vec{\mathbf{T}}) = -\nabla\cdot(\vec{\mathbf{T}}\times\mathbf{r})$, we can define $\vec{\mathbf{M}} = (\vec{\mathbf{T}}\times\mathbf{r})$ and $\mathbf{L}_{mech} = \int_V \mathscr{L}_{mech}\, d^3r$ to

obtain

$$\frac{\partial \mathscr{L}_{mech}}{\partial t} + \frac{\partial \mathscr{L}_{field}}{\partial t} + \nabla\cdot\vec{\mathbf{M}} = 0$$

Poynting's theorem for Harmonic fields

For harmonic time variation of the fields, all fields and sources are assumed to be have a time dependence $e^{-i\omega t}$; therefore we write

$$\mathbf{E}(\vec{\mathbf{r}},t) = \mathrm{Re}\left[\mathbf{E}(\vec{\mathbf{r}})e^{-i\omega t}\right] = \frac{1}{2}\left[\mathbf{E}(\vec{\mathbf{r}})e^{-i\omega t} + \mathbf{E}^*(\vec{\mathbf{r}})e^{i\omega t}\right]$$

The field $\mathbf{E}(\vec{\mathbf{r}})$ is in general complex, with a magnitude and phase that change with position. For product forms, such as $\mathbf{J}(\vec{\mathbf{r}},t)\cdot\mathbf{E}(\vec{\mathbf{r}},t)$, we have

$$\begin{aligned}\mathbf{J}(\vec{\mathbf{r}},t)\cdot\mathbf{E}(\vec{\mathbf{r}},t) &= \frac{1}{4}\left[\mathbf{J}(\vec{\mathbf{r}})e^{-i\omega t} + \mathbf{J}^*(\vec{\mathbf{r}})e^{i\omega t}\right]\left[\mathbf{E}(\vec{\mathbf{r}})e^{-i\omega t} + \mathbf{E}^*(\vec{\mathbf{r}})e^{i\omega t}\right] \\ &= \frac{1}{2}\mathrm{Re}\left[\mathbf{J}^*(\vec{\mathbf{r}})\mathbf{E}(\vec{\mathbf{r}}) + \mathbf{J}(\vec{\mathbf{r}})\mathbf{E}(\vec{\mathbf{r}})e^{-2i\omega t}\right]\end{aligned}.$$

For time averages of products, the convention is therefore to take one-half of the real part of the product of one complex quantity with the complex conjugate of the other. For harmonic fields, the Maxwell equations become

$$\nabla\cdot\mathbf{B} = 0 \quad ; \quad \nabla\times\mathbf{E} - i\omega\mathbf{B} = 0$$
$$\nabla\cdot\mathbf{D} = \rho \quad ; \quad \nabla\times\mathbf{H} + i\omega\mathbf{D} = \mathbf{J}\ ,$$

where all quantities are complex functions of **r**, according to the form $\mathbf{E}(\vec{\mathbf{r}},t) = \mathrm{Re}\left[\mathbf{E}(\vec{\mathbf{r}})e^{-i\omega t}\right]$.

Now considering the average of the product $\int_V \mathbf{J}^*\cdot\mathbf{E}\, d^3x$ and using the time-harmonic

Maxwell equations, we obtain

$$\frac{1}{2}\int_V \mathbf{J}^*\cdot\mathbf{E}\, d^3x = \frac{1}{2}\int_V \left(\nabla\times\mathbf{H}^* - i\omega\mathbf{D}^*\right)\cdot\mathbf{E}\, d^3r\,.$$

Adding a null term $(\nabla\times\mathbf{E} - i\omega\mathbf{B})\cdot\mathbf{H}^* = 0$ to the integrand of the right-hand side of the equation, we obtain

$$\begin{aligned}\frac{1}{2}\int_V \mathbf{J}^*\cdot\mathbf{E}\, d^3x &= \frac{1}{2}\int_V \left(\nabla\times\mathbf{H}^* - i\omega\mathbf{D}^*\right)\cdot\mathbf{E} - (\nabla\times\mathbf{E} - i\omega\mathbf{B})\cdot\mathbf{H}^*\, d^3r \\ &= -\frac{1}{2}\int_V \nabla\cdot(\mathbf{E}\times\mathbf{H}^*) + i\omega(\mathbf{E}\cdot\mathbf{D}^* - \mathbf{B}\cdot\mathbf{H}^*)\, d^3r\end{aligned}$$

We now define the complex Poynting vector

$$\mathbf{S} = \frac{1}{2}\left(\mathbf{E} \times \mathbf{H}^*\right)$$

and the harmonic electric and magnetic energy densities

$$w_e = \frac{1}{4}\left(\mathbf{E} \cdot \mathbf{D}^*\right),\ w_m = \frac{1}{4}\left(\mathbf{B} \cdot \mathbf{H}^*\right).$$

Then we can obtain

$$\frac{1}{2}\int_V \mathbf{J}^* \cdot \mathbf{E}\, d^3x + 2i\omega \int_V \;(w_e - w_m)\, d^3r + \oint_S \mathbf{S} \cdot \mathbf{n}\, da = 0$$

It is a complex equation whose real part gives the conservation of energy for the time-averaged quantities and whose imaginary part relates to the reactive or stored energy and its alternating flow. If the energy densities w_e and w_m have real volume integrals, as occurs for systems with lossless dielectrics and perfect conductors, the real part of the equation becomes

$$\frac{1}{2}\int_V \mathrm{Re}(\mathbf{J}^* \cdot \mathbf{E})\, d^3x + \oint_S \mathrm{Re}(\mathbf{S} \cdot \mathbf{n})\, da = 0\,.$$

This result shows that the steady-state, time-averaged rate of doing work on the sources in V by the fields is equal to the average flow of power into the volume V through the boundary surface S, as calculated from the normal component of Re(**S**).

The question of magnetic monopoles

So far, there is no experimental evidence for the existence of magnetic charges or monopoles. Nevertheless, there is a great interest in magnetic monopoles, because on the one hand, electrodynamics would become very symmetric in it structure and on the other hand, the mere existence of a simple monopole would lead necessarily to the quantization of the electric charge. The quantization of the electric charge in units of the elementary charge e (the charge of the electron) is a great mystery of physics, and its relationship to the existence of magnetic monopoles was pointed out for the first time by Dirac (Phy. Rev. 74, 817, 1984). Let us suppose that there exist magnetic charge and current densities, $_m$ and $\mathbf{J}_m$, in addition to the electric densities, $_e$ and $\mathbf{J}_e$. The Maxwell equations are then given by

$$\nabla \cdot \mathbf{D} = \rho_e \quad ; \quad \nabla \times \mathbf{H} = \mathbf{J}_e + \frac{\partial \mathbf{D}}{\partial t}$$

$$\nabla \cdot \mathbf{B} = \rho_m \quad ; \quad -\nabla \times \mathbf{E} = \mathbf{J}_m + \frac{\partial \mathbf{B}}{\partial t}$$

where there are two continuity equations

$$\nabla \cdot \mathbf{J}_e + \frac{\partial \rho_e}{\partial t} = 0 \quad ; \quad \nabla \cdot \mathbf{J}_m + \frac{\partial \rho_m}{\partial t} = 0$$

In other words, the charge conservation holds for both kinds of charges. Although one might expect that the new Maxwell equations can have some physical effects different from those of the known, this is not the case. Consider the following duality transformation:

$$\mathbf{E} = \mathbf{E}'\cos\xi + Z_o\mathbf{H}'\sin\xi \quad ; \quad Z_o\mathbf{D} = Z_o\mathbf{D}'\cos\xi + \mathbf{B}'\sin\xi$$
$$Z_o\mathbf{H} = -\mathbf{E}'\sin\xi + Z_o\mathbf{H}'\cos\xi \quad ; \quad \mathbf{B} = -Z_o\mathbf{D}'\sin\xi + \mathbf{B}'\cos\xi \, ,$$

For a real angle , such a transformation leaves quadratic forms like $\mathbf{E}\times\mathbf{H}$, $\mathbf{E}\cdot\mathbf{D}+\mathbf{H}\cdot\mathbf{B}$, and the components of the Maxwell stress tensor T_{uv} to be invariant. If the sources are transformed in the same way,

$$Z_o\rho_e = Z_o\rho_e'\cos\xi + \rho_m'\sin\xi \quad ; \quad Z_o\mathbf{J}_e = Z_o\mathbf{J}_e'\cos\xi + \mathbf{J}_m'\sin\xi$$
$$\rho_m = -Z_o\rho_e'\sin\xi + \rho_m'\cos\xi \quad ; \quad \mathbf{J}_m = -Z_o\mathbf{J}_e'\sin\xi + \mathbf{J}_m'\cos\xi$$

one can find that the Maxwell equations are valid unchanged for $(\mathbf{E}',\mathbf{B}',\mathbf{D}',\mathbf{H}')$.

$$\nabla\cdot\mathbf{D}' = \rho_e' \quad ; \quad \nabla\times\mathbf{H}' = \mathbf{J}_e' + \frac{\partial\mathbf{D}'}{\partial t}$$
$$\nabla\cdot\mathbf{B}' = \rho_m' \quad ; \quad -\nabla\times\mathbf{E}' = \mathbf{J}_m' + \frac{\partial\mathbf{B}'}{\partial t}$$

This demonstrates the invariance of the new Maxwell equations mentioned above under dual transformations. This also indicates that to a great extent it is a question of convention what one calls the magnetic charge of a particle. If all particles had the same ratio of magnetic to electric charge, the angle could be chosen such that

$$\rho_m = Z_o\rho_e'\left(-\sin\xi + \frac{\rho_m'}{Z_o\rho_e'}\cos\xi\right) = 0$$

$$\mathbf{J}_m = Z_o\mathbf{J}_e'\left(-\sin\xi + \frac{\mathbf{J}_m'}{Z_o\mathbf{J}_e'}\cos\xi\right) = Z_o\mathbf{J}_e'\left(-\sin\xi + \frac{\rho_m'}{Z_o\rho_e'}\cos\xi\right) = 0$$

For this special angle , the new Maxwell equations turn into the old, well-known Maxwell equations. Then, one would fix the charge of the electron, by convention: q_e(electron)= e and q_m(electron)=0. The charges of all other elementary particles could then be measured with these units. For example, for the proton on finds with a precision of ~10^{-20}: q_e(proton)= e and q_m(proton)=0. This high precision for the vanishing magnetic charge is based on the fact that the magnetic field intensity of the earth at its surface amounts to 1 gauss only. Otherwise, it would have to be much larger if q_m(proton)≠0

Now, we discuss the Dirac's idea on the relation between charge quantization and the existence of magnetic monopoles. Let a magnetic charge *g* at the origin produce the magnetic induction

$$\mathbf{B}(\mathbf{r}) = \frac{g}{4\pi}\frac{\mathbf{r}}{r^3} .$$

at the point **r**. The $\mathbf{B}(\mathbf{r})$ field follows from $\nabla \cdot \mathbf{B} = \rho_m$ if $\rho_m = g\delta(r)$ is a point-like density. Let a particle with the charge *e* and the velocity $\mathbf{v} = v\,\mathbf{e}_z$ pass by along a straight line with an impact parameter *b*. This particle experiences the Lorentz force

$$F_y = ev\frac{g}{4\pi}\frac{b}{\left[b^2 + (vt)^2\right]^{3/2}} ,$$

The momentum transmitted by this force is

$$\Delta p_y = \int_{-\infty}^{\infty} ev\frac{g}{4\pi}\frac{b}{\left[b^2 + (vt)^2\right]^{3/2}} = \frac{eg}{2\pi b} .$$

This momentum transfer leads to a change in the angular momentum

$$\Delta L_z = b\,\Delta p_y = \frac{eg}{2\pi} .$$

Now, according to quantum mechanics, the orbital angular momentum is always quantized: $L_z = n\hbar$, $n = 0, \pm 1, \pm 2, \cdots$,

Consequently,

$$\frac{eg}{4\pi\hbar} = \frac{\alpha g}{Z_o e} = \frac{n}{2}$$

$$\nabla \cdot \mathbf{B} = 0 \quad ; \quad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$$

$$\nabla \cdot \mathbf{D} = \rho \quad ; \quad \nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

$$\mathbf{D} = \varepsilon_o \mathbf{E} + \mathbf{P} \quad ; \quad \mathbf{H} = \frac{\mathbf{B}}{\mu_o} - \mathbf{M}$$

$$\nabla \cdot \mathbf{b} = 0 \quad ; \quad \nabla \times \mathbf{e} + \frac{\partial \mathbf{b}}{\partial t} = 0$$

$$\nabla \cdot \mathbf{e} = \eta / \varepsilon_o \quad ; \quad \nabla \times \mathbf{b} - \frac{1}{c^2}\frac{\partial \mathbf{e}}{\partial t} = \mu_o \mathbf{j}$$

The spatial average of a function F with respect to a test function f is given by $\langle F(\vec{r},t)\rangle = \int f(\vec{r}\,')F(\vec{r}-\vec{r}\,',t)\, d^3r'$. Based on the definition, it can be shown that

$$\frac{\partial \langle F(\vec{r},t)\rangle}{\partial x_i} = \int f(\vec{r}\,')\frac{\partial F(\vec{r}-\vec{r}\,',t)}{\partial x_i}\, d^3r' = \left\langle \frac{\partial F}{\partial x_i}\right\rangle$$

$$\frac{\partial \langle F(\vec{r},t)\rangle}{\partial t} = \int f(\vec{r}\,')\frac{\partial F(\vec{r}-\vec{r}\,',t)}{\partial t}\, d^3r' = \left\langle \frac{\partial F}{\partial t}\right\rangle$$

With the property that the operations of space and time differentiation can commute with average operation, we can obtain

$$\nabla\cdot\langle \mathbf{b}\rangle = 0 \quad ; \quad \nabla\times\langle \mathbf{e}\rangle + \frac{\partial \langle \mathbf{b}\rangle}{\partial t} = 0$$

$$\nabla\cdot\langle \mathbf{e}\rangle = \langle \eta\rangle/\varepsilon_o \quad ; \quad \nabla\times\langle \mathbf{b}\rangle - \frac{1}{c^2}\frac{\partial \langle \mathbf{e}\rangle}{\partial t} = \mu_o\,\langle \mathbf{j}\rangle$$

Therefore, $\langle \mathbf{b}\rangle = \mathbf{B}$, $\langle \mathbf{e}\rangle = \mathbf{E}$.

$$\eta\,(\vec{r},t) = \sum_j q_j\ \delta[\vec{r}-\vec{r}_j(t)]$$

$$\eta(\vec{r},t) = \eta_{free}(\vec{r},t) + \eta_{bound}(\vec{r},t)$$

$$\eta_{free}(\vec{r},t) = \sum_{j(free)} q_j\ \delta(\vec{r}-\vec{r}_j)\,;\, \eta_{bound}(\vec{r},t) = \sum_{n(molecules)} \eta_n(\vec{r},t)$$

$$\eta_n(\vec{r},t) = \sum_{j(n)} q_j\ \delta(\vec{r}-\vec{r}_n-\vec{r}_{j\,n})$$

$$\langle \eta_n(\vec{r},t)\rangle = \int \sum_{j(n)} q_j\ \delta(\vec{r}-\vec{r}\,'-\vec{r}_n-\vec{r}_{j\,n}) f(\vec{r}\,')\, d^3r' = \sum_{j(n)} q_j\ f(\vec{r}-\vec{r}_n-\vec{r}_{j\,n})$$

$$f(\vec{r}-\vec{r}_n-\vec{r}_{j\,n}) = f(\vec{r}-\vec{r}_n) - \vec{r}_{j\,n}\cdot\nabla f(\vec{r}-\vec{r}_n) + \frac{1}{2}\sum_{\alpha\beta}\left(\vec{r}_{j\,n}\right)_\alpha\left(\vec{r}_{j\,n}\right)_\beta \frac{\partial^2 f(\vec{r}-\vec{r}_n)}{\partial r_\alpha \partial r_\beta} + \cdots$$

$$q_n = \sum_{j(n)} q_j\,;\quad \vec{p}_n = \sum_{j(n)} q_j \vec{r}_{j\,n}\,;\quad \left(Q_n\right)_{\alpha\beta} = 3\sum_{j(n)} q_j\left(\vec{r}_{j\,n}\right)_\alpha\left(\vec{r}_{j\,n}\right)_\beta$$

$$\langle \eta_n(\vec{r},t)\rangle = q_n f(\vec{r}-\vec{r}_n) - \vec{p}_n\cdot\nabla f(\vec{r}-\vec{r}_n) + \frac{1}{6}\sum_{\alpha\beta}\left(Q_n\right)_{\alpha\beta}\frac{\partial^2 f(\vec{r}-\vec{r}_n)}{\partial r_\alpha \partial r_\beta} + \cdots$$

$$\langle \eta_n(\vec{r},t)\rangle = \langle q_n\delta(\vec{r}-\vec{r}_n)\rangle - \nabla\cdot\langle \vec{p}_n\delta(\vec{r}-\vec{r}_n)\rangle + \frac{1}{6}\sum_{\alpha\beta}\frac{\partial^2}{\partial r_\alpha \partial r_\beta}\left\langle \left(Q_n\right)_{\alpha\beta}\delta(\vec{r}-\vec{r}_n)\right\rangle + \cdots$$

As far as the result of the averaging process is concerned, the molecule can be viewed as a collection of point multipoles located at one fixed point in the molecule. The detailed extent of the molecular charge distribution is important at the microscopic level, of course, but is replaced in its effect by a sum of multipoles for macroscopic phenomena.

$$\langle \eta(\vec{r},t) \rangle = \rho(\vec{r},t) - \nabla \cdot \mathbf{P}(\vec{r},t) + \sum_{\alpha\beta} \frac{\partial^2}{\partial r_\alpha \partial r_\beta} Q_{\alpha\beta}(\vec{r},t) + \cdots$$

where is macroscopic charge density

$$\rho(\vec{r},t) = \left\langle \sum_{j(free)} q_j \ \delta(\vec{r} - \vec{r}_j) + \sum_{n(molecules)} q_n \ \delta(\vec{r} - \vec{r}_n) \right\rangle,$$

$$\mathbf{P}(\vec{r},t) = \left\langle \sum_{n(molecules)} \mathbf{p}_n \ \delta(\vec{r} - \vec{r}_n) \right\rangle$$

$$Q_{\alpha\beta}(\vec{r},t) = \frac{1}{6} \left\langle \sum_{n(molecules)} (Q_n)_{\alpha\beta} \ \delta(\vec{r} - \vec{r}_n) \right\rangle$$

$$\sum_\alpha \frac{\partial}{\partial r_\alpha} \left[\varepsilon_o E_\alpha + P_\alpha - \frac{\partial}{\partial r_\beta} Q_{\alpha\beta} \right] = \rho$$

Macroscopic displacement vector is defined as

$$D_\alpha = \varepsilon_o E_\alpha + P_\alpha - \frac{\partial}{\partial r_\beta} Q_{\alpha\beta}$$

Chapter Eight: Plane Electromagnetic Waves and Propagation

Plane wave in uniform materials

To separate **E** and **B** from Maxwell's equations, we employ the usual trick by taking curl of the curl equations:

$$\nabla\times\nabla\times\mathbf{E} = -\frac{\partial}{\partial t}(\nabla\times\mathbf{B}) = -\mu\frac{\partial}{\partial t}\left(\mathbf{J}+\frac{\partial\mathbf{D}}{\partial t}\right) \quad . \tag{1}$$

Using the constitutive relations for linear materials with $\mathbf{J}=\sigma\mathbf{E}$ and $\mathbf{D}=\varepsilon\mathbf{E}$, Eq. (1) can be rewritten as

$$\nabla(\nabla\cdot\mathbf{E}) - \nabla^2\mathbf{E} = -\mu\frac{\partial}{\partial t}\left(\sigma\mathbf{E}+\varepsilon\frac{\partial\mathbf{E}}{\partial t}\right) \quad . \tag{2}$$

In the absence of free charges, $\nabla\cdot\mathbf{E}=0$, and assuming time harmonic varying fields $\mathbf{E}=\mathbf{E}_o(\mathbf{r})e^{-i\omega t}$, the derivative $\partial/\partial t$ can be replaced with $-i\omega$ to obtain

$$\nabla^2\mathbf{E}_o + \mu\varepsilon\omega^2\left(1+\frac{i\sigma}{\omega\varepsilon}\right)\mathbf{E}_o = 0 \quad . \tag{3}$$

Following the same steps, we can derive the same equation for **H**:

$$\nabla^2\mathbf{H}_o + \mu\varepsilon\omega^2\left(1+\frac{i\sigma}{\omega\varepsilon}\right)\mathbf{H}_o = 0 \quad . \tag{4}$$

Plane wave in linear isotropic conductors

In linear isotropic conductors, the fields satisfy Eqs. (3) and (4) together with the constitutive relations $\mathbf{D}=\varepsilon\,\mathbf{E}$, $\mathbf{B}=\mu\,\mathbf{H}$, and $\mathbf{J}=\sigma\,\mathbf{E}$. The solutions can be expressed as the form of a damped plane wave

$$\mathbf{E} = \mathbf{E}_o e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \quad , \tag{5}$$

$$\mathbf{H} = \mathbf{H}_o e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \quad . \tag{6}$$

where the wave vector **k** is a complex vector with the dispersion relation

$$k^2 = \mu\varepsilon\omega^2\left(1+\frac{i\sigma}{\omega\varepsilon}\right) \quad . \tag{7}$$

We set the amplitude of the complex wave vector to be

$$k = \beta + i\frac{\alpha}{2} \quad . \tag{8}$$

where the parameter β is related to the effective wavelength for an electromagnetic wave propagating through the conductor and the parameter α is known as the attenuation constant or absorption coefficient. Then, after some algebra, we can show that

$$\beta = \frac{\sqrt{\mu\varepsilon}\,\omega}{\sqrt{2}}\sqrt{\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}+1} \quad , \tag{9}$$

$$\frac{\alpha}{2} = \frac{\sqrt{\mu\varepsilon}\,\omega}{\sqrt{2}}\sqrt{\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}-1} \quad . \tag{10}$$

For good conductors, $\sigma >> \omega\varepsilon$, the parameters α and β can be given by

$$\beta = \frac{\sqrt{\mu\sigma\omega}}{\sqrt{2}} \quad , \tag{11}$$

$$\alpha = \sqrt{2\mu\sigma\omega} \quad . \tag{12}$$

Substituting Eq. (8) into the solutions in Eqs. (5) and (6), we obtain

$$\mathbf{E} = \mathbf{E}_o\, e^{-\boldsymbol{\alpha}\cdot\mathbf{r}/2}\, e^{i(\boldsymbol{\beta}\cdot\mathbf{r}-\omega t)} \quad , \tag{13}$$

$$\mathbf{H} = \mathbf{H}_o\, e^{-\boldsymbol{\alpha}\cdot\mathbf{r}/2}\, e^{i(\boldsymbol{\beta}\cdot\mathbf{r}-\omega t)} \quad . \tag{14}$$

The wave decays exponentially along its path. The penetration depth known as the skin depth is given by $\delta = 2/\alpha$. At microwave frequencies (10 GHz), the skin depth can be approximately given by $\delta = \sqrt{2/\mu\sigma\omega} = 0.92\ \mu\text{m}$, where $\sigma = 3\times10^7\ \Omega^{-1}\text{m}^{-1}$. Therefore, constructing microwave components of silver is usually to thinly electroplate them with silver to a thickness of several skin depths.

The electric and magnetic fields are not independent but are related by Maxwell's equations. The divergence equations lead the plane waves to be subject to

$$\mathbf{k}\cdot\mathbf{E}_o = 0, \tag{15}$$

$$\mathbf{k}\cdot\mathbf{H}_o = 0. \tag{16}$$

These two equations imply that the wave vector **k** is perpendicular to each of **E** and **H**. The curl equations give

$$\mathbf{k}\times\mathbf{E}_o = \omega\mu\,\mathbf{H}_o, \tag{17}$$

$$\mathbf{k}\times\mathbf{H}_o = -\omega\varepsilon\left(1+\frac{i\sigma}{\omega\varepsilon}\right)\mathbf{E}_o. \tag{18}$$

In terms of **E**, we can take **H** as the form

$$\mathbf{H} = \frac{\mathbf{k}\times\mathbf{E}_o}{\mu\omega}\, e^{-\boldsymbol{\alpha}\cdot\mathbf{r}/2}\, e^{i(\boldsymbol{\beta}\cdot\mathbf{r}-\omega t)} \quad . \tag{19}$$

The time averaged Poynting vector can be evaluated from the complex fields as

$$\begin{aligned}\langle\mathbf{S}\rangle &= \mathrm{Re}\left(\frac{\mathbf{E}\times\mathbf{H}^*}{2}\right) = \frac{\mathbf{E}\times\mathbf{H}^* + \mathbf{E}^*\times\mathbf{H}}{4} \\ &= \frac{1}{4}\left[\frac{\mathbf{E}_o\times(\mathbf{k}^*\times\mathbf{E}_o^*)}{\mu\omega} + \frac{\mathbf{E}_o^*\times(\mathbf{k}\times\mathbf{E}_o)}{\mu\omega}\right] e^{-\boldsymbol{\alpha}\cdot\mathbf{r}} = \frac{1}{2}|E_o|^2\,\frac{\mathrm{Re}(\mathbf{k})}{\mu\omega}\, e^{-\boldsymbol{\alpha}\cdot\mathbf{r}}\end{aligned} \quad . \tag{20}$$

Similarly, the energy density is calculated as

$$\begin{aligned}\langle U\rangle &= \frac{\mathbf{E}\times\mathbf{D}^* + \mathbf{E}^*\times\mathbf{D} + \mathbf{B}\times\mathbf{H}^* + \mathbf{B}^*\times\mathbf{H}}{8} \\ &= \frac{1}{2}|E_o|^2\left(\varepsilon + \frac{k^2}{\mu\omega^2}\right) e^{-\boldsymbol{\alpha}\cdot\mathbf{r}}\end{aligned} \quad . \tag{21}$$

The wave properties for waves in a homogeneous conductor can now be expressed in terms of the medium's properties as:

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{\omega\sqrt{\mu\varepsilon}}\frac{\sqrt{2}}{\sqrt{\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}+1}} \approx 2\pi\sqrt{\frac{2}{\mu\sigma\omega}} \quad , \tag{22}$$

$$\delta = \frac{2}{\alpha} = \frac{1}{\omega\sqrt{\mu\varepsilon}}\frac{\sqrt{2}}{\sqrt{\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}-1}} \approx \sqrt{\frac{2}{\mu\sigma\omega}} \quad , \tag{23}$$

$$v_{phase} = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu\varepsilon}}\frac{\sqrt{2}}{\sqrt{\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}+1}} \approx \sqrt{\frac{2\omega}{\mu\sigma}} \quad , \tag{24}$$

$$n = \frac{c}{v_{phase}} = \frac{c\sqrt{\mu\varepsilon}}{\sqrt{2}}\sqrt{\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}+1} \approx c\sqrt{\frac{\mu\sigma}{2\omega}} \quad , \tag{25}$$

where the near equality in each of the four expressions above holds for good conductors only ($\sigma >> \omega\varepsilon$). Under the circumstance of $\sigma >> \omega\varepsilon$, the time averaged Poynting vector in Eq. (20) can be used to express the power transported by a plane wave through a conductor:

$$\frac{dP}{da} = < S > = \frac{1}{2\sigma\delta}|H_o|^2\, e^{-\boldsymbol{\alpha}\cdot\mathbf{r}} \quad . \tag{26}$$

Plane wave in linear isotropic dielectrics

In dielectrics, the fields satisfy the homogeneous wave equations

$$\nabla^2 \mathbf{E} - \mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0 \quad , \tag{1}$$

$$\nabla^2 \mathbf{H} - \mu\varepsilon \frac{\partial^2 \mathbf{H}}{\partial t^2} = 0 \quad . \tag{2}$$

Here we use H instead of B because of the increased symmetry in the constitutive relations, $\mathbf{D} = \varepsilon \mathbf{E}$ and $\mathbf{B} = \mu \mathbf{H}$, and because of the more parallel boundary conditions: $E_\parallel$ and $H_\parallel$ are continuous; $D_\perp$ and $B_\perp$ are continuous. The simplest solutions for the pair of Eqs. (1) and (2) are

$$\mathbf{E} = \mathbf{E}_o e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} \quad , \tag{3}$$

$$\mathbf{H} = \mathbf{H}_o e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} \quad , \tag{4}$$

where $k^2 = \mu\varepsilon\omega^2$. The solutions in Eqs. (3) and (4) are not independent but are linked by Maxwell's equations. The divergence equations lead the plane waves to be subject to

$$\mathbf{k} \cdot \mathbf{D}_o = 0 , \tag{5}$$

$$\mathbf{k} \cdot \mathbf{B}_o = 0 . \tag{6}$$

These two equations imply that the wave vector **k** is perpendicular to each of **D** and **B**. The curl equations give

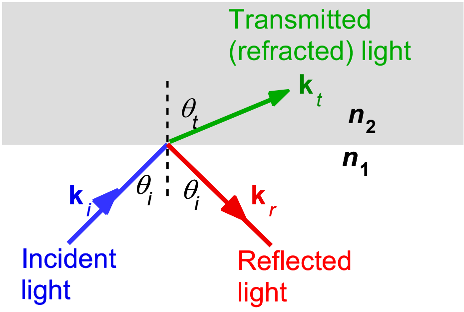
$$\mathbf{k} \times \mathbf{E}_o = \omega\mu \mathbf{H}_o , \tag{7}$$

$$\mathbf{k} \times \mathbf{H}_o = -\omega\varepsilon \mathbf{E}_o . \tag{8}$$

These two equations imply that **E** and **H** are perpendicular to each other. For isotropic media **E** is parallel to **D**, and **B** is parallel to **H**. We may therefore conclude that the wave vector **k**, the normal to the surface of constant phase, is parallel to the Poynting vector $\mathbf{S} = \mathbf{E} \times \mathbf{H}$. In other words, for isotropic media the direction of energy propagation is along **k**. It is worth while to note that this conclusion does not hold for anisotropic media, in which **S** is perpendicular to **E** whereas **k** is perpendicular to **D**.

Snell's law for reflection and refraction

Let us consider a plane wave with wave vector $\mathbf{k}_i$ incident on a plane interface, giving rise to a reflected wave with wave vector $\mathbf{k}_r$ and a transmitted wave with wave vector $\mathbf{k}_t$, as shown in Figure.



At a point **r** on the interface, the parallel component of the electric field must be the same on both sides of the interface. Therefore

$$n \times \left[\mathbf{E}_{o,i}\, e^{i(\mathbf{k}_i \cdot \mathbf{r} - \omega t)} + \mathbf{E}_{o,r}\, e^{i(\mathbf{k}_r \cdot \mathbf{r} - \omega t)} \right] = n \times \mathbf{E}_{o,t}\, e^{i(\mathbf{k}_t \cdot \mathbf{r} - \omega t)} . \tag{9}$$

Equation (9) can be reduced as

$$n \times \mathbf{E}_{o,i} = n \times \left[\mathbf{E}_{o,t}\, e^{i(\mathbf{k}_t - \mathbf{k}_i)\cdot \mathbf{r}} - \mathbf{E}_{o,r}\, e^{i(\mathbf{k}_r - \mathbf{k}_i)\cdot \mathbf{r}} \right] . \tag{10}$$

Obviously, the phase factor needs to be zero for satisfying the continuous condition at all **r** on the interface; that is,

$$(\mathbf{k}_t - \mathbf{k}_i) \cdot \mathbf{r} = 0 \quad , \tag{11}$$

$$(\mathbf{k}_r - \mathbf{k}_i) \cdot \mathbf{r} = 0 \quad . \tag{12}$$

Equation (12) implies that

$$k_i \sin\theta_i = k_r \sin\theta_r \quad . \tag{13}$$

Since the incident and reflected waves are both in the same medium, the magnitudes $\mathbf{k}_i$ and $\mathbf{k}_r$ are equal,

$$\sin\theta_i = \sin\theta_r \quad \Rightarrow \quad \theta_i = \theta_r \quad . \tag{14}$$

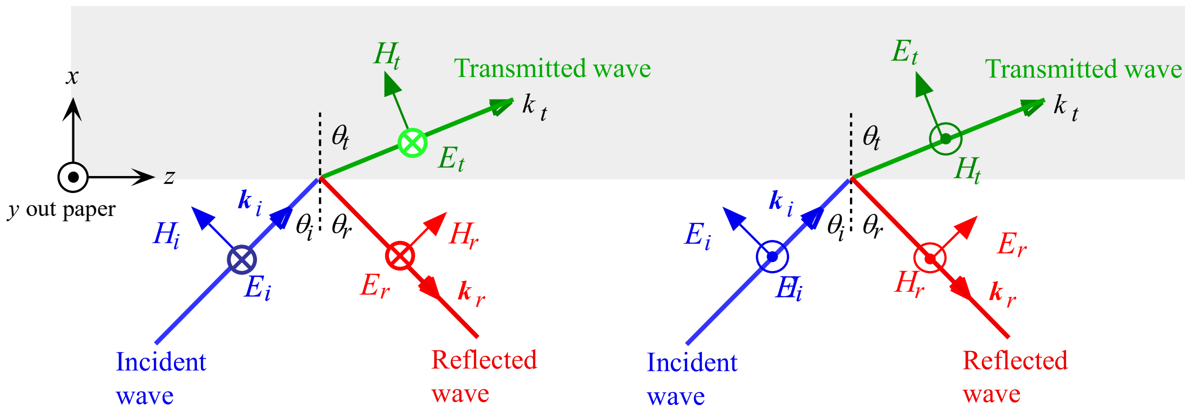
The angle of incidence is equal to the angle of reflection. Equation (11) implies that

$$k_i \sin\theta_i = k_t \sin\theta_t \quad . \tag{15}$$

In optics, Eq. (15) can be in terms of the refractive index to be given by

$$n_i \sin\theta_i = n_t \sin\theta_t \quad . \tag{16}$$

It is apparent that Snell's law is a consequence only of the plane wave nature of the disturbance and the requirement of continuity.



(a) The electric vector of *s*-polarized light is perpendicular to the plane of incidence.

(b) The electric vector of *p*-polarized light lies in plane of incidence.

Fresnel's equation

To derive the amplitude $E_{o,i}$, $E_{o,r}$, and $E_{o,t}$ of the incident, reflected, and transmitted wave, we need to use the boundary conditions in more detail. First of all, it is necessary to decompose **E** into two components, one labelled E_p, parallel to the plane of incidence, and the other labelled E_s, perpendicular to the plane of incidence. The plane of incidence is defined by **n** and **k**; the subscript *s* comes from *senkrecht*, German for perpendicular, and the *p* stands for parallel. The two components are now handled separately.

When the wave is *s*-polarized, the electric field is perpendicular to the plane of incidence and the magnetic field lies in the plane of incidence, as shown in Fig. (a). The boundary conditions for $E_{\parallel}$ and $H_{\parallel}$ to be continuous lead to

$$E_i + E_r = E_t \quad , \tag{17}$$

$$H_i \cos\theta_i - H_r \cos\theta_r = H_t \cos\theta_t \quad . \tag{18}$$

From Eq. (7), the magnitudes of H and E in dielectrics are related by an $H = kE / \omega\mu = E / \mu v$. Using $v = c / n$ and substituting for H in Eq. (18), we obtain

$$\frac{n_i}{\mu_i}(E_i - E_r)\cos\theta_i = \frac{n_t}{\mu_t} E_t \cos\theta_t \quad . \tag{19}$$

Equations (17) and (18) can be solved for E_r and E_t to lead to

$$\left(\frac{E_r}{E_i}\right)_s = r_s = \frac{\dfrac{n_i}{\mu_i}\cos\theta_i - \dfrac{n_t}{\mu_t}\cos\theta_t}{\dfrac{n_i}{\mu_i}\cos\theta_i + \dfrac{n_t}{\mu_t}\cos\theta_t} \quad , \tag{20}$$

$$\left(\frac{E_t}{E_i}\right)_s = t_s = \frac{2\frac{n_i}{\mu_i}\cos\theta_i}{\frac{n_i}{\mu_i}\cos\theta_i + \frac{n_t}{\mu_t}\cos\theta_t} \quad . \tag{21}$$

When the wave is *p*-polarized, the electric and magnetic fields are parallel and perpendicular to the plane of incidence, respectively, as shown in Fig. (b). The boundary conditions for $E_{\|}$ and $H_{\|}$ to be continuous lead to

$$H_i + H_r = H_t \ , \tag{22}$$

$$E_i \cos\theta_i - E_r \cos\theta_r = E_t \cos\theta_t \quad . \tag{23}$$

Using the same derivation, we have

$$\left(\frac{E_r}{E_i}\right)_p = r_p = \frac{\frac{n_t}{\mu_t}\cos\theta_i - \frac{n_i}{\mu_i}\cos\theta_t}{\frac{n_t}{\mu_t}\cos\theta_i + \frac{n_i}{\mu_i}\cos\theta_t} \quad , \tag{24}$$

$$\left(\frac{E_t}{E_i}\right)_p = t_p = \frac{2\frac{n_i}{\mu_i}\cos\theta_i}{\frac{n_t}{\mu_t}\cos\theta_i + \frac{n_i}{\mu_i}\cos\theta_t} \quad . \tag{25}$$

Note the difference between r_s, t_s and r_p, t_p. For dielectrics of optical interest, $\mu = \mu_o$. As a result, the permeabilities μ in Eqs. (20), (21), (24) and (25) can be cancelled. Using Snell's law to eliminate the ratios of refractive indices, Eqs. (20), (21), (24) and (25) can be simplified as the Fresnel equations:

$$r_s = -\frac{\sin(\theta_i - \theta_t)}{\sin(\theta_i + \theta_t)} \ , \qquad r_p = \frac{\tan(\theta_i - \theta_t)}{\tan(\theta_i + \theta_t)}$$

$$t_s = \frac{2\sin\theta_t \ \cos\theta_i}{\sin(\theta_i + \theta_t)} , \qquad t_p = \frac{2\sin\theta_t \ \cos\theta_i}{\sin(\theta_i + \theta_t)\cos(\theta_i - \theta_t)} \quad . \tag{26}$$

Another expressions for r_s, t_s, r_p, and t_p are obtained by eliminating the angle of transmission θ_t with $\cos\theta_t = \sqrt{1-\sin^2\theta_t} = \sqrt{n^2 - \sin^2\theta_i}\,/\,n$, where $n = n_t / n_i$. Consequently,

$$r_s = \frac{\cos\theta_i - \sqrt{n^2 - \sin^2\theta_i}}{\cos\theta_i + \sqrt{n^2 - \sin^2\theta_i}} \quad , \tag{27}$$

$$t_s = \frac{2\cos\theta_i}{\cos\theta_i + \sqrt{n^2 - \sin^2\theta_i}} \quad , \tag{28}$$

$$r_p = \frac{n^2\cos\theta_i - \sqrt{n^2 - \sin^2\theta_i}}{n^2\cos\theta_i + \sqrt{n^2 - \sin^2\theta_i}} \quad , \tag{29}$$

$$t_p = \frac{2n\cos\theta_i}{n^2\cos\theta_i + \sqrt{n^2 - \sin^2\theta_i}} \quad . \tag{30}$$

At normal incidence the amplitude reflection coefficients reduce to

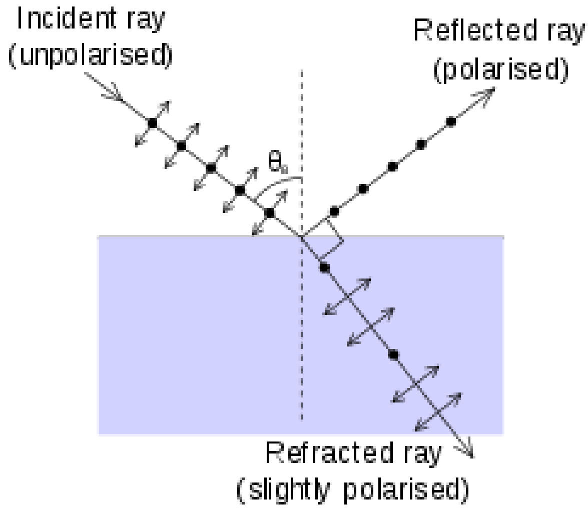
$$r_s = \frac{1-n}{1+n} \text{ and } r_p = -\frac{1-n}{1+n} \quad . \tag{31}$$

As there is no difference between an s and p wave at normal incidence, the difference in sign appears due to the difference of the choice for positive E_i and E_r in Figures (a) and (b). As usual, for $n > 1$, both correspond to a change in sign of the electric field when a wave is reflected.

Form Eq. (26), it is clear that if the incident angle of the *p*-wave satisfies $\theta_i + \theta_t = \pi/2$, then the amplitude of the reflected wave is zero. This implies that $\sin\theta_t = \cos\theta_i$. With Snell's law, it can be shown that this incident angle is called Brewster's angle θ_B and is given by

$$\tan\theta_B = \frac{n_t}{n_i} \quad . \tag{32}$$

Brewster's angle (also known as the polarization angle) is an angle of incidence at which light with a particular polarization is perfectly transmitted through a transparent dielectric surface, with no reflection. When unpolarized light is incident at this angle, the light that is reflected from the surface is therefore perfectly polarized. This special angle of incidence is named after the Scottish physicist Sir David Brewster (1781–1868).



The physical mechanism for this can be qualitatively understood from the manner in which electric dipoles in the media respond to p-polarized light. One can imagine that light incident on the surface is absorbed, and then reradiated by oscillating electric dipoles at the interface between the two media. The polarization of freely propagating light is always perpendicular to the direction in which the light is travelling. The dipoles that produce the transmitted (refracted) light oscillate in the polarization direction of that light. These same oscillating dipoles also generate the reflected light. However, dipoles do not radiate any energy in the direction of the dipole moment. Consequently, if the direction of the refracted light is perpendicular to the direction in which the light is predicted to be specularly reflected, the dipoles cannot create any reflected light.

When $n_i > n_t$, i.e. $n < 1$, there is a critical angle beyond which the sine of θ_i exceed n and both r_s and r_p become complex numbers of unit magnitude. The coefficients now give the phase change of the wave on reflection. This phase change can be evaluated by re-writing Eqs. (27) and (29) as

$$r_s = \frac{\cos\theta_i - i\sqrt{\sin^2\theta_i - n^2}}{\cos\theta_i + i\sqrt{\sin^2\theta_i - n^2}} \quad , \tag{33}$$

$$r_p = \frac{n^2\cos\theta_i - i\sqrt{\sin^2\theta_i - n^2}}{n^2\cos\theta_i + i\sqrt{\sin^2\theta_i - n^2}} \quad . \tag{34}$$

We can express $r_s = e^{-i\varphi_s}$ and $r_p = e^{-i\varphi_p}$ with

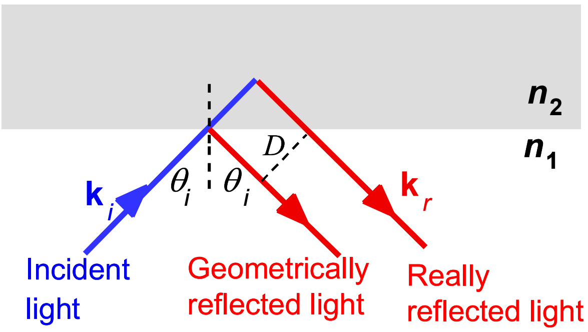
$$\varphi_s = 2\tan^{-1}\frac{\sqrt{\sin^2\theta_i - n^2}}{\cos\theta_i} \quad , \tag{35}$$

$$\varphi_p = 2\tan^{-1}\frac{\sqrt{\sin^2\theta_i - n^2}}{n^2\cos\theta_i} \quad . \tag{36}$$

Each of the waves is totally internally reflected but suffers a phase shift that differs for the two polarizations.

Goos and Hänchen Effect

In 1943, Goos and Hänchen (Ann. Physik 1, 333, 1947) devised an experiment to show that what happens at total reflection if the incident wave penetrates into the medium of lower index and reemerges into the medium of higher index. Goos and Hänchen reported the first experimental evidence of the displacement of a light beam upon total internal reflection. This displacement, called the Goos-Hänchen (GH) shift, is perpendicular to the direction of propagation of the reflected beam in the plane of incidence, as shown in Fig. .



If we took the incident beam to be a single plane wave, then it would be impossible to determine the GH shift. We use the same theoretical formulation as Hora (H. Hora, Optik (Stuttg.) 17, 409, 1960) and Renard (R. H. Renard, J. Opt. Soc. Am. 54, 1190, 1964) using a Debye-Picht wave packet (P. Debye, Ann. Phys. (Leipz.) 30, 755, 1909; J. Picht, Ann. Phys. (Leipz.) 77, 685, 785, 1925) for the incident beam. Thus, we require that the incident beam (1) have a half-width W which is as small and as nearly constant as possible along the beam, (2) make an angle θ_o with respect to the x axis, and (3) be focused on the origin (x =0, z = 0). The Debye-Picht wave packet meets these requirements accidentally well. We take as the wave packet for the incident beam

$$E_i(x,z;\theta_o,\xi) = \frac{1}{2\xi}E_{io}\int_{\theta_o-\xi}^{\theta_o+\xi} e^{i\frac{2\pi n_1}{\lambda}(x\cos\theta + z\sin\theta)}\, d\theta \ ,$$

where ξ should be chosen sufficiently small and θ_o sufficiently large such that θ_o-ξ exceeds the critical angle. The reflected wave function is given by

$$E_r(x,z;\theta_o,\xi)=\frac{1}{2\xi}E_{io}\int_{\theta_o-\xi}^{\theta_o+\xi} e^{-i\varphi(\theta)}\, e^{i\frac{2\pi n_1}{\lambda}(-x\cos\theta+z\sin\theta)}\, d\theta \quad ,$$

where $\varphi(\theta)$ is given by Eq. (35) and Eq. (36) for s-wave and p-wave, respectively. Changing the variable of $\eta=\theta-\theta_o$ and using small angle approximation, we can express the following terms as

$$\cos\theta=\cos\theta_o-\eta\sin\theta_o \quad ,$$

$$\sin\theta=\sin\theta_o+\eta\cos\theta_o \quad ,$$

$$\varphi(\theta)=\varphi(\theta_o)+\eta\left.\frac{d\varphi}{d\theta}\right|_{\theta=\theta_o} \quad .$$

With these expressions, the reflected wave function can be rewritten as

$$E_r(x,z;\theta_o,\xi)=\frac{1}{2\xi}E_{io}\, e^{-i\varphi(\theta_o)}e^{i\frac{2\pi n_1}{\lambda}(-x\cos\theta_o+z\sin\theta_o)}\int_{-\xi}^{\xi} e^{i\eta\, k_1[x\sin\theta_o+z\cos\theta_o-\varphi'(\theta_o)/k_1]}\, d\eta \ ,$$

where $k_1=2\pi\, n_1/\lambda$. Carrying out the integration, the reflected wave function is given by

$$E_r(x,z;\theta_o,\xi)=E_{io}\, e^{-i\varphi(\theta_o)}e^{i\frac{2\pi n_1}{\lambda}(-x\cos\theta_o+z\sin\theta_o)}\,\frac{\sin\{\xi k_1[x\sin\theta_o+z\cos\theta_o-\varphi'(\theta_o)/k_1]\}}{\{\xi k_1[x\sin\theta_o+z\cos\theta_o-\varphi'(\theta_o)/k_1]\}} \ .$$

The property of the function sin(x)/x indicates that the reflected intensity is mainly concentrated on the line equation of

$$x\sin\theta_o+z\cos\theta_o=\varphi'(\theta_o)/k_1 \quad .$$

Therefore, the GH shift can be evaluated as

$$D=\varphi'(\theta_o)/k_1 \quad .$$

With Eq. (35), we can show that

$$D_s=\frac{1}{k_1}\left.\frac{d\varphi_s}{d\theta}\right|_{\theta=\theta_o}=\frac{2}{k_1}\frac{d}{d\theta}\left[\tan^{-1}\frac{\sqrt{\sin^2\theta-n^2}}{\cos\theta}\right]_{\theta=\theta_o}$$

$$=\frac{2}{k_1}\frac{\cos^2\theta_o}{1-n^2}\,\frac{\frac{\sin\theta_o\cos\theta_o}{\sqrt{\sin^2\theta_o-n^2}}\cos\theta_o+\sqrt{\sin^2\theta_o-n^2}\sin\theta_o}{\cos^2\theta_o},$$

$$=\frac{\lambda_1}{\pi}\frac{\sin\theta_o}{\sqrt{\sin^2\theta_o-n^2}}$$

where $n=n_2/n_1$ and we have used the property of $d[\tan^{-1}u]/du=1/(1+u^2)$ and the chain rule for the result of $d\tan^{-1}u(y)/dy=\left(1/[1+u^2(y)]\right)\!\left(du/dy\right)$. Note that $\lambda_1=\lambda/n_1$ is the wavelength in the medium of higher index of refraction. Similarly, we can use Eq. (36) to show that

$$
\begin{aligned}
D_p &= \frac{1}{k_1}\frac{d\varphi_p}{d\theta}\bigg|_{\theta=\theta_o} = \frac{2}{k_1}\frac{d}{d\theta}\left[\tan^{-1}\frac{\sqrt{\sin^2\theta - n^2}}{n^2\cos\theta}\right]_{\theta=\theta_o} \\
&= \frac{2}{k_1}\frac{n^4\cos^2\theta_o}{n^4\cos^2\theta_o + \sin^2\theta_o - n^2}\frac{\dfrac{\sin\theta_o\cos\theta_o}{\sqrt{\sin^2\theta_o - n^2}}n^2\cos\theta_o + \sqrt{\sin^2\theta_o - n^2}\,n^2\sin\theta_o}{n^4\cos^2\theta_o} \\
&= \frac{\lambda_1}{\pi}\frac{1}{\sin^2\theta_o(1-n^2) - n^2\cos^2\theta_o(1-n^2)}\frac{(1-n^2)n^2\sin\theta_o}{\sqrt{\sin^2\theta_o - n^2}} \\
&= \frac{n^2}{(\sin^2\theta_o - n^2\cos^2\theta_o)}D_s
\end{aligned} .
$$

The GH effects originate from the dispersion of the reflection or transmission coefficients, as was first shown by Artmann in 1948 (K. Artmann, Ann. Phys. 2, 87, 1948). For the past few decades, the spatial GH shift has been studied in a variety of systems (Nature Photon. 3, 337, 2009; PRA 25, 2099, 1982; PRL 68, 931, 1992; PRL 70, 2281, 1993; PRL 75, 1511, 1995; PRL 91, 133903, 2003; PRL 92, 193902, 2004; PRL 102, 146804, 2009; PRL 104, 010401, 2010) embracing plasmonics, metamaterials, and quantum systems.

Frequency dispersion characteristics of dielectrics and conductors

In reality all media reveal some dispersion. Only over a limited range of frequencies, or in vacuum, can the velocity of propagation be treated as constant. There are no dispersive effects for a single frequency component. However, when a superposition of a range of frequencies occurs, dispersive effects arise as a result of the frequency dependence of ε and μ. Here a simple model of dispersion is developed to examine some of these consequences. For simplicity, we neglect the difference between the applied electric field and the local field.

The equation of motion for an electron of charge $-e$ bound by a harmonic force and acted on by an electric field $\mathbf{E}(\mathbf{r},t)$ is

$$
m\left(\ddot{\mathbf{r}} + \gamma\,\dot{\mathbf{r}} + \omega_o^2\,\mathbf{r}\right) = -e\mathbf{E}(\mathbf{r},t) \quad ,
$$

where γ measures the phenomenological damping force and magnetic force effects are neglected. If the field varies harmonically in time with frequency ω as $e^{-i\omega t}$, the dipole moment contributed by one electron is

$$
\mathbf{p} = -e\mathbf{r} = \frac{e^2}{m}\frac{1}{\omega_o^2 - \omega^2 - i\gamma\omega}\mathbf{E}(\mathbf{r}) \quad .
$$

If we assume that there are N molecules per unit volume with Z electrons per molecule, and that, instead of a single binding frequency for all, there are f_j electrons per molecule with binding frequency ω_j and damping constant γ_j, then the dielectric function is given by

$$\frac{\varepsilon(\omega)}{\varepsilon_o} = 1 + \chi_e = 1 + \frac{Ne^2}{\varepsilon_o m}\sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j\omega} \quad ,$$

where the oscillator strengths f_j satisfy the sum rule

$$\sum_j f_j = Z \quad .$$

With suitable quantum-mechanical definitions of f_j, ω_j, and γ_j, this simple model is an accurate description of atomic contribution to the dielectric function.

We employ this dielectric model to discuss the anomalous dispersion and resonant absorption. Since the damping constants γ_j are generally small compared with the binding or resonant frequencies ω_j, the factor $\omega_j^2 - \omega^2$ is positive for $\omega < \omega_j$ and negative for $\omega > \omega_j$. Therefore, at low frequencies, below the smallest ω_j, all the terms in the sum in the dielectric function contribute with the same positive sign and $\varepsilon(\omega)/\varepsilon_o$ is greater than unity. As successive ω_j values are passed, more and more negative terms occur in the sum, until finally the whole sum is negative and $\varepsilon(\omega)/\varepsilon_o$ is less than one. In the neighborhood of any ω_j, there is rather violent behavior. The real part of the denominator in the dielectric function vanishes for that term at $\omega = \omega_j$ and the term is large and purely imaginary. Normal dispersion is associated with an increase in $\mathrm{Re}[\varepsilon(\omega)]$ with ω, anomalous dispersion with the reverse. Normal dispersion is seen to occur everywhere except in the neighborhood of a resonant frequency. Only where there is anomalous dispersion is the imaginary part of ε appreciable. Since a positive imaginary part to ε represents dissipation of energy from the electromagnetic wave into the medium, the region where $\mathrm{Im}[\varepsilon(\omega)]$ is large are called regions of resonant absorption.

The attenuation of a plane wave is most directly expressed in terms of the real and imaginary parts of the wave number *k*. If the wave number is given by

$$k = \beta + i\frac{\alpha}{2},$$

then the parameter α is known as the attenuation constant or absorption coefficient. The intensity of the wave falls off as $\exp(-\alpha z)$. With $k^2 = \mu\varepsilon\omega^2$, the connection between (α, β) and $(\mathrm{Re}\,\varepsilon, \mathrm{Im}\,\varepsilon)$ is given by

$$\beta^2 - \frac{\alpha^2}{4} = \frac{\omega^2}{c^2}\mathrm{Re}\left(\frac{\varepsilon}{\varepsilon_o}\right) ,$$

$$\beta\alpha = \frac{\omega^2}{c^2}\mathrm{Im}\left(\frac{\varepsilon}{\varepsilon_o}\right) .$$

If $\alpha << \beta$, as occurs unless the absorption is very large or $\mathrm{Re}\,\varepsilon$ is negative, the attenuation constant α can be written approximately as

$$\alpha = \frac{\mathrm{Im}(\varepsilon)}{\mathrm{Re}(\varepsilon)}\,\beta ,$$

where $\beta = \mathrm{Re}(\varepsilon/\varepsilon_o)\omega/c$. The fractional decrease in intensity per wavelength divided by 2π is then given by the ratio $\mathrm{Im}(\varepsilon)/\mathrm{Re}(\varepsilon)$.

In the limit $\omega \to 0$ there is a qualitative difference in the response of the medium depending on whether the lowest resonant frequency is zero or not. For insulators the lowest resonant frequency is different from zero. If some fractions of the electrons per molecule are free in the sense of having $\omega_j = 0$, the dielectric function is singular at $\omega = 0$. If the contribution of the free electrons is exhibited separately, the dielectric function becomes

$$\varepsilon(\omega) = \varepsilon_b(\omega) + i\frac{Ne^2}{m\omega}\frac{f_o}{(\gamma_o - i\omega)} ,$$

where $\varepsilon_b(\omega)$ is the contribution of all the other dipoles. The singular behavior can be understood as follows. Assuming that the medium obeys Ohm's law $\mathbf{J} = \sigma\mathbf{E}$ and has a normal dielectric constant ε_b, one of the Maxwell equations with harmonic time dependence is given by

$$\nabla \times \mathbf{H}_o = -i\omega\left(\varepsilon_b + \frac{i\sigma}{\omega}\right)\mathbf{E}_o .$$

If we did not insert Ohm's law explicitly but attributed instead all the properties of the medium to the dielectric function, this equation can be expressed as

$$\nabla \times \mathbf{H}_o = -i\omega\varepsilon(\omega)\mathbf{E}_o .$$

Compared with these two representations, the conductivity can be deduced as

$$\sigma = \frac{Ne^2}{m}\frac{f_o}{(\gamma_o - i\omega)} \quad .$$

This is originally the model of Drude (1900) for the electrical conductivity, with Nf_o being the number of free electrons per unit volume in the medium. The damping constant γ_o / f_o can be determined empirically from experimental data on the conductivity. For copper, $N = 8\times10^{28}$ atoms/m^3 and at normal temperatures the low-frequency conductivity is $\sigma = 5.9\times10^7$ $(\Omega\cdot\mathrm{m})^{-1}$. This gives $\gamma_o / f_o = 4\times10^{13}$ s^{-1}. Assuming that $f_o = 1$, this shows that up to frequencies well beyond the microwave region ($\omega \le 10^{11}$ s^{-1}) conductivities of metals are essentially real (implying current in phase with the field) and independent of frequency.

At frequencies far above the highest resonant frequency the dielectric function takes on the simple form

$$\frac{\varepsilon(\omega)}{\varepsilon_o} \approx 1 - \frac{\omega_p^2}{\omega^2} \quad ,$$

where $\omega_p^2 = ZNe^2 / \varepsilon_o m$. The frequency ω_p, which depends only on the total number NZ of electrons per unit volume, is called the plasma frequency of the medium. For a typical metal like copper ($NZ = 8\times10^{28}$ atoms/m^3), we get $f_p = \omega_p / 2\pi = 2.5\times10^{15}$ Hz, corresponding to a wavelength of 0.12 μm. Experimentally, light in the visible region are reflected from metal surfaces. Above the plasma frequency, the metal becomes transparent to ultraviolet light with wave length shorter than 0.12 μm. For semiconductor such as indium antimonide (InSb) with $NZ = 2\times10^{24}$ atoms/m^3, there are drastic changes in reflectance at the expected plasma frequency of $f_p = \omega_p / 2\pi = 2\times10^{13}$. In the high frequency limit $\omega > \omega_p$, $\varepsilon / \varepsilon_o$ is real and there in no attenuation for wave propagation. The wave number and wavelength can be expressed as

$$2\pi c / \lambda = ck = \sqrt{\varepsilon / \varepsilon_o}\,\omega = \sqrt{\omega^2 - \omega_p^2} \quad .$$

With this result, the dispersion relation is known as

$$\omega^2 = \omega_p^2 + c^2 k^2 \quad .$$

An important application is the ionosphere above the surface of the earth. In the ionosphere, the electrons are free and the damping is negligible. The height and intensity of ionosphere change with the hour of the day, the season of the year, and the sunspot cycle, etc. It is approximately a layer (known as the F-layer) extending from 100 km to 300 km above the earth. Inside the F-layer, the electron density is about 2×10^{12} m^{-3} in the daytime and 3×10^{11} m^{-3} at night. The plasma frequencies are 13 MHz and 5 MHz. in daytime and at night. There is also a lower layer, known as the E-layer, extending from approximately 60 to 100 km above the earth with electron density lower by about a factor of 10. AM radio stations broadcast in the frequency range of 0.5 to 1.6 MHz. These radio waves are reflected by the ionosphere. At night, the electron density is lower due to the absence of solar radiation and solar wind bombardment, and the ionosphere is higher above the surface of the earth. It is possible to pick up faraway radio stations since the radio signal can travel longer distances by reflecting off higher ionosphere layers.

Manifestation of the pulse spreading in a dispersive medium

We consider a specific model to derive the property of a pulse propagating in a dispersive medium. Considering a real 1D time-dependent wave packet $\psi(x,t)$, it can be expanded with the pane waves as

$$\psi(x,t) = \frac{1}{2}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} A(k)\, e^{ikx-i\omega(k)t} dk + c.c. \quad .$$

It can be shown that $A(k)$ is given in terms of the initial values by

$$A(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}\left[\psi(x,0) + \frac{i}{\omega(k)}\frac{\partial\psi}{\partial t}(x,0)\right]dx \quad .$$

For simplicity, we assume $\partial\psi(x,0)/\partial t = 0$ and consider a Gaussian modulated oscillation

$$\psi(x,0) = e^{-x^2/2L^2}\cos(k_o x)$$

as the initial shape of the pulse. This Gaussian modulated wave function means that at times immediately before t = 0 the wave consisted of two pulses, both moving toward the origin, such that at t = 0 they coalesced into the modulated shape. It is clearly expected that at later times each pulse will remerge on the other side of the origin. The normalization can be found to be

$$\int_{-\infty}^{\infty} |\psi(x,0)|^2\, dx = \frac{\sqrt{\pi}L}{2}(1 + e^{-k_o^2 L^2})$$

The Fourier component $A(k)$ for this Gaussian modulated wave is

$$A(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \, e^{-x^2/2L^2} \cos(k_o x) \, dx = \frac{L}{2}\left[e^{-L^2(k-k_o)^2/2} + e^{-L^2(k+k_o)^2/2} \right]$$

The symmetry $A(k) = A(-k)$ is a reflection of the presence of two pulses traveling away from the origin. To consider the essential dispersive effects exactly, we assume

$$\omega(k) = \omega_o \left(1 + \frac{a^2 k^2}{2} \right) ,$$

where ω_o is a constant frequency and a is a constant length that is a typical wavelength where dispersive effects become important. The time-dependent wave function is given by

$$\psi(x,t) = \frac{L}{2}\frac{1}{\sqrt{2\pi}} \mathrm{Re}\left\{ \int_{-\infty}^{\infty} \left[e^{-L^2(k-k_o)^2/2} + e^{-L^2(k+k_o)^2/2} \right] e^{ikx - i\omega_o\left(1+\frac{a^2k^2}{2}\right)t} dk \right\} .$$

The integration can be performed as follows:

$$\psi(x,t) = \frac{L}{2}\frac{1}{\sqrt{2\pi}} \mathrm{Re}\left\{ \int_{-\infty}^{\infty} \left[e^{-L^2(k-k_o)^2/2} + e^{-L^2(k+k_o)^2/2} \right] e^{ikx - i\omega_o\left(1+\frac{a^2k^2}{2}\right)t} dk \right\}$$

$$= \frac{L}{2\sqrt{2\pi}} \mathrm{Re}\left\{ \int_{-\infty}^{\infty} \left[e^{-\left(L^2 + i\omega_o a^2 t\right)(k-k_o)^2/2} e^{i(k-k_o)(x-\omega_o a^2 k_o t)} \right] e^{ik_o x} e^{-i\omega_o\left(1+\frac{a^2k_o^2}{2}\right)t} dk + \left[k_o \to -k_o\right] \right\}$$

$$= \frac{L}{2\sqrt{2\pi}} \mathrm{Re}\left\{ \int_{-\infty}^{\infty} \left[e^{-\left(L^2 + i\omega_o a^2 t\right)(k-k_o)^2/2} e^{i(k-k_o)(x-\omega_o a^2 k_o t) + \frac{(x-\omega_o a^2 k_o t)^2}{2\left(L^2 + i\omega_o a^2 t\right)}} \right] e^{ik_o x - i\omega_o\left(1+\frac{a^2k_o^2}{2}\right)t} e^{-\frac{(x-\omega_o a^2 k_o t)^2}{2\left(L^2 + i\omega_o a^2 t\right)}} dk \right.$$

$$\left. + \left[k_o \to -k_o\right] \right\}$$

$$= \frac{L}{2\sqrt{2\pi}} \mathrm{Re}\left\{ \frac{\sqrt{2\pi}}{\sqrt{(L^2 + i\omega_o a^2 t)}} e^{ik_o x - i\omega_o\left(1+\frac{a^2k_o^2}{2}\right)t} e^{-\frac{(x-\omega_o a^2 k_o t)^2}{2\left(L^2 + i\omega_o a^2 t\right)}} + \left[k_o \to -k_o\right] \right\}$$

$$= \frac{1}{2} \mathrm{Re}\left\{ \frac{e^{-i\frac{1}{2}\tan^{-1}(\omega_o a^2 t/L^2)}}{\left[1 + (\omega_o a^2 t/L^2)^2\right]^{1/4}} e^{ik_o x - i\omega_o\left(1+\frac{a^2k_o^2}{2}\right)t} e^{-\frac{(x-\omega_o a^2 k_o t)^2}{2L^2[1+(\omega_o a^2 t/L^2)^2]}} e^{i\frac{(x-\omega_o a^2 k_o t)^2(\omega_o a^2 t/L^2)}{2L^2[1+(\omega_o a^2 t/L^2)^2]}} + \left[k_o \to -k_o\right] \right\} .$$

This equation represents two pulses traveling in opposite directions. The peak amplitude of each pulse travels with the group velocity $v_g = \omega_o a^2 k_o$, while the modulation envelop remains Gaussian in shape. The width of the Gaussian is not constant, however, but increases with time. The width of the envelop is

$$L_{eff}(t) = L\left[1+(\omega_o a^2 t/L^2)^2\right]^{1/2} \quad .$$

Thus the dispersive effects on the pulse are greater for a given elapsed time, the sharper the envelope. The criterion for a small change in shape is that *L*>>*a*. Even so, at long time the width of the Gaussian increases linearly with time $L_{eff}(t) \to \omega_o a^2 t/L$. Note that the time of attainment of this asymptotic form depends on the ratio *L*/*a*.

There is an interesting analogy with the paraxial propagation of a Gaussian beam. The Fourier transform of a Gaussian beam is expressed as

$$A(k) = \frac{1}{\sqrt{2\sqrt{\pi}w_o}}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ik_x x}\, e^{-x^2/w_o^2}\, dx = \frac{\sqrt{w}}{2\sqrt{\sqrt{\pi}}} e^{-w_o^2 k_x^2/4} ,$$

where w_o is the beam waist of the Gaussian beam. Considering the paraxial propagation of a Gaussian beam, the space evolution is given by

$$\psi(x,z) = \frac{\sqrt{w_o}}{2\sqrt{\sqrt{\pi}}}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-k_x^2 w_o^2/4}\, e^{ik_x x} e^{ikz - i\left(\frac{k_x^2}{2k}\right)z}\, dk_x \quad .$$

Analogous to the time evolution, we can make the replacement of $L \to w_o/\sqrt{2}$, $t \to z$, $\omega_o a^2 \to 1/k$, $k_o \to 0$ to obtain the variation of the beam spot with z

$$w_{eff}(z) = w_o\left[1+(2z/kw_o^2)^2\right]^{1/2} \quad .$$

Causality in the connection between D and E (nonlocality in time)

First of all, there are two properties to remind for discussing the causality. One is that if the function $f(t)$ is real, then the complex conjugate of the Fourier transform of $f(t)$ satisfies

$$F^*(\omega) = \left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{i\omega t}\, dt\right]^* = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}\, dt = F(-\omega) \quad .$$

The other is that if the transform $H(\omega) = F(\omega)G(\omega)$ is the product of the two transforms *F* and *G*, then the inverse transform of *H* is given by the convolution of the two inverse transforms of *F* and *G*. Suppose *f*, *g*, and *h* are the inverse transforms of *F*, *G*, and *H*. Then we can derive this property as follows:

$$
\begin{aligned}
h(t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) G(\omega) e^{-i\omega t}\, d\omega \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\tau) e^{i\omega\tau}\, d\tau \right] e^{-i\omega t}\, d\omega \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\tau) \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) e^{-i\omega(t-\tau)}\, d\omega \right] d\tau \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\tau) f(t-\tau)\, d\tau
\end{aligned}
.
$$

Based on the linear response, the relationship between the Fourier transforms of $\mathbf{D}(\mathbf{r},\omega)$ and $\mathbf{E}(\mathbf{r},\omega)$ can be given by

$$\mathbf{D}(\mathbf{r},\omega) = \varepsilon(\omega)\mathbf{E}(\mathbf{r},\omega) \quad .$$

In terms of the susceptibility $\chi_e(\omega) = [\varepsilon(\omega)/\varepsilon_o] - 1$, the relationship can be rewritten as

$$\mathbf{D}(\mathbf{r},\omega) = [\varepsilon_o + \varepsilon_o \chi_e(\omega)]\mathbf{E}(\mathbf{r},\omega) .$$

Taking the inverse Fourier transform, the relationship between $\mathbf{D}(\mathbf{r},t)$ and $\mathbf{E}(\mathbf{r},t)$ can be shown to include the form of the convolution

$$\mathbf{D}(\mathbf{r},t) = \varepsilon_o \left[\mathbf{E}(\mathbf{r},t) + \int_{-\infty}^{\infty} G(\tau)\mathbf{E}(\mathbf{r},t-\tau)\, d\tau \right] \quad ,$$

where the susceptibility kernel $G(\tau)$ is given by

$$G(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \chi_e(\omega)\; e^{-i\omega\tau}\, d\omega \quad .$$

The form of the convolution gives a nonlocal connection between $\mathbf{D}(\mathbf{r},t)$ and $\mathbf{E}(\mathbf{r},t)$, in which $\mathbf{D}(\mathbf{r},t)$ at time *t* depends on the electric field $\mathbf{E}(\mathbf{r},t)$ at times other than *t*. If $\varepsilon(\omega)$ is independent of ω for all ω, the susceptibility kernel $G(\tau)$ becomes a constant times a delta function $\delta(\tau)$ and the instantaneous connection is obtained. In contrast, if $\varepsilon(\omega)$ varies with ω, $G(\tau)$ is nonvanishing for some values of τ different from zero.

To manifest the character of the connection implied by $G(\tau)$ and the form of the convolution, we consider a one-resonance version of the index of refraction:

$$\chi_e(\omega) = [\varepsilon(\omega)/\varepsilon_o] - 1 = \frac{\omega_p^2}{\omega_o^2 - \omega^2 - i\gamma\omega} \quad .$$

The susceptibility kernel $G(\tau)$ for this model is given by

$$G(\tau) = \frac{\omega_p^2}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-i\omega\tau}}{\omega_o^2 - \omega^2 - i\gamma\omega}\, d\omega \quad .$$

The integral can be evaluated by contour integration. The integrand has poles in the lower half-plane at

$$\omega_{\pm} = \pm\omega_R - i\gamma/2 \quad ,$$

where $\omega_R^2 = \omega_o^2 - \gamma^2/4$. For $\tau < 0$ the contour can be closed in the upper half-plane without affecting the value of the integral since the integrand is regular inside the closed contour, the integral vanishes. This means that at time t only values of the electric field prior to that time enter in determining the displacement, in accord with our fundamental ideas of causality in physical phenomenon. For $\tau > 0$ the contour is closed in the lower half-plane and the integral is given by $-2\pi i$ times the residues at the two poles. To be brief, causality requires that the transform have no poles in the upper half-plane. For a single oscillator, the susceptibility kernel $G(\tau)$ can be evaluated as follows:

$$\begin{aligned}
G(\tau) &= \frac{\omega_p^2}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-i\omega\tau}}{\omega_o^2 - \omega^2 - i\gamma\omega}\, d\omega \\
&= \frac{\omega_p^2}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-i\omega\tau}}{-(\omega-\omega_-)(\omega-\omega_+)}\, d\omega \\
&= (-2\pi i)\frac{\omega_p^2}{2\pi}\left[\frac{e^{-i\omega_+\tau}}{(\omega_- - \omega_+)} - \frac{e^{-i\omega_-\tau}}{(\omega_- - \omega_+)}\right] u(\tau) \quad . \\
&= i\omega_p^2\left[\frac{e^{-i\omega_+\tau} - e^{-i\omega_-\tau}}{(\omega_+ - \omega_-)}\right] u(\tau) \\
&= \omega_p^2\, e^{-\gamma\tau/2}\,\frac{\sin(\omega_R\tau)}{\omega_R}\, u(\tau)
\end{aligned}$$

where $u(\tau)$ is the unit step function; $u(\tau) = 0$ for $\tau < 0$ and $u(\tau) = 1$ for $\tau > 0$. It can be seen that the kernel $G(\tau)$ is oscillatory with the characteristic frequency of the medium and damped in time with the damping constant of electronic oscillators.

Kramers-Kronig (K-K) relations (Dispersion relations)

In the 1920s, H. A. Kramers and R. de L. Kronig discovered how to use the property of analyticity to relate the real and imaginary parts of the dielectric function of a material, thus deriving relations that relate the dispersive and absorptive properties of a material in its interaction with electromagnetic waves. The *K-K* relations are a specific example of a more general class of relations called dispersion relations. In recent years, these relations have proved important in other branches of physics as well such as in particle physics.

Assume a function $f(z)$ is analytic everywhere in the upper half-plane. The first

Cauchy formula can be used to express the value of the function at a point z_o in terms of an integral around a curve *C* that surrounds z_o:

$$f(z_o)=\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_o}dz \quad ,$$

where both z_o and C are in the upper half-plane. Now if $|f(z)|\to 0$ as $z\to\infty$, we may choose *C* to be a large semicircle with its flat side along the real axis. Since the integral along the curved part is zero, we have

$$f(z_o)=\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{f(x)}{x-z_o}dx \quad .$$

Now we let the point z_o approach the *x*-axis from above. The path of integration must remain below the pole, so we put a small semicircle under the pole and obtain

$$\begin{aligned} f(x_o)&=\frac{1}{2\pi i}\left[P\int_{-\infty}^{\infty} \frac{f(x)}{x-z_o}dx+\lim_{r\to 0}\int_{-\pi}^{0} \frac{f(x_o+re^{i\theta})}{re^{i\theta}}ire^{i\theta}d\theta\right] \\ &=\frac{1}{2\pi i}\left[P\int_{-\infty}^{\infty} \frac{f(x)}{x-z_o}dx+i\pi f(x_o)\right] \end{aligned} .$$

Consequently,

$$f(x_o)=-\frac{i}{\pi}P\int_{-\infty}^{\infty} \frac{f(x)}{x-x_o}dx \quad .$$

Similarly, we can obtain the same result by letting the point z_o approach the *x*-axis from below. The path of integration is above the pole, so we put a small semicircle above the pole and obtain

$$\begin{aligned} 0&=\frac{1}{2\pi i}\left[P\int_{-\infty}^{\infty} \frac{f(x)}{x-z_o}dx+\lim_{r\to 0}\int_{\pi}^{0} \frac{f(x_o+re^{i\theta})}{re^{i\theta}}ire^{i\theta}d\theta\right] \\ &=\frac{1}{2\pi i}\left[P\int_{-\infty}^{\infty} \frac{f(x)}{x-z_o}dx-i\pi f(x_o)\right] \end{aligned} .$$

Therefore, it is clear that the real and imaginary parts of $f(x)$ are related by

$$\mathrm{Re}[f(x_o)]=\frac{1}{\pi}P\int_{-\infty}^{\infty} \frac{\mathrm{Im}[f(x)]}{x-x_o}dx \quad ,$$

$$\mathrm{Im}[f(x_o)]=-\frac{1}{\pi}P\int_{-\infty}^{\infty} \frac{\mathrm{Re}[f(x)]}{x-x_o}dx \quad .$$

Since the dielectric function of any material approaches ε_o at very high frequencies, the function $f(\omega)=\varepsilon(\omega)/\varepsilon_o-1$ definitely approaches zero at high frequencies. The real and imaginary parts can be related by

$$\mathrm{Re}\left[\varepsilon(\omega)/\varepsilon_o\right] = 1 + \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\mathrm{Im}\left[\varepsilon(\omega')/\varepsilon_o\right]}{\omega' - \omega} d\omega' \quad ,$$

$$\mathrm{Im}\left[\varepsilon(\omega)/\varepsilon_o\right] = -\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\mathrm{Re}\left[\varepsilon(\omega')/\varepsilon_o - 1\right]}{\omega' - \omega} d\omega' \quad .$$

These relations are called Kramers-Kronig relations or dispersion relations. They were fisrt derived by H. A. Kramers (1927) and R. de L. Kronig (1926) independently. The symmetry property $\varepsilon(-\omega) = \varepsilon^*(\omega)$ shows the $\mathrm{Re}\,\varepsilon(\omega)$ is even in ω, while $\mathrm{Im}\,\varepsilon(\omega)$ is odd. The integrals can thus be transformed to span only positive frequencies:

$$\begin{aligned}\mathrm{Re}\left[\varepsilon(\omega)/\varepsilon_o\right] &= 1 + \frac{1}{\pi} P \left\{ \int_{-\infty}^{0} \frac{\mathrm{Im}\left[\varepsilon(\omega')/\varepsilon_o\right]}{\omega' - \omega} d\omega' + \int_{0}^{\infty} \frac{\mathrm{Im}\left[\varepsilon(\omega')/\varepsilon_o\right]}{\omega' - \omega} d\omega' \right\} \\ &= 1 + \frac{1}{\pi} P \left\{ \int_{0}^{\infty} \frac{\mathrm{Im}\left[\varepsilon(\omega')/\varepsilon_o\right]}{\omega' + \omega} d\omega' + \int_{0}^{\infty} \frac{\mathrm{Im}\left[\varepsilon(\omega')/\varepsilon_o\right]}{\omega' - \omega} d\omega' \right\} \quad , \\ &= 1 + \frac{2}{\pi} P \int_{0}^{\infty} \frac{\omega' \,\mathrm{Im}\left[\varepsilon(\omega')/\varepsilon_o\right]}{\omega'^2 - \omega^2} d\omega'\end{aligned}$$

$$\begin{aligned}\mathrm{Im}\left[\varepsilon(\omega)/\varepsilon_o\right] &= -\frac{1}{\pi} P \left\{ \int_{-\infty}^{0} \frac{\mathrm{Re}\left[\varepsilon(\omega')/\varepsilon_o - 1\right]}{\omega' - \omega} d\omega' + \int_{0}^{\infty} \frac{\mathrm{Re}\left[\varepsilon(\omega')/\varepsilon_o - 1\right]}{\omega' - \omega} d\omega' \right\} \\ &= -\frac{1}{\pi} P \left\{ \int_{0}^{\infty} -\frac{\mathrm{Re}\left[\varepsilon(\omega')/\varepsilon_o - 1\right]}{\omega' + \omega} d\omega' + \int_{0}^{\infty} \frac{\mathrm{Re}\left[\varepsilon(\omega')/\varepsilon_o - 1\right]}{\omega' - \omega} d\omega' \right\} \quad . \\ &= -\frac{2\omega}{\pi} P \int_{0}^{\infty} \frac{\mathrm{Re}\left[\varepsilon(\omega')/\varepsilon_o - 1\right]}{\omega'^2 - \omega^2} d\omega'\end{aligned}$$

These relations indicate that measurements of the absorption properties of a material determine the dispersion and vice versa.

Chapter Nine: Waveguide and resonators

Practically, it is not possible to maintain the electric field and the magnetic field each perpendicular to the direction of propagation. Bounded electromagnetic waves are usually classified as transverse electric (TE) and transverse magnetic (TM). Travelling waves are bounded either by being enclosed within metallic waveguides or guided by dielectric rods. Furthermore, standing waves are usually confined in metallic cavities or dielectric resonators. In the case that the dimensions of the guide are much larger than the wavelength, the propagation of waves can be regarded as simply a plane wave reflecting successively from the walls either by metallic reflection for metal waveguides, or total internal reflection for dielectric waveguides.

We now consider electromagnetic waves in the hollow space enclosed by a waveguide. Assuming time harmonic varying fields $\mathbf{E}, \mathbf{H} \propto e^{-i\omega t}$ and considering the source free region with $\mathbf{D} = \varepsilon \mathbf{E}$, we have

$$\nabla \cdot \mathbf{E} = 0 , \quad (1)$$

$$\nabla \cdot \mathbf{B} = 0 , \quad (2)$$

$$\nabla \times \mathbf{E} = i\omega \mathbf{B} , \quad (3)$$

$$\nabla \times \mathbf{B} = -i\mu\varepsilon\omega \mathbf{E} . \quad (4)$$

Here it is worthwhile to note the symmetry of the equations in exchanges $\mathbf{E} \leftrightarrow \mathbf{B}$ with $\omega \leftrightarrow -\omega\mu\varepsilon$. As seen earlier, the curl-curl operation on Maxwell's equations can lead to

$$\nabla^2 \mathbf{E} + \mu\varepsilon\omega^2 \mathbf{E} = 0 , \quad (5)$$

$$\nabla^2 \mathbf{B} + \mu\varepsilon\omega^2 \mathbf{B} = 0 . \quad (6)$$

Assuming that waves propagate in the z direction, the fields may be written in Cartesian coordinates as

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}(x,y) e^{i(k_z z - \omega t)} , \quad (7)$$

$$\mathbf{B}(\mathbf{r},t) = \mathbf{B}(x,y) e^{i(k_z z - \omega t)} . \quad (8)$$

For convenience, we set

$$\nabla^2 = \nabla_t^2 + \nabla_z^2 , \quad (9)$$

$$\nabla_t^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} . \quad (10)$$

In terms of ∇_t^2 and substituting Eqs. (7) and (8) into Eqs. (5) and (6), we obtain

$$\left(\nabla_t^2 + \mu\varepsilon\omega^2 - k_z^2\right)\mathbf{E}(x,y) = 0 \quad , \tag{11}$$

$$\left(\nabla_t^2 + \mu\varepsilon\omega^2 - k_z^2\right)\mathbf{B}(x,y) = 0 \, . \tag{12}$$

Next, the two curl equations in (3) and (4) are used to show that the electromagnetic field can be entirely determined by the longitudinal components E_z and B_z from which the transverse components $\mathbf{E}_t$ and $\mathbf{B}_t$ can be derived. The longitudinal components E_z and B_z can be expressed to satisfy a simple scalar wave equation and appropriate boundary conditions. We use $\nabla = \nabla_t + \nabla_z$, $\mathbf{E} = \mathbf{E}_t + \mathbf{E}_z$, and $\mathbf{B} = \mathbf{B}_t + \mathbf{B}_z$ to express Eq. (3) as

$$\left(\nabla_t + \nabla_z\right)\times\left(\mathbf{E}_t + \mathbf{E}_z\right) = i\omega\left(\mathbf{B}_t + \mathbf{B}_z\right) \quad . \tag{13}$$

In the z-direction, we have

$$\nabla_t \times \mathbf{E}_t = i\omega\,\mathbf{B}_z \quad . \tag{14}$$

In the xy-plane, we have

$$\nabla_t \times \mathbf{E}_z + \nabla_z \times \mathbf{E}_t = i\omega\,\mathbf{B}_t \quad . \tag{15}$$

Similarly, we can use Eq. (4) to obtain

$$\nabla_t \times \mathbf{B}_z + \nabla_z \times \mathbf{B}_t = -i\omega\mu\varepsilon\,\mathbf{E}_t \quad . \tag{16}$$

We multiply Eq. (15) from the left by $\nabla_z \times$ and obtain

$$\nabla_z \times \left(\nabla_t \times \mathbf{E}_z + \nabla_z \times \mathbf{E}_t\right) = i\omega\,\nabla_z \times \mathbf{B}_t \quad . \tag{17}$$

Using the relation "*curl curl = grad div minus div grad*" with a triple vector product

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C} \tag{18}$$

taking care of the ordering, Eq. (17) can be rewritten as

$$\nabla_t\left(\nabla_z \cdot \mathbf{E}_z\right) - \left(\nabla_z \cdot \nabla_t\right)\mathbf{E}_z + \nabla_z\left(\nabla_z \cdot \mathbf{E}_t\right) - \nabla_z^2\,\mathbf{E}_t = i\omega\,\nabla_z \times \mathbf{B}_t \quad . \tag{19}$$

The second and third terms on the left of Eq. (19) are scalar products of orthogonal vectors and therefore have no contributions. Consequently, we can use Eq. (16) to eliminate the term $\nabla_z \times \mathbf{B}_t$ and obtain

$$\nabla_t\left(\nabla_z \cdot \mathbf{E}_z\right) - \nabla_z^2\,\mathbf{E}_t = -i\omega\,\nabla_t \times \mathbf{B}_z + \omega^2\mu\varepsilon\,\mathbf{E}_t \quad . \tag{20}$$

To replace $\nabla_z = ik_z$, Eq. (20) can be expressed as

$$(\omega^2\mu\varepsilon - k_z^2)\mathbf{E}_t = \left(ik_z\nabla_t E_z - i\omega\,\mathbf{a}_z \times \nabla_t B_z\right) \quad . \tag{21}$$

Due to the symmetry of $\mathbf{E} \leftrightarrow \mathbf{B}$ and $\omega \leftrightarrow -\omega\mu\varepsilon$ in the Maxwell equations (3) and (4), we also have

$$(\omega^2\mu\varepsilon - k_z^2)\mathbf{B}_t = \left(ik_z\nabla_t B_z + i\omega\mu\varepsilon\ \mathbf{a}_z \times \nabla_t E_z\right) \ . \tag{22}$$

Equations (21) and (22) clearly indicate that the transverse components are entirely determined by the longitudinal components. Therefore, we have to consider only the z-component of Eqs. (11) and (12):

$$\left(\nabla_t^2 + \mu\varepsilon\omega^2 - k_z^2\right) E_z(x,y) = 0 \ \ , \tag{23}$$

$$\left(\nabla_t^2 + \mu\varepsilon\omega^2 - k_z^2\right) B_z(x,y) = 0 \, . \tag{24}$$

The solutions of Eqs. (23) and (24) are solved with the boundary conditions corresponding to a perfect conductor-dielectric interface, namely $\mathbf{n}\cdot\mathbf{B} = 0$ and $\mathbf{n}\times\mathbf{E} = 0$, where n is the unit vector normal to the surface. Note that since surface charges and surface currents are allowed to occur, we are not able to make direct statements on D_{n} and H_{t}. The boundary condition $\mathbf{n}\times\mathbf{E} = 0$ at the walls is just $E_z\big|_S = 0$. On the other hand, the boundary condition $\mathbf{n}\cdot\mathbf{B} = 0$ at the walls is equivalent to $\mathbf{n}\cdot\mathbf{B}_t\big|_S = 0$. Substituting Eq. (22) into $\mathbf{n}\cdot\mathbf{B}_t\big|_S = 0$, we obtain

$$\mathbf{n}\cdot\mathbf{B}_t\ \big|_S = \frac{1}{(\omega^2\mu\varepsilon - k_z^2)}\mathbf{n}\cdot\left(ik_z\nabla_t B_z + i\omega\mu\varepsilon\ \mathbf{a}_z\times\nabla_t E_z\right)\big|_S = \frac{ik_z}{(\omega^2\mu\varepsilon - k_z^2)}\mathbf{n}\cdot\nabla_t B_z\ \big|_S = 0 \, . \tag{25}$$

Here we have used the fact that $\mathbf{a}_z \times \nabla_t E_z$ is tangential to the surface S and therefore $\mathbf{n}\cdot(\mathbf{a}_z\times\nabla_t E_z) = 0$. The boundary condition in Eq. (25) can be manifestly expressed as

$$\mathbf{n}\cdot\nabla_t B_z\ \big|_S = \left.\frac{\partial B_z}{\partial n}\right|_S = 0 \ \ . \tag{26}$$

The two-dimensional wave equations (23) and (24) for E_z and B_z, together with the boundary conditions at the cylindrical surface, form an eigenvalue problem. For a given frequency ω, only definite longitudinal wave numbers k_z obey the differential equation and boundary conditions. According to the boundary conditions, we can distinguish the fields as transverse magnetic (TM) modes and transverse electric (TE) modes. For TM waves, the longitudinal electric field E_z is a solution of Eq. (23) subject to the boundary condition $E_z\big|_S = 0$, while the longitudinal magnetic field B_z vanishes everywhere (this trivially satisfies the wave equation in Eq. (24) for B_z as well as the boundary condition $\partial B_z/\partial n\big|_S = 0$) . **B** has only transverse components. For TE waves, the longitudinal magnetic field B_z is a solution of Eq. (24) subject to the boundary condition $\partial B_z / \partial n|_S = 0$, while the longitudinal electric field E_z vanishes everywhere. **E** has only transverse components. In terms of **E** and **H**, Eqs. (21) and (22) are usually expressed as

$$\mathbf{E}_t = \frac{1}{(\omega^2 \mu\varepsilon - k_z^2)} \left(ik_z \nabla_t E_z - i\omega\mu \, \mathbf{a}_z \times \nabla_t H_z \right) \ ,$$

$$\mathbf{H}_t = \frac{1}{(\omega^2 \mu\varepsilon - k_z^2)} \left(ik_z \nabla_t H_z + i\omega\varepsilon \, \mathbf{a}_z \times \nabla_t E_z \right) \ .$$

Example

Find the dispersion and cutoff frequencies for the TM and TE modes of a rectangular cross section waveguide of sides *a* and *b*.

Solution

For a TM mode, $B_z = 0$ and E_z satisfies

$$\left(\nabla_t^2 + k_t^2 \right) E_z(x, y) = 0$$

where $k_t^2 = \mu\varepsilon\omega^2 - k_z^2$ is used for brevity. The general solution can be expressed as

$$E_z = \begin{Bmatrix} \cos(k_x x) \\ \sin(k_x x) \end{Bmatrix} \begin{Bmatrix} \cos(k_y y) \\ \sin(k_y y) \end{Bmatrix} \ ,$$

with $k_x^2 + k_y^2 = k_t^2 = \mu\varepsilon\omega^2 - k_z^2$ and the braces are used to indicate the arbitrary linear combination of terms enclosed. Applying the boundary conditions $E_z|_S = 0$, we can reduce the solution to

$$E_z(x, y) = \sum_{m,n} A_{m,n} \sin\left(\frac{m\pi x}{a} \right) \sin\left(\frac{n\pi y}{b} \right) \ .$$

The dispersion relation for the *m*, *n* mode then becomes

$$\omega^2 = c^2 \left(k_x^2 + k_y^2 + k_z^2 \right) = c^2 \left[\left(\frac{m\pi}{a} \right)^2 + \left(\frac{n\pi}{b} \right)^2 + k_z^2 \right] \ .$$

When $\omega^2 < c^2 \, [(m\pi / a)^2 + (n\pi / b)^2]$, k_z becomes imaginary, leading to an exponentially damped wave. The cutoff frequency for the waveguide carrying the *m*, *n* mode is

$$\omega_c = c \sqrt{\left(\frac{m\pi}{a} \right)^2 + \left(\frac{n\pi}{b} \right)^2} \ .$$

In terms of E_z and using $B_z = 0$, the remaining components of the field are easily obtained with

$$\mathbf{E}_t = \frac{ik_z}{(\omega^2 \mu\varepsilon - k_z^2)} \nabla_t E_z$$

and

$$\mathbf{H}_t = \frac{i\omega\varepsilon}{(\omega^2 \mu\varepsilon - k_z^2)} \mathbf{a}_z \times \nabla_t E_z \quad .$$

For TE modes, the boundary conditions $\partial B_z / \partial n|_S = 0$ can be used to obtain

$$H_z(x, y) = \sum_{m,n} A_{m,n} \cos\left(\frac{m\pi\, x}{a}\right) \cos\left(\frac{n\pi\, y}{b}\right) \quad .$$

All other parts can be evaluated with the same procedures for deriving the TM modes.

Isosceles right triangular waveguide

A square waveguide, $a = b$, has a further type of degeneracy, since the TM*mn* and TM*nm* modes have the same transverse wavenumber, as do the TE*mn* and TE*nm* modes. By suitable linear combinations of these degenerate modes, it is possible to construct the mode functions appropriate to a guide with a cross section in the form of an isosceles right triangle. The mode function,

$$\psi_{mn}(x, y) = \sin\left(\frac{m\pi\, x}{a}\right) \sin\left(\frac{n\pi\, y}{a}\right) - \sin\left(\frac{n\pi\, x}{a}\right) \sin\left(\frac{m\pi\, y}{a}\right) \quad ,$$

describes a possible TM mode in a square waveguide, with the transverse wavenumber

$$k_t = \frac{\pi}{a}\sqrt{m^2 + n^2} \quad .$$

The function thus constructed vanishes on the line $y = x$ as well as the boundaries $y = 0$, $x = a$, and therefore satisfies all boundary conditions for a TM mode in an isosceles right triangular guide, as shown in Fig. . Note that the function $\psi_{mn}(x, y)$ vanishes if $m = n$, and that therefore an interchange of the integers produces a trivial change in sign of the function. Hence the possible TM modes of an isosceles right triangular waveguide are obtained from $\psi_{mn}(x, y)$ with the integers restricted by $0 < m < n$. Thus the dominate TM mode corresponds to $m = 1$, $n = 2$.

The mode function

$$\varphi_{mn}(x, y) = \cos\left(\frac{m\pi\, x}{a}\right) \cos\left(\frac{n\pi\, y}{a}\right) + \cos\left(\frac{n\pi\, x}{a}\right) \cos\left(\frac{m\pi\, y}{a}\right) \quad ,$$

describing a possible TE mode in the square waveguide, has a vanishing derivative normal to the line $y = x$:

$$\frac{\partial}{\partial n}\varphi_{mn}(x,y) = \frac{1}{2}\left(-\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)\varphi_{mn}(x,y) = 0 \quad , \quad y = x ,$$

and therefore satisfies all boundary conditions for a TE mode in an isosceles right triangular guide under consideration with the transverse wavenumber

$$k_t = \frac{\pi}{a}\sqrt{m^2+n^2} \quad .$$

Note that the function $\varphi_{mn}(x,y)$ is symmetrical in the integers *m* and *n*, and therefore the possible TE modes of an isosceles right triangular waveguide are obtained from $\varphi_{mn}(x,y)$ with the integers restricted by $0 < m \le n$, but with $m = n = 0$ excluded. Thus the dominate TE mode corresponds to $m = 0$, $n = 1$.

Circular waveguide

For cylindrical waveguides we express the *z* component of the fields in terms of the polar coordinates, ρ and ϕ. As before, the *z* component of the fields satisfies the equation

$$\left(\nabla_t^2 + k_t^2\right)\psi(\rho,\phi) = 0 \quad .$$

where ψ represents E_z for TM modes or H_z for TE modes. In polar coordinates, the equation is given by

$$\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}+k_t^2\right]\psi(\rho,\phi) = 0 \quad .$$

Separating variables and applying a periodic boundary condition to the azimuthal component, it can be found that

$$\psi(\rho,\phi) = \sum_m R_m(\rho)e^{\pm im\phi} \quad ,$$

with *m* an integer. The radial function $R_m(\rho)$ satisfies Bessel's equation

$$\left[\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{d}{d\rho}\right)-\frac{m^2}{\rho^2}+k_t^2\right]R_m(\rho) = 0 \quad .$$

The general solution is given by the form

$$R_m(\rho) = A_m J_m(k_t\rho) + B_m N_m(k_t\rho) \quad .$$

The radial equation can be put in a standard form by the change of variable $x = k\rho$. The equation becomes

$$\frac{d^2R}{dx^2}+\frac{1}{x}\frac{dR}{dx}+\left(1-\frac{m^2}{x^2}\right)R=0 \quad . \tag{6}$$

This is the Bessel equation and the solutions are called Bessel functions of the order *m*. With the approach of power series solution, the Bessel function can be found to be given by

$$J_m(x)=\sum_{j=0}^{\infty}\frac{(-1)^j}{\Gamma(j+m+1)j!}\left(\frac{x}{2}\right)^{2j+m} \tag{7}$$

$$J_{-m}(x)=\sum_{j=0}^{\infty}\frac{(-1)^j}{\Gamma(j-m+1)j!}\left(\frac{x}{2}\right)^{2j-m} . \tag{8}$$

These solutions are called Bessel functions of the first kind of order $\pm m$. The series converge for all finite values of *x*. If *m* is not an integer, these two solutions $J_{\pm m}(x)$ form a pair of linearly independent solutions to the second-order Bessel equation. For the potential to be single-valued when the full azimuthal is allowed, *m* must be an integer. Under this circumstance, it is well known that the solutions are linearly dependent. Actually it can be shown that

$$J_{-m}(x)=(-1)^m J_m(x) . \tag{9}$$

In general, no matter what *m* is, the second solution is replaced by the *Neumann function*:

$$N_m(x)=\frac{J_m(x)\cos m\pi - J_{-m}(x)}{\sin m\pi} . \tag{10}$$

The solutions $J_m(x)$ and $N_m(x)$ are called Bessel functions of the second kind.

The Bessel functions of the third kind, called *Hankel functions*, are defined as linear combinations of $J_m(x)$ and $N_m(x)$:

$$\begin{aligned} H_m^{(1)}(x) &= J_m(x)+iN_m(x) \\ H_m^{(2)}(x) &= J_m(x)-iN_m(x) \end{aligned} \quad . \tag{11}$$

The Hankel functions form a fundamental set of solutions to the Bessel equation, just as do $J_m(x)$ and $N_m(x)$.

The other solution in the separation of The Laplace equation is given by Eq. (4). The function $Z(z)$ would have been $\sin kz$ or $\cos kz$ and the equation for $R(\rho)$ would have been:

$$\frac{d^2R}{d\rho^2}+\frac{1}{\rho}\frac{dR}{d\rho}-\left(k^2+\frac{m^2}{\rho^2}\right)R=0 \quad . \tag{12}$$

With $x = k\rho$. The equation becomes

$$\frac{d^2R}{dx^2} + \frac{1}{x}\frac{dR}{dx} - \left(1 + \frac{m^2}{x^2}\right)R = 0 \quad . \tag{13}$$

The solutions of this equation are called modified Bessel functions. It is evident that they are just Bessel functions of pure imaginary argument. The usual choices of linearly independent solutions are denoted by $I_m(x)$ and $K_m(x)$. They are defined by

$$I_m(x) = i^{-m} J_m(ix) \quad , \tag{14}$$

$$K_m(x) = \frac{\pi}{2} i^{m+1} H_m^{(1)}(ix) \tag{15}$$

and are real functions for real x and m.

Another representation for the Bessel functions is based on the generating function. The generating function of the Bessel functions of integral order is given by

$$e^{i\rho\sin\phi} = \sum_{m=-\infty}^{\infty} J_m(\rho) e^{im\phi} \, . \tag{1}$$

Using $i\sin\phi = (e^{i\phi} - e^{-i\phi})/2$, the left-hand side of Eq. (1) can be derived as

$$e^{i\rho\sin\phi} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{\rho}{2}\right)^n \left(e^{i\phi} - e^{-i\phi}\right)^n = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{\rho}{2}\right)^n \sum_{j=0}^{n} \frac{(-1)^j n!}{j!\,(n-j)!} e^{i(n-2j)\phi} \, , \tag{2}$$

where the binominal expansion has applied to the term $\left(e^{i\phi} - e^{-i\phi}\right)^n$ in the derivation. Changing the index n in Eq. (2) as $m = n - 2j$, the expansion in Eq. (2) can be written as

$$e^{i\rho\sin\phi} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{\rho}{2}\right)^n \sum_{j=0}^{n} \frac{(-1)^j n!}{j!\,(n-j)!} e^{i(n-2j)\phi} = \sum_{m=-\infty}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\,(m+j)!}\left(\frac{\rho}{2}\right)^{m+2j} e^{im\phi} \, . \tag{3}$$

In comparison with Eq. (1), the Bessel function is given by

$$J_m(\rho) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!(m+j)!}\left(\frac{\rho}{2}\right)^{m+2j} \quad . \tag{4}$$

This result is the same as Eq. (7) in the above section. Differentiating Eq. (1) partially with respect to ϕ, we have

$$\frac{\partial}{\partial\phi} e^{i\rho\sin\phi} = \frac{\rho}{2}\left(e^{i\phi} + e^{-i\phi}\right) e^{i\rho\sin\phi} = \sum_{m=-\infty}^{\infty} m J_m(\rho) e^{im\phi} \quad . \tag{5}$$

Substituting in Eq. (1) and equating the coefficients of like terms of $e^{im\phi}$, we obtain the result

$$\frac{\rho}{2}\left(e^{i\phi} + e^{-i\phi}\right) \sum_{m=-\infty}^{\infty} J_m(\rho) e^{im\phi} = \sum_{m=-\infty}^{\infty} \frac{\rho}{2}\left[J_{m-1}(\rho) + J_{m+1}(\rho)\right] e^{im\phi} = \sum_{m=-\infty}^{\infty} m J_m(\rho) e^{im\phi} \, .$$

Consequently the first recurrence relation can be expressed as

$$J_{m-1}(\rho)+J_{m+1}(\rho)=\frac{2m}{\rho}J_m(\rho) \quad . \tag{6}$$

Similarly, differentiating Eq. (1) partially with respect to ρ, we have

$$\frac{d}{d\rho}e^{i\rho\sin\phi}=\frac{1}{2}\left(e^{i\phi}-e^{-i\phi}\right)e^{i\rho\sin\phi}=\sum_{m=-\infty}^{\infty}J'_m(\rho)e^{im\phi} \quad . \tag{7}$$

Substituting in Eq. (1) and equating the coefficients of like terms of $e^{im\phi}$, we obtain the result

$$\frac{1}{2}\left(e^{i\phi}-e^{-i\phi}\right)\sum_{m=-\infty}^{\infty}J_m(\rho)e^{im\phi}=\sum_{m=-\infty}^{\infty}\frac{1}{2}\left[J_{m-1}(\rho)-J_{m+1}(\rho)\right]e^{im\phi}=\sum_{m=-\infty}^{\infty}J'_m(\rho)e^{im\phi} .$$

Consequently the second recurrence relation can be expressed as

$$J_{m-1}(\rho)-J_{m+1}(\rho)=2J'_m(\rho) \quad . \tag{8}$$

Suppose we consider a set of function $\Omega_v(x)$ which satisfies the basic recurrence relations (Eqs. (6) and (8)), but with v not necessarily integer and $\Omega_v(x)$ not necessarily given by the series Eq. (4). Subtracting Eq. (8) from Eq. (6) and dividing by 2 yields

$$x\,\Omega'_v(x)-v\Omega_v(x)+x\Omega_{v+1}(x)=0 \quad , \tag{9}$$

where the index has been changed as $(m\rightarrow v)$. Adding Eq. (6) and Eq. (8) and dividing by 2, the result can be rewritten $(m\rightarrow v)$ as

$$x\,\Omega'_v(x)+v\Omega_v(x)-x\Omega_{v-1}(x)=0 \quad . \tag{10}$$

On differentiating with respect to x, we have

$$x\Omega''_v(x)+(v+1)\Omega'_v(x)-\Omega_{v-1}(x)-x\Omega'_{v-1}(x)=0 \quad . \tag{11}$$

Multiplying by x and then subtracting Eq. (10) multiplied by v gives us

$$x^2\,\Omega''_v(x)+x\Omega'_v(x)-v^2\,\Omega_v(x)+x(v-1)\Omega_{v-1}(x)-x^2\,\Omega'_{v-1}(x)=0 \quad . \tag{12}$$

Now we write Eq. (9) and replace v by v-1:

$$x\,\Omega'_{v-1}(x)-(v-1)\Omega_{v-1}(x)+x\Omega_v(x)=0 \, . \tag{13}$$

Adding Eqs. (12) and (13) for eliminating $\Omega'_{v-1}(x)$ and $\Omega_{v-1}(x)$, we finally get

$$x^2\,\Omega''_v(x)+x\Omega'_v(x)+\left(x^2-v^2\right)\Omega_v(x)=0 \, . \tag{14}$$

This is just Bessel's equation. Hence any functions, $\Omega_v(x)$, that satisfy the recurrence relations Eqs. (6) and (8) satisfy Bessel's equation. In other words, the unknown $\Omega_v(x)$ are Bessel functions. In particular, we have shown that the functions $J_m(\rho)$, defined by the generating function, satisfy Bessel's equation. If the argument is $k\rho$ rather than x, Eq. (14) becomes

$$\rho^2 \frac{d^2}{d\rho^2}\Omega_\nu(k\rho) + \rho \frac{d}{d\rho}\Omega_\nu(k\rho) + \left(k^2\rho^2 - \nu^2\right)\Omega_\nu(k\rho) = 0 \,. \tag{15}$$

The generating function in Eq. (1) can be linked to the 2D plane wave:

$$e^{i\left(k_x x + k_y y\right)} = e^{i k\rho \cos(\phi-\theta)} = \sum_{m=-\infty}^{\infty} (i)^m J_m(k\rho) e^{i\,m(\phi-\theta)} \,. \tag{16}$$

From the Fourier transform, we have

$$\left(\frac{1}{2\pi}\right)^2 \iint e^{i\left[k_x(x-x')+k_y(y-y')\right]} dk_x dk_y = \delta(x-x')\delta(y-y') = \frac{\delta(\rho-\rho')}{\rho}\delta(\phi-\phi') \,. \tag{17}$$

Combining Eqs. (16) and (17), we have

$$\begin{aligned}&\left(\frac{1}{2\pi}\right)^2 \int_0^\infty \int_0^{2\pi} \sum_{m=-\infty}^{\infty} (i)^m J_m(k\rho) e^{i\,m(\phi-\theta)} \sum_{m'=-\infty}^{\infty} (i)^{-m'} J_{m'}(k\rho') e^{-i\,m'(\phi'-\theta)} d\theta\, k dk \\ &= \frac{\delta(\rho-\rho')}{\rho}\delta(\phi-\phi')\end{aligned} \,. \tag{18}$$

Using the orthonormal property

$$\left(\frac{1}{2\pi}\right) \int_0^{2\pi} e^{i\,\theta(m'-m)} d\theta = \delta_{m,m'} \quad , \tag{19}$$

we can obtain

$$\left(\frac{1}{2\pi}\right) \int_0^\infty \sum_{m=-\infty}^{\infty} J_m(k\rho) J_m(k\rho') e^{i\,m(\phi-\phi')}\, k dk = \frac{\delta(\rho-\rho')}{\rho}\delta(\phi-\phi') \quad . \tag{20}$$

Since the Fourier series gives

$$\left(\frac{1}{2\pi}\right) \sum_{m=-\infty}^{\infty} e^{i\,m(\phi-\phi')} = \delta(\phi-\phi') \,, \tag{21}$$

we have

$$\int_0^\infty J_m(k\rho) J_m(k\rho')\, k\, dk = \frac{\delta(\rho-\rho')}{\rho} \quad . \tag{22}$$

Equivalently, we have

$$\int_0^\infty J_m(k\rho) J_m(k'\rho)\, \rho\, d\rho = \frac{\delta(k-k')}{k} \quad . \tag{23}$$

If there is a boundary condition $J_m(ka) = 0$ for a finite interval $0 \le \rho \le a$, then the parameter k should be quantized as

$$k_{mn} = x_{mn} / a \quad , \tag{24}$$

where x_{mn} is the nth zero of J_m. The solutions are expected to be orthogonal. The demonstration starts with the differential equation satisfied by $J_m(x_{mn}\rho / a)$:

$$\frac{1}{\rho}\frac{d}{d\rho}\left[\rho\frac{d\,J_m\left(x_{mn}\frac{\rho}{a}\right)}{d\rho}\right]+\left(\frac{x_{mn}^2}{a^2}-\frac{m^2}{\rho^2}\right)J_m\left(x_{mn}\frac{\rho}{a}\right)=0 \quad . \tag{25}$$

Changing the parameter x_{mn} to $x_{mn'}$, we find that $J_m(x_{mn'}\rho/a)$ satisfies

$$\frac{1}{\rho}\frac{d}{d\rho}\left[\rho\frac{d\,J_m\left(x_{mn'}\frac{\rho}{a}\right)}{d\rho}\right]+\left(\frac{x_{mn'}^2}{a^2}-\frac{m^2}{\rho^2}\right)J_m\left(x_{mn'}\frac{\rho}{a}\right)=0 \quad . \tag{26}$$

We multiply Eq. (25) by $\rho J_m(x_{mn'}\rho/a)$ and Eq. (26) by $\rho J_m(x_{mn}\rho/a)$ and subtract, obtaining

$$\begin{aligned}&J_m\left(x_{mn'}\frac{\rho}{a}\right)\frac{d}{d\rho}\left[\rho\frac{d\,J_m\left(x_{mn}\frac{\rho}{a}\right)}{d\rho}\right]-J_m\left(x_{mn}\frac{\rho}{a}\right)\frac{d}{d\rho}\left[\rho\frac{d\,J_m\left(x_{mn'}\frac{\rho}{a}\right)}{d\rho}\right]\\&=\left(\frac{x_{mn'}^2}{a^2}-\frac{x_{mn}^2}{a^2}\right)\rho J_m\left(x_{mn}\frac{\rho}{a}\right)J_m\left(x_{mn'}\frac{\rho}{a}\right)\end{aligned} \quad . \tag{27}$$

Integrating from $\rho=0$ to $\rho=a$, we obtain

$$\begin{aligned}&\frac{x_{mn'}^2-x_{mn}^2}{a^2}\int_0^a \rho\,J_m\left(x_{mn}\frac{\rho}{a}\right)J_m\left(x_{mn'}\frac{\rho}{a}\right)d\rho\\&=\int_0^a\left\{J_m\left(x_{mn'}\frac{\rho}{a}\right)\frac{d}{d\rho}\left[\rho\frac{d\,J_m\left(x_{mn}\frac{\rho}{a}\right)}{d\rho}\right]-J_m\left(x_{mn}\frac{\rho}{a}\right)\frac{d}{d\rho}\left[\rho\frac{d\,J_m\left(x_{mn'}\frac{\rho}{a}\right)}{d\rho}\right]\right\}d\rho\end{aligned} \quad . \tag{28}$$

Upon integrating by parts in the right-hand side of Eq. (28), we have

$$\begin{aligned}&\frac{x_{mn'}^2-x_{mn}^2}{a^2}\int_0^a \rho\,J_m\left(x_{mn}\frac{\rho}{a}\right)J_m\left(x_{mn'}\frac{\rho}{a}\right)d\rho\\&=\rho\frac{d\,J_m\left(x_{mn}\frac{\rho}{a}\right)}{d\rho}J_m\left(x_{mn'}\frac{\rho}{a}\right)\Bigg|_0^a-\rho\frac{d\,J_m\left(x_{mn'}\frac{\rho}{a}\right)}{d\rho}J_m\left(x_{mn}\frac{\rho}{a}\right)\Bigg|_0^a\end{aligned} \quad . \tag{29}$$

For $m\ge 0$ the factor ρ guarantees a zero at the lower limit, $\rho=0$. At $\rho=a$, each expression on the right-hand side of Eq. (29) vanishes because the parameters x_{mn} and $x_{mn'}$ are roots of J_m. Therefore, for $n\neq n'$

$$\int_0^a \rho\,J_m\left(x_{mn}\frac{\rho}{a}\right)J_m\left(x_{mn'}\frac{\rho}{a}\right)d\rho=0 \quad . \tag{30}$$

This gives us orthogonality over the interval [0, a]. The normalization integral may be developed by rewriting Eq. (29) as

$$\begin{aligned}&\int_0^a \rho\, J_m\left(x_{mn}\frac{\rho}{a}\right)J_m\left(x_{mn'}\frac{\rho}{a}\right)d\rho \\ &=\left(\frac{a^2}{x_{mn'}^2-x_{mn}^2}\right)\left[x_{mn}\frac{d\,J_m(x_{mn})}{d\,x}J_m(x_{mn'})-x_{mn'}\frac{d\,J_m(x_{mn'})}{d\,x}J_m(x_{mn})\right]\end{aligned} \quad . \tag{30}$$

Setting $x_{mn'}=x_{mn}+\varepsilon$, and taking the limit $\varepsilon\to 0$, we have

$$\int_0^a \rho\, J_m\left(x_{mn}\frac{\rho}{a}\right)J_m\left(x_{mn}\frac{\rho}{a}\right)d\rho=\left(\frac{a^2}{2}\right)\left[\frac{d\,J_m(x_{mn})}{d\,x}\right]^2 \quad . \tag{31}$$

With the aid of the recurrence relation Eq. (9), this result can be also written as

$$\int_0^a \rho\, J_m\left(x_{mn}\frac{\rho}{a}\right)J_m\left(x_{mn}\frac{\rho}{a}\right)d\rho=\left(\frac{a^2}{2}\right)\left[J_{m+1}(x_{mn})\right]^2 \quad . \tag{32}$$

The behavior of $J_m(\rho)$ near the origin is indicated by the first term of the form of the power series:

$$J_m(\rho)\approx\frac{1}{\Gamma(m+1)}\left(\frac{\rho}{2}\right)^m,\ \ |\rho|<<1 \quad .$$

The form of the Neumann functions near the origin can be given by

$$N_m(\rho)\approx-\frac{\Gamma(m)}{\pi}\left(\frac{2}{\rho}\right)^m,\ \ |\rho|<<1,\ \ m>0 \quad ,$$

$$N_0(\rho)\approx\frac{2}{\pi}\left[\ln\left(\frac{\rho}{2}\right)+0.5772\cdots\right]\ ,\ \ |\rho|<<1 \quad .$$

The values of the Bessel and Neumann functions for sufficiently large magnitudes of ρ are can be expressed as

$$J_m(\rho)\approx\sqrt{\frac{2}{\pi\rho}}\left[\cos\left(\rho-\frac{m\pi}{2}-\frac{\pi}{4}\right)P_m(\rho)-\sin\left(\rho-\frac{m\pi}{2}-\frac{\pi}{4}\right)Q_m(\rho)\right]\ ,$$

$$N_m(\rho)\approx\sqrt{\frac{2}{\pi\rho}}\left[\sin\left(\rho-\frac{m\pi}{2}-\frac{\pi}{4}\right)P_m(\rho)+\cos\left(\rho-\frac{m\pi}{2}-\frac{\pi}{4}\right)Q_m(\rho)\right]\ ,\ \ |\rho|>>1,$$

where

$$P_m(\rho)=1-\frac{(4m^2-1)(4m^2-9)}{2!(8\rho)^2}+\frac{(4m^2-1)(4m^2-9)(4m^2-25)(4m^2-49)}{4!(8\rho)^4}-\cdots,$$

$$Q_m(\rho) = \frac{(4m^2-1)}{8\rho} - \frac{(4m^2-1)(4m^2-9)(4m^2-25)}{3!(8\rho)^3} - \cdots$$

Since the radial function must be well behaved at $\rho = 0$, the coefficients B_m need to be zero, giving

$$\psi(\rho,\phi) = \sum_m A_m J_m(k_t \rho)\, e^{\pm im\phi} \quad .$$

For TM modes, $H_z = 0$ and $E_z = \psi(\rho,\phi)$. At $\rho = a$, the waveguide wall, E_z must vanish, implying that $J_m(k_t a) = 0$. The argument $k_t a$ must therefore be a root of the Bessel function J_m. Let the *n*th root be x_{mn} . Substituting for k_t, we obtain the *z* component of the electric field for the (m,n) mode

$$\psi_{mn}(\rho,\phi) = E_o J_m\left(\frac{x_{mn}}{a}\rho\right) e^{\pm im\phi} \quad .$$

The transverse fields can be obtained as

$$\mathbf{E}_t = \frac{ik_z}{(\omega^2\mu\varepsilon - k_z^2)}\nabla_t E_z = ik_z\left(\frac{a}{x_{mn}}\right)^2 \nabla_t E_z$$

and

$$\mathbf{H}_t = \frac{i\varepsilon\omega}{(\omega^2\mu\varepsilon - k_z^2)}\left(\mathbf{a}_z \times \nabla_t E_z\right) = i\varepsilon\omega\left(\frac{a}{x_{mn}}\right)^2\left(\mathbf{a}_z \times \nabla_t E_z\right) \quad .$$

The dispersion relation is given by

$$k_z^2 = \omega^2\mu\varepsilon - k_t^2 = \omega^2\mu\varepsilon - (x_{mn}/a)^2 \quad .$$

When $\omega^2\mu\varepsilon < (x_{mn}/a)^2$, k_z becomes imaginary and the wave no longer propagates. The various field components of a $TM_{m,n}$ mode are given by

$$\mathbf{E}_t(\rho,\phi) = \frac{ik_z}{(x_{m,n}/a)^2} E_o\left[\left(\frac{x_{m,n}}{a}\right)J'_m\left(\frac{x_{m,n}}{a}\rho\right)e^{\pm im\phi}\,\mathbf{a}_\rho \pm \frac{(im)}{\rho}J_m\left(\frac{x_{m,n}}{a}\rho\right)e^{\pm im\phi}\,\mathbf{a}_\phi\right],$$

$$\mathbf{H}_t(\rho,\phi) = \frac{i\varepsilon\omega}{(x_{m,n}/a)^2} E_o\left[\left(\frac{x_{m,n}}{a}\right)J'_m\left(\frac{x_{m,n}}{a}\rho\right)e^{\pm im\phi}\,\mathbf{a}_\phi \mp \frac{(im)}{\rho}J_m\left(\frac{x_{m,n}}{a}\rho\right)e^{\pm im\phi}\,\mathbf{a}_\rho\right].$$

The $TM_{0,1}$ mode in particular has $x_{0,1} = 2.405$. With $J'_0 = -J_1$, we have

$$E_z(\rho,\phi) = E_o J_0\left(\frac{x_{0,1}}{a}\rho\right)$$

$$\mathbf{E}_t(\rho,\phi) = -\frac{ik_z\, a}{x_{0,1}} E_o J_1\left(\frac{x_{0,1}}{a}\rho\right)\mathbf{a}_\rho$$

$$\mathbf{H}_t(\rho,\phi) = \frac{i\omega\varepsilon\, a}{x_{0,1}} E_o J_1\left(\frac{x_{0,1}}{a}\rho\right)\mathbf{a}_\phi \quad .$$

The cutoff wavelength can be found to be $\lambda_c = 2\pi/(x_{0,1}/a) = 2.61\, a$.

The TE modes have $E_z = 0$ and $H_z = A_m J_m(k_t\rho)\, e^{\pm im\phi}$. The boundary condition requires that $J_m'(k_t a) = 0$. The argument $k_t\, a$ must therefore be a root of the first derivative Bessel function J_m'. Let the *n*th root be x_{mn}' . Substituting for k_t, we obtain the *z* component of the magnetic field for the (m,n) mode

$$\psi_{mn}(\rho,\phi) = H_o J_m\left(\frac{x_{mn}'}{a}\rho\right)e^{\pm im\phi} \quad .$$

The transverse fields can be obtained as

$$\mathbf{H}_t = \frac{ik_z}{(\omega^2\mu\varepsilon - k_z^2)}\nabla_t H_z = ik_z\left(\frac{a}{x_{mn}'}\right)^2 \nabla_t H_z$$

and

$$\mathbf{E}_t = -\frac{i\mu\omega}{(\omega^2\mu\varepsilon - k_z^2)}\left(\mathbf{a}_z \times \nabla_t H_z\right) = -i\mu\omega\left(\frac{a}{x_{mn}'}\right)^2\left(\mathbf{a}_z \times \nabla_t H_z\right) \quad .$$

The various field components of a TEm,n mode are then given by

$$\mathbf{E}_t(\rho,\phi) = \frac{-i\mu\omega}{(x_{m,n}/a)^2} H_o\left[\left(\frac{x_{m,n}'}{a}\right)J_m'\left(\frac{x_{m,n}'}{a}\rho\right)e^{\pm im\phi}\,\mathbf{a}_\phi \mp \frac{(im)}{\rho}J_m\left(\frac{x_{m,n}'}{a}\rho\right)e^{\pm im\phi}\,\mathbf{a}_\rho\right],$$

$$\mathbf{H}_t(\rho,\phi) = \frac{ik_z}{(x_{m,n}/a)^2} H_o\left[\left(\frac{x_{m,n}'}{a}\right)J_m'\left(\frac{x_{m,n}}{a}\rho\right)e^{\pm im\phi}\,\mathbf{a}_\rho \pm \frac{(im)}{\rho}J_m\left(\frac{x_{m,n}'}{a}\rho\right)e^{\pm im\phi}\,\mathbf{a}_\phi\right].$$

The roots for a few values of *m* and *n* are tabulated below:

$$\begin{aligned}
x_{0,n}' &= 3.832,\ 7.016,\ 10.173,\ldots \\
x_{1,n}' &= 1.841,\ 5.331,\ 8.536,\ldots \\
x_{2,n}' &= 3.054,\ 6.706,\ 9.970,\ldots \\
x_{3,n}' &= 4.201,\ 8.015,\ 11.336,\ldots
\end{aligned} .$$

It can be seen that the smallest root of the Bessel function derivatives J'_m, is the first root of J'_1, $x'_{1,1} = 1.841$. The mode with the lowest cut-off frequency is therefore the TE1,1 mode. The fields for the TE1,1 mode are given by

$$H_z(\rho,\phi) = H_o\, J_1\!\left(\frac{x'_{1,1}}{a}\rho\right)e^{\pm i\phi}$$

$$\mathbf{E}_t(\rho,\phi) = -\frac{i\mu\omega a^2}{x'^{\,2}_{1,1}}H_o\left[\mp\frac{i}{\rho}J_1\!\left(\frac{x'_{1,1}}{a}\rho\right)e^{\pm i\phi}\,\mathbf{a}_\rho + \frac{x'_{1,1}}{a}J'_1\!\left(\frac{x'_{1,1}}{a}\rho\right)e^{\pm i\phi}\,\mathbf{a}_\phi\right]$$

$$\mathbf{H}_t(\rho,\phi) = \frac{ik_z a^2}{x'^{\,2}_{1,1}}H_o\left[\frac{x'_{1,1}}{a}J'_1\!\left(\frac{x'_{1,1}}{a}\rho\right)e^{\pm i\phi}\,\mathbf{a}_\rho \pm \frac{i}{\rho}J_1\!\left(\frac{x'_{1,1}}{a}\rho\right)e^{\pm i\phi}\,\mathbf{a}_\phi\right] .$$

The cutoff wavelength can be found to be $\lambda_c = 2\pi/(x'_{1,1}/a) = 3.41a$. Because $J'_0 = -J_1$, the roots of J'_0 coincide with those of J_1, leading to a degeneracy of the TM$_{1,n}$ modes and modes TE$_{0,n}$ modes.

Unlike the rectangular guide, the transverse wavenumbers of the TM and TE modes do not coincide. Explicit formula for the two types of roots can be obtained from the Bessel function asymptotic formulae:

$$x_{mn} = \pi\left(n + \frac{m}{2} - \frac{1}{4}\right)\left[1 - \frac{\left(m^2 - \frac{1}{4}\right)}{2\pi^2\left(n + \frac{m}{2} - \frac{1}{4}\right)^2} - \frac{\left(m^2 - \frac{1}{4}\right)\left(28m^2 - 31\right)}{96\pi^4\left(n + \frac{m}{2} - \frac{1}{4}\right)^4} - \cdots\right] ,$$

$$x'_{mn} = \pi\left(n + \frac{m}{2} - \frac{3}{4}\right)\left[1 - \frac{\left(m^2 + \frac{3}{4}\right)}{2\pi^2\left(n + \frac{m}{2} - \frac{3}{4}\right)^2} - \frac{\left(m^2 + \frac{3}{4}\right)\left(28m^2 + 61\right) - 192}{96\pi^4\left(n + \frac{m}{2} - \frac{3}{4}\right)^4} - \cdots\right]$$

Energy flow and group velocity in wave guides

The expressions for both TM and TE waves can be used to discuss the flow of energy along the guide and the attenuation of the waves due to losses in the walls having finite conductivity. For TM modes, the fields can be expressed as

$$\mathbf{E} = \frac{ik_z}{k_t^2}\nabla_t\psi + \mathbf{a_z}\psi$$

$$\mathbf{H} = \frac{i\varepsilon\omega}{k_t^2}\left(\mathbf{a_z}\times\nabla_t\psi\right).$$

Substituting these expressions into the complex Poynting vector, we can obtain

$$\mathbf{S}_{TM} = \frac{1}{2}\left(\mathbf{E}\times\mathbf{H}^*\right) = \frac{1}{2}\left(\frac{ik_z}{k_t^2}\nabla_t\psi + \mathbf{a_z}\psi\right)\times\left[\frac{-i\varepsilon\omega}{k_t^2}\left(\mathbf{a_z}\times\nabla_t\psi^*\right)\right]$$
$$= \frac{1}{2}\left(\mathbf{a_z}\frac{k_z\varepsilon\omega}{k_t^4}\left|\nabla_t\psi\right|^2 + \frac{i\varepsilon\omega}{k_t^2}\psi\nabla_t\psi^*\right)$$

Similarly, the fields and the complex Poynting vector for TE modes can be given by

$$\mathbf{H} = \frac{ik_z}{k_t^2}\nabla_t\psi + \mathbf{a_z}\psi$$
$$\mathbf{E} = \frac{-i\mu\omega}{k_t^2}\left(\mathbf{a_z}\times\nabla_t\psi\right).$$

$$\mathbf{S}_{TE} = \frac{1}{2}\left(\mathbf{E}\times\mathbf{H}^*\right) = \frac{1}{2}\left[\frac{-i\mu\omega}{k_t^2}\left(\mathbf{a_z}\times\nabla_t\psi\right)\right]\times\left(\frac{-ik_z}{k_t^2}\nabla_t\psi^* + \mathbf{a_z}\psi^*\right)$$
$$= \frac{1}{2}\left(\mathbf{a_z}\frac{k_z\mu\omega}{k_t^4}\left|\nabla_t\psi\right|^2 - \frac{i\mu\omega}{k_t^2}\psi\nabla_t\psi^*\right).$$

Since the wave function ψ is generally real, the transverse component of **S** represents reactive energy flow and does not contribute to the time averaged flux of energy. On the other hand, the axial component of **S** gives the time-averaged flow of energy along the guide. The total power flow over the cross-section area A can evaluated with integrating the axial component of **S**:

$$P_{TM} = \int_A \mathbf{S}_{TM}\cdot\mathbf{a_z}\,da = \frac{1}{2}\frac{k_z\varepsilon\omega}{k_t^4}\int_A (\nabla_t\psi^*\cdot\nabla_t\psi)\,da$$
$$P_{TE} = \int_A \mathbf{S}_{TE}\cdot\mathbf{a_z}\,da = \frac{1}{2}\frac{k_z\mu\omega}{k_t^4}\int_A (\nabla_t\psi^*\cdot\nabla_t\psi)\,da \quad .$$

Using Green's first identity applied to two dimensions, the total power flow can be written as

$$P_{TM} = \frac{1}{2}\frac{k_z\varepsilon\omega}{k_t^4}\left[-\int_A (\psi^*\cdot\nabla_t^2\psi)\,da + \oint_C (\psi^*\cdot\partial\psi/\partial n)dl\right]$$
$$P_{TE} = \frac{1}{2}\frac{k_z\mu\omega}{k_t^4}\left[-\int_A (\psi^*\cdot\nabla_t^2\psi)\,da + \oint_C (\psi^*\cdot\partial\psi/\partial n)dl\right]$$

where the second integral is around the curve C, which defines the boundary surface of the cylinder. This integral vanishes for both types of field because of boundary conditions. By means of the wave equation, we have $\nabla_t^2\psi = -k_t^2\psi$ and $k_z = \sqrt{k^2 - k_t^2}$. the first integral can be reduced to the normalization integral for ψ:

$$P_{TM} = \frac{1}{2}\frac{\varepsilon\omega k_z}{k_t^2}\left[\int_A (\psi^* \cdot \psi)\, da\right] = \frac{1}{2\sqrt{\mu\varepsilon}}\varepsilon\frac{\omega^2}{\omega_t^2}\sqrt{1-\frac{\omega_t^2}{\omega^2}}\left[\int_A (\psi^* \cdot \psi)\, da\right]$$

$$P_{TE} = \frac{1}{2}\frac{\mu\omega k_z}{k_t^2}\left[\int_A (\psi^* \cdot \psi)\, da\right] = \frac{1}{2\sqrt{\mu\varepsilon}}\mu\frac{\omega^2}{\omega_t^2}\sqrt{1-\frac{\omega_t^2}{\omega^2}}\left[\int_A (\psi^* \cdot \psi)\, da\right]$$

The field energy per unit length of the guided wave can be evaluated in the same way. The energy density is given by

$$u_{TM} = \frac{1}{4}\left(\varepsilon\,\mathbf{E}\cdot\mathbf{E}^* + \mu\,\mathbf{H}\cdot\mathbf{H}^*\right) = \frac{1}{4}\left[\varepsilon\left(\frac{k_z^2}{k_t^4}|\nabla_t\psi|^2 + |\psi|^2\right) + \mu\left(\frac{\varepsilon^2\omega^2}{k_t^4}|\nabla_t\psi|^2\right)\right]$$

$$u_{TE} = \frac{1}{4}\left(\varepsilon\,\mathbf{E}\cdot\mathbf{E}^* + \mu\,\mathbf{H}\cdot\mathbf{H}^*\right) = \frac{1}{4}\left[\varepsilon\left(\frac{\mu^2\omega^2}{k_t^4}|\nabla_t\psi|^2\right) + \mu\left(\frac{k_z^2}{k_t^4}|\nabla_t\psi|^2 + |\psi|^2\right)\right] .$$

Integrating the density for all the transverse plane, the field energy per unit length of the guided wave along z-axis for TM mode is then given by

$$\begin{aligned} U_{TM} &= \int_A u_{TM}\, da = \int_A \frac{1}{4}\left[\varepsilon\left(\frac{k_z^2}{k_t^4}|\nabla_t\psi|^2 + |\psi|^2\right) + \mu\left(\frac{\varepsilon^2\omega^2}{k_t^4}|\nabla_t\psi|^2\right)\right] da \\ &= \int_A \frac{1}{4}\left[\varepsilon\left(\frac{k_z^2}{k_t^2}|\psi|^2 + |\psi|^2\right) + \mu\left(\frac{\varepsilon^2\omega^2}{k_t^2}|\psi|^2\right)\right] da \\ &= \frac{1}{2}\varepsilon\frac{k^2}{k_t^2}\int_A |\psi|^2\, da \end{aligned} .$$

Similarly, we have

$$U_{TE} = \int_A u_{TM}\, da = \frac{1}{2}\mu\frac{k^2}{k_t^2}\int_A |\psi|^2\, da$$

It can be found that *P* and *U* are proportional with the constant to be just the group velocity:

$$\frac{P}{U} = \frac{\omega}{k^2}k_z = \frac{\omega}{k}\sqrt{1-\frac{k_t^2}{k^2}} = \frac{k_z}{\omega\mu\varepsilon} = \frac{\omega}{k}\sqrt{1-\frac{\omega_t^2}{\omega^2}} = \frac{1}{\sqrt{\mu\varepsilon}}\sqrt{1-\frac{\omega_t^2}{\omega^2}} = v_g = \frac{d\omega}{dk_z} .$$

Note that $k_z = \sqrt{\mu\varepsilon}\sqrt{\omega^2 - \omega_t^2}$. It is clear that v_g is always less than the velocity of waves in an infinite medium and falls to zero at cutoff. The product of phase velocity and group velocity is constant:

$$v_g v_p = 1/\mu\varepsilon .$$

This is an immediate consequence of the fact that $\omega\Delta\omega \propto k_z\,\Delta k_z$.

Example

Calculate the power transported by a TM$_{m,n}$ wave in a rectangular waveguide of sides a and b. Estimate the power loss in the walls, assuming ohmic losses. Use these results to find the attenuation length of the waveguide.

Solution

For a TM$_{m,n}$ mode, $B_z = 0$ and E_z is given by

$$\psi(x,y) = E_o \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

The power transported by a TM$_{m,n}$ wave can be evaluated as

$$P_{TM} = \frac{1}{2}\frac{\varepsilon\omega\, k_z}{k_t^2}\left[\int_0^b\int_0^a (\psi^* \cdot \psi)\, dx\, dy\right] = \frac{\varepsilon\omega\, k_z\, E_o^2\, ab}{8\, k_t^2}.$$

We proceed to find the power dissipated in traveling along the waveguide. To do so, we require $H_{\parallel}$ at each of the walls. The magnetic field in the cavity is given by

$$\mathbf{H} = \frac{i\varepsilon\omega}{k_t^2}\left(\mathbf{a_z}\times\nabla_t\psi\right)$$
$$= \frac{i\varepsilon\omega}{k_t^2}E_o\left[-\left(\frac{n\pi}{b}\right)\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)\mathbf{a}_x + \left(\frac{m\pi}{a}\right)\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\mathbf{a}_y\right].$$

Along the bottom wall, $y = 0$, $H_{\parallel}$ is given by

$$\mathbf{H}_{\parallel} = -\frac{i\varepsilon\omega}{k_t^2}E_o\left(\frac{n\pi}{b}\right)\sin\left(\frac{m\pi x}{a}\right)\mathbf{a}_x\ .$$

The power dissipation along the bottom wall is

$$-\left.\frac{dP}{dz}\right|_{y-0} = \frac{1}{2\sigma\delta}\int_0^a |H_{\parallel}|^2\, dx$$
$$= \frac{1}{2\sigma\delta}\frac{\varepsilon^2\omega^2}{k_t^4}E_o^2\left(\frac{n\pi}{b}\right)^2\left[\int_0^a \sin^2\left(\frac{m\pi x}{a}\right)dx\right] = \frac{1}{2\sigma\delta}\frac{\varepsilon^2\omega^2}{k_t^4}E_o^2\left(\frac{n\pi}{b}\right)^2\left(\frac{a}{2}\right).$$

The top wall gives an identical result. On the side wall at $x = 0$, we have

$$\mathbf{H}_{\parallel} = -\frac{i\varepsilon\omega}{k_t^2}E_o\left(\frac{m\pi}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\mathbf{a}_y$$

and the power dissipated on the side wall is

$$-\left.\frac{dP}{dz}\right|_{x-0} = \frac{1}{2\sigma\delta}\int_0^b |H_{\parallel}|^2\, dy$$
$$= \frac{1}{2\sigma\delta}\frac{\varepsilon^2\omega^2}{k_t^4}E_o^2\left(\frac{m\pi}{a}\right)^2\left[\int_0^b \sin^2\left(\frac{n\pi y}{b}\right)dy\right] = \frac{1}{2\sigma\delta}\frac{\varepsilon^2\omega^2}{k_t^4}E_o^2\left(\frac{m\pi}{a}\right)^2\left(\frac{b}{2}\right).$$

Again, the right hand wall at $x = a$ gives the same result. Adding all four terms, we have

$$-\frac{dP}{dz} = \frac{1}{2\sigma\delta}\frac{\varepsilon^2\omega^2}{k_t^4}E_o^2\left[\left(\frac{m\pi}{a}\right)^2 b + \left(\frac{n\pi}{b}\right)^2 a\right].$$

Finally, the attenuation constant can be derived as

$$-\frac{1}{2P}\frac{dP}{dz} = \frac{\dfrac{1}{2\sigma\delta}\dfrac{\varepsilon^2\omega^2}{k_t^4}E_o^2\left[\left(\dfrac{m\pi}{a}\right)^2 b + \left(\dfrac{n\pi}{b}\right)^2 a\right]}{2\dfrac{\varepsilon\omega\, k_z\, E_o^2\, ab}{8\, k_t^2}} = \frac{2}{\sigma\delta}\frac{\varepsilon\omega}{k_t^2\, k_z ab}\left[\left(\frac{m\pi}{a}\right)^2 b + \left(\frac{n\pi}{b}\right)^2 a\right]$$

$$= \frac{1}{\sigma\delta_t}\sqrt{\frac{\varepsilon}{\mu}}\frac{\sqrt{\dfrac{\omega}{\omega_t}}}{\sqrt{1-\dfrac{\omega_t^2}{\omega^2}}}\frac{C}{2A}\,\xi$$

where

$$\xi = \frac{4}{C}\frac{\left[\left(\dfrac{m\pi}{a}\right)^2 b + \left(\dfrac{n\pi}{b}\right)^2 a\right]}{\left(\dfrac{m\pi}{a}\right)^2 + \left(\dfrac{n\pi}{b}\right)^2}$$

is a dimensionless number of the order unity, $\delta_t = \sqrt{2/\mu\sigma\omega_t}$ is the skin depth at the cutoff frequency, $C = 2(a+b)$ is the circumference, and $A = ab$ is the area of cross section.

Example

Calculate the power transported by a $TE_{m,n}$ wave in a rectangular waveguide of sides *a* and *b*. Estimate the power loss in the walls, assuming ohmic losses. Use these results to find the attenuation length of the waveguide.

Solution

For a $TE_{m,n}$ mode, $E_z = 0$ and B_z is given by

$$\psi(x,y) = H_o\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)$$

The power transported by a $TE_{m,n}$ wave can be evaluated as

$$P_{TE} = \frac{1}{2}\frac{\mu\omega\, k_z}{k_t^2}\left[\int_0^b\int_0^a (\psi^*\cdot\psi)\,dx\,dy\right] = \frac{\mu\omega\, k_z\, H_o^2\, ab}{8\, k_t^2}.$$

We proceed to find the power dissipated in traveling along the waveguide. To do so, we require $H_\parallel$ at each of the walls. The magnetic field in the cavity is given by

$$\mathbf{H} = \frac{ik_z}{k_t^2}\nabla_t \psi + \mathbf{a_z}\psi$$

$$= H_o\left\{-\frac{ik_z}{k_t^2}\left[\left(\frac{m\pi}{a}\right)\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)\mathbf{a}_x + \left(\frac{n\pi}{b}\right)\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\mathbf{a}_y\right].\right.$$

$$\left. + \cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)\mathbf{a}_z\right\}$$

Along the bottom wall, $y = 0$, $H_\parallel$ is given by

$$\mathbf{H}_\parallel = -\frac{ik_z}{k_t^2}H_o\left(\frac{m\pi}{a}\right)\sin\left(\frac{m\pi x}{a}\right)\mathbf{a}_x + H_o\cos\left(\frac{m\pi x}{a}\right)\mathbf{a}_z \quad .$$

The power dissipation along the bottom wall is

$$-\left.\frac{dP}{dz}\right|_{y-0} = \frac{1}{2\sigma\delta}\int_0^a |H_\parallel|^2\, dx$$

$$= \frac{1}{2\sigma\delta}H_o^2\left\{\frac{k_z^2}{k_t^4}\left(\frac{m\pi}{a}\right)^2\left[\int_0^a \sin^2\left(\frac{m\pi x}{a}\right)dx\right] + \int_0^a \cos^2\left(\frac{m\pi x}{a}\right)dx\right\}.$$

$$= \frac{1}{2\sigma\delta}H_o^2\left(\frac{a}{2}\right)\left[1 + \frac{k_z^2}{k_t^4}\left(\frac{m\pi}{a}\right)^2\right]$$

The top wall gives an identical result. On the side wall at $x = 0$, we have

$$\mathbf{H}_\parallel = -\frac{ik_z}{k_t^2}H_o\left(\frac{n\pi}{b}\right)\sin\left(\frac{n\pi y}{b}\right)\mathbf{a}_y + H_o\cos\left(\frac{n\pi y}{b}\right)\mathbf{a}_z$$

and the power dissipated on the side wall is

$$-\left.\frac{dP}{dz}\right|_{x-0} = \frac{1}{2\sigma\delta}\int_0^b |H_\parallel|^2\, dy$$

$$= \frac{1}{2\sigma\delta}H_o^2\left\{\frac{k_z^2}{k_t^4}\left(\frac{n\pi}{b}\right)^2\left[\int_0^b \sin^2\left(\frac{n\pi y}{b}\right)dy\right] + \int_0^b \cos^2\left(\frac{n\pi y}{b}\right)dy\right\}.$$

$$= \frac{1}{2\sigma\delta}H_o^2\left(\frac{b}{2}\right)\left[1 + \frac{k_z^2}{k_t^4}\left(\frac{n\pi}{b}\right)^2\right]$$

Again, the right hand wall at $x = a$ gives the same result. Adding all four terms, we have

$$-\frac{dP}{dz} = \frac{1}{2\sigma\delta}H_o^2\left\{(a+b) + \frac{k_z^2}{k_t^4}\left[\left(\frac{m\pi}{a}\right)^2 a + \left(\frac{n\pi}{b}\right)^2 b\right]\right\} \quad .$$

Thus, the attenuation constant can be derived as

$$-\frac{1}{2P}\frac{dP}{dz}=\frac{\frac{1}{2\sigma\delta}H_o^2\left\{(a+b)+\frac{k_z^2}{k_t^4}\left[\left(\frac{m\pi}{a}\right)^2 a+\left(\frac{n\pi}{b}\right)^2 b\right]\right\}}{2\frac{\mu\omega k_z H_o^2 ab}{8k_t^2}}.$$

$$=\frac{2}{\sigma\delta}\frac{k_z}{\mu\omega}\frac{k_t^2}{k_z^2}\frac{1}{ab}\left\{(a+b)+\frac{k_z^2}{k_t^4}\left[\left(\frac{m\pi}{a}\right)^2 a+\left(\frac{n\pi}{b}\right)^2 b\right]\right\}$$

This result can be further rewritten as

$$-\frac{1}{2P}\frac{dP}{dz}=\frac{2}{\sigma\delta}\frac{k_z}{\mu\omega}\frac{k^2}{k_z^2}\frac{1}{ab}\left\{\frac{k_t^2}{k^2}(a+b)+\frac{k_z^2}{k^2}\frac{\left[\left(\frac{m\pi}{a}\right)^2 a+\left(\frac{n\pi}{b}\right)^2 b\right]}{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2}\right\}$$

$$=\frac{2}{\sigma\delta}\frac{k_z}{\mu\omega}\frac{k^2}{k_z^2}\frac{1}{ab}\left\{\frac{k_t^2}{k^2}(a+b)+\left(1-\frac{k_t^2}{k^2}\right)\frac{\left[\left(\frac{m\pi}{a}\right)^2 a+\left(\frac{n\pi}{b}\right)^2 b\right]}{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2}\right\}$$

$$=\frac{2}{\sigma\delta}\frac{k_z}{\mu\omega}\frac{k^2}{k_z^2}\frac{1}{ab}\left\{\left(\frac{k_t^2}{k^2}\right)\frac{\left[\left(\frac{m\pi}{a}\right)^2 b+\left(\frac{n\pi}{b}\right)^2 a\right]}{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2}+\frac{\left[\left(\frac{m\pi}{a}\right)^2 a+\left(\frac{n\pi}{b}\right)^2 b\right]}{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2}\right\}$$

$$=\frac{1}{\sigma\delta_t}\sqrt{\frac{\varepsilon}{\mu}}\frac{\sqrt{\frac{\omega}{\omega_t}}}{\sqrt{1-\frac{\omega_t^2}{\omega^2}}}\frac{C}{2A}\left[\left(\frac{\omega_t^2}{\omega^2}\right)\eta+\xi\right]$$

where

$$\eta=\frac{4}{C}\frac{\left(\frac{m\pi}{a}\right)^2 b+\left(\frac{n\pi}{b}\right)^2 a}{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2}$$

and

$$\xi=\frac{4}{C}\frac{\left(\frac{m\pi}{a}\right)^2 a+\left(\frac{n\pi}{b}\right)^2 b}{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2}$$

are dimensionless numbers of the order unity, $\delta_t=\sqrt{2/\mu\sigma\omega_t}$ is the skin depth at the cutoff frequency, $C=2(a+b)$ is the circumference, and $A=ab$ is the area of cross section.

Resonant cavities

The modes of resonant cavities are easily obtained by using linear combinations of waves traveling in the $\pm z$ direction. For perfectly conducting plates at $z=0$ and $z=d$, there are additional boundary conditions on **H** and **E**, namely $\mathbf{E}_t=0$ and $H_z=0$. The latter implies for TE modes that

$$H_z=\psi(x,y)\sin(p\pi z/d).$$

For TM modes, $\mathbf{E}_t=0$ on the end walls give $\partial E_z/\partial n=\pm\partial E_z/\partial z=0$ using an argument similar to that used for treating $\mathbf{H}_t=0$ on the side walls for TE modes. Thus, the TM cavity mode is given by

$$E_z=\psi(x,y)\cos(p\pi z/d).$$

For the cavity mode, the formula

$$\mathbf{E}_t=\frac{1}{(\omega^2\mu\varepsilon-k_z^2)}\left(ik_z\nabla_t E_z-i\omega\mu\,\mathbf{a}_z\times\nabla_t H_z\right) \quad ,$$

$$\mathbf{H}_t=\frac{1}{(\omega^2\mu\varepsilon-k_z^2)}\left(ik_z\nabla_t H_z+i\omega\varepsilon\,\mathbf{a}_z\times\nabla_t E_z\right) \quad .$$

need to be modified with replacing k_z by $\partial/\partial z$:

$$\mathbf{E}_t=\frac{1}{(\omega^2\mu\varepsilon-k_z^2)}\left(\nabla_t\frac{\partial E_z}{\partial z}-i\omega\mu\,\mathbf{a}_z\times\nabla_t H_z\right) \quad ,$$

$$\mathbf{H}_t=\frac{1}{(\omega^2\mu\varepsilon-k_z^2)}\left(\nabla_t\frac{\partial H_z}{\partial z}+i\omega\varepsilon\,\mathbf{a}_z\times\nabla_t E_z\right) \quad .$$

As a result, the transverse fields for TM modes with $E_z=\psi(x,y)\cos(p\pi z/d)$ in the cavity become:

$$\mathbf{E}_t=-\frac{p\pi}{d\,k_t^2}\sin\left(\frac{p\pi z}{d}\right)\nabla_t\psi(x,y) \quad ,$$

$$\mathbf{H}_t=\frac{i\omega\varepsilon}{k_t^2}\cos\left(\frac{p\pi z}{d}\right)\left[\mathbf{a}_z\times\nabla_t\psi(x,y)\right] \quad .$$

Similarly, the transverse fields for TE modes with $H_z=\psi(x,y)\sin(p\pi z/d)$ in the cavity become:

$$\mathbf{E}_t=-\frac{i\mu\omega}{k_t^2}\sin\left(\frac{p\pi z}{d}\right)\left[\mathbf{a}_z\times\nabla_t\psi(x,y)\right] \quad ,$$

$$\mathbf{H}_t = \frac{p\pi}{d\,k_t^2}\cos\left(\frac{p\pi z}{d}\right)\nabla_t \psi(x,y).$$

Resonant frequencies for a cylindrical cavity

The resonant frequency for the TM mode in a cylindrical waveguide is given by

$$\omega_{m,n,p} = \frac{1}{\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{x_{m,n}}{a}\right)^2 + \left(\frac{p\pi}{d}\right)^2} \quad .$$

The lowest frequency is given by $m=0$, $n=1$, $p=0$. This is called the TM_{010} mode. The corresponding frequency is

$$\omega_{0,1,0} = \frac{1}{\sqrt{\mu\varepsilon}}\frac{2.405}{a} \quad .$$

Note that this frequency is independent of *d*. Thus one cannot tune such a mode with a piston which changes the length of the cavity.

The resonant frequency for the TE mode in a cylindrical waveguide is given by

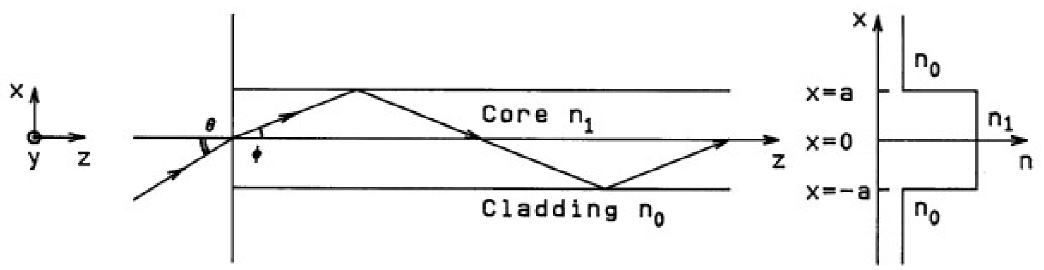
$$\omega_{m,n,p} = \frac{1}{\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{x'_{m,n}}{a}\right)^2 + \left(\frac{p\pi}{d}\right)^2} \quad .$$

The lowest frequency is given by $m=1$, $n=1$, $p=1$. This is called the TE_{111} mode. The corresponding frequency is

$$\omega_{1,1,1} = \frac{1}{\sqrt{\mu\varepsilon}}\frac{1.841}{a}\sqrt{1+\left(\frac{\pi}{1.841}\right)^2\left(\frac{a}{d}\right)^2} = \frac{1}{\sqrt{\mu\varepsilon}}\frac{1.841}{a}\sqrt{1+2.912\left(\frac{a}{d}\right)^2} \quad .$$

For $a/d < 1/2.03$ this TE_{111} mode is the overall lowest or fundamental mode. Note also that this frequency depends on a/d and can therefore be easily tune by using a piston.

Formation of guided waves



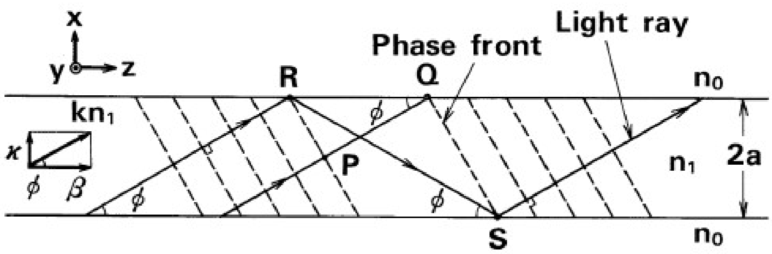
Optical fibers and optical waveguides consist of a core, in which light is confined, and a cladding, or substrate surrounding the core, as shown in Fig.. The refractive index of the core n_1 is higher than that of the cladding n_o also higher than that of the substrate n_s. Therefore the light beam that is coupled to the end face of the waveguide is confined in the core by the total internal reflection. Assuming $n_1 > n_s \geq n_o$, the condition for the total internal reflection at the core-substrate interface is given by $n_1 \sin(\pi/2-\phi) \geq n_s$, where the angle ϕ is related to the incident angle θ by $\sin\theta = n_1 \sin\phi$. As a result, we obtain $\sin\theta \leq \sqrt{n_1^2 - n_s^2}$ and the critical condition for the total internal reflection as

$$\sin\theta_{\max} = \sqrt{n_1^2 - n_s^2} \quad . \tag{1}$$

For the case of optical fibers, $n_s = n_o$, the refractive-index difference between core and cladding is of the order of $n_1 - n_o = 0.01$. Then $\theta_{\max}$ in Eq. (1) can be approximated by

$$\theta_{\max} \cong \sqrt{n_1^2 - n_o^2} \quad . \tag{2}$$

The value of $\theta_{\max}$ denotes the maximum light acceptance angle of the waveguide and is known as the numerical aperture (NA).



We have accounted for the mechanism of the mode confinement and have indicated that the angle ϕ must not exceed the critical angle. Although the angle ϕ is smaller than the critical angle, light rays with arbitrary angles are not able to propagate in the waveguide. Each mode is associated with light rays at a discrete angle of propagation, as given by electromagnetic wave analysis. Let us consider a plane wave propagating along the z-direction with inclination angle ϕ. The wavelength and the wavenumber of light in the core are λ / n_1 and $n_1 k$, respectively, where λ is the wavelength of light in vacuum. The propagation constants along z and x are expressed by

$$k_z = n_1 k \cos\phi \quad , \tag{3}$$

$$k_t = n_1 k \sin\phi \quad . \tag{4}$$

From the Fresnel's law, the reflection coefficient of the reflected TE wave is given by

$$\left(\frac{E_r}{E_i}\right)_s = r = \frac{n_1\cos\theta_i - n_s\cos\theta_t}{n_1\cos\theta_i + n_s\cos\theta_t} = \frac{n_1 k\sin\phi - n_s k\sin\phi_t}{n_1 k\sin\phi + n_s k\sin\phi_t} \quad , \tag{5}$$

where ϕ_t is the inclination angle of the refracted beam. The Snell's law gives $n_s\, k\cos\phi_t = n_1\, k\cos\phi = k_z$. As a consequence, $n_s k\sin\phi_t$ for the totally reflected light can be expressed as

$$n_s\, k\sin\phi_t = \sqrt{(n_s k)^2 - k_z^2} = i\sqrt{k_z^2 - (n_s k)^2} = i\sqrt{(n_1^2 - n_s^2)k^2 - k_t^2} \quad . \tag{6}$$

Substituting Eq. (6) into Eq. (5), the reflection coefficient of the reflected TE wave is given by

$$r = \frac{k_t - i\sqrt{(n_1^2 - n_s^2)k^2 - k_t^2}}{k_t + i\sqrt{(n_1^2 - n_s^2)k^2 - k_t^2}} \quad . \tag{7}$$

Expressing the complex reflection coefficient *r* as $r = \exp(-i\Phi_s)$, the amount of phase shift Φ_s is given by

$$\Phi_s = 2\tan^{-1}\left[\frac{\sqrt{(n_1^2 - n_s^2)k^2 - k_t^2}}{k_t}\right] \quad . \tag{8}$$

Now the phase difference between the two light rays belonging to the same plane wave in Fig. 2 is considered. Light ray *PQ*, which propagates from point *P* to *Q*, does not suffer the influence of reflection. On the other hand, light ray *RS*, propagating from point *R* to *S*, is reflected two times (at the upper and lower core-cladding interfaces). Since points *P* and *R* or points *Q* and *S* are on the same phase front, optical paths *PQ* and *RS* should be equal, or their difference should be an integral multiple of 2π. Using the fact that the distance between points *Q* and *R* is $2a/\tan\phi - 2a\tan\phi$, the distance between points *P* and *Q* is given by

$$d_1 = (2a/\tan\phi - 2a\tan\phi)\cos\phi = 2a\left(\frac{1}{\sin\phi} - 2\sin\phi\right) \quad . \tag{9}$$

The distance between points *R* and *S* is given by

$$d_2 = \frac{2a}{\sin\phi} \quad . \tag{10}$$

The phase matching condition for the optical paths *PQ* and *RS* then becomes

$$(kn_1 d_2 - \Phi_s - \Phi_o) - kn_1 d_1 = 2m\pi \quad , \tag{11}$$

where *m* is an integer. Substituting Eqs. (4) and (8-10) into Eq. (11) we obtain the condition for the propagation angle ϕ as

$$k_t\, a = \frac{m\pi}{2} + \frac{1}{2}\tan^{-1}\left(\frac{\sqrt{(n_1^2 - n_s^2)k^2 - k_t^2}}{k_t}\right) + \frac{1}{2}\tan^{-1}\left(\frac{\sqrt{(n_1^2 - n_o^2)k^2 - k_t^2}}{k_t}\right) \quad . \tag{12}$$

Planar Optical Waveguides

Planar optical waveguides are the fundamental elements to construct integrated optical circuits and semiconductor lasers. In general, rectangular waveguides consist of a square or rectangular core surrounded by a cladding with lower refractive index than that of the core. Although three-dimensional analysis is necessary to explore the transmission characteristics, two-dimensional slab waveguides are often used to acquire a clear insight into optical waveguides.

Taking into account the effect of refractive index in dielectric optical waveguides, the two curl Maxwell's equations for **E** and **H** are given by

$$\nabla \times \mathbf{E} = -\mu_o \frac{\partial \mathbf{H}}{\partial t} \quad , \tag{1}$$

$$\nabla \times \mathbf{H} = \varepsilon\, n^2 \frac{\partial\, \mathbf{E}}{\partial\, t} \quad . \tag{2}$$

Assuming that waves propagate in the z direction of the slab waveguide, the fields may be written in Cartesian coordinates as

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}(x,y)e^{\,i(k_z z - \omega t)} \quad , \tag{3}$$

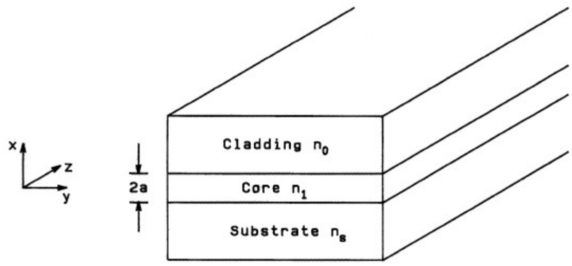
$$\mathbf{H}(\mathbf{r},t) = \mathbf{H}(x,y)e^{\,i(k_z z - \omega t)} \quad . \tag{4}$$

Substituting Eqs. (3) and (4) into Eqs. (1) and (2), we obtain the following set of equations:

$$\begin{aligned} &\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = i\omega\mu_o H_z \\ &\frac{\partial E_z}{\partial y} - ik_z E_y = i\omega\mu_o H_x \\ &ik_z E_x - \frac{\partial E_z}{\partial x} = i\omega\mu_o H_y \end{aligned} \tag{5}$$

and

$$\begin{aligned} &\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = -i\omega\varepsilon_o n^2 E_z \\ &\frac{\partial H_z}{\partial y} - ik_z H_y = -i\omega\varepsilon_o n^2 E_x \quad . \\ &ik_z H_x - \frac{\partial H_z}{\partial x} = -i\omega\varepsilon_o n^2 E_y \end{aligned} \tag{6}$$



For the slab waveguide, as shown in Fig., **E** and **H** do not have y-axis dependency. Therefore, we can set $\partial \mathbf{E}/\partial y = 0$ and $\partial \mathbf{H}/\partial y = 0$. Substituting these relations into Eqs. (5) and (6), electromagnetic waves inside the waveguide can be decomposed into two independent modes that are denoted as TE mode with $E_z = 0$ and TM mode with $H_z = 0$. The TE mode can be found to satisfy the wave equation only in terms of E_y:

$$\frac{d^2 E_y}{d x^2} + \left(k^2 n^2 - k_z^2\right) E_y = 0 \quad , \tag{7}$$

where $\omega^2 \mu_o \varepsilon_o = k^2$

$$H_x = -\frac{k_z}{\omega \mu_o} E_y \quad , \tag{8}$$

$$H_z = \frac{-i}{\omega \mu_o} \frac{d E_y}{d x} \quad , \tag{9}$$

and

$$E_z = E_x = H_y = 0 \quad . \tag{10}$$

Since the electric field lies in the plane that is perpendicular to the *z*-axis, this electromagnetic field distribution is called transverse electric (TE) mode. Furthermore, the tangential components E_y and H_z should be continuous at the boundaries of two different media.

On the other hand, the TM modes satisfies the wave equation only in terms of H_y:

$$\frac{d}{d x}\left(\frac{1}{n^2} \frac{dH_y}{d x}\right) + \left(k^2 - \frac{k_z^2}{n^2}\right) H_y = 0 \quad , \tag{11}$$

where $\omega^2 \mu_o \varepsilon_o = k^2$

$$E_x = \frac{k_z}{\omega\varepsilon_o n^2} H_y \quad , \tag{12}$$

$$E_z = \frac{i}{\omega\varepsilon_o n^2}\frac{dH_y}{dx} \quad , \tag{13}$$

and

$$H_z = H_x = E_y = 0 \quad . \tag{14}$$

Since the magnetic field lies in the plane that is perpendicular to the *z*-axis, this electromagnetic field distribution is called transverse magnetic (TM) mode.

Propagation constants and electromagnetic fields for TE and TM modes can be obtained by solving Eq. (7) or (11). Let us consider the slab waveguide with uniform refractive index profile in the core, as shown in Fig.. Since the guided electromagnetic fields are confined in the core and exponentially decay in the cladding, the electric field distribution for TE mode is expressed as

$$E_y = \begin{cases} A\cos(k_t a - \varphi)e^{-\alpha_o(x-a)} & , \quad x > a \\ A\cos(k_t x - \varphi) \quad , & -a \le x \le a \quad , \\ A\cos(k_t a + \varphi)e^{\alpha_s(x+a)} & , \quad x < -a \end{cases} \tag{15}$$

where k_t , α_o , and α_s are wavenumbers along the *x*-axis in the core and cladding regions and are given by

$$k_t = \sqrt{k^2 n_1^2 - k_z^2} \quad , \tag{16}$$

$$\alpha_s = \sqrt{k_z^2 - k^2 n_s^2} = \sqrt{k^2(n_1^2 - n_s^2) - k_t^2} \quad , \tag{17}$$

$$\alpha_o = \sqrt{k_z^2 - k^2 n_o^2} = \sqrt{k^2(n_1^2 - n_o^2) - k_t^2} \quad . \tag{18}$$

The electric field component E_y in Eq. (15) is continuous at the boundaries of core-cladding interfaces $x = \pm a$. Another boundary condition is that the magnetic field component H_z should be continuous at the boundaries. As given by Eq. (9), the boundary condition for H_z is equivalently treated by the continuity condition of dE_y / dx as

$$\frac{dE_y}{dx} = \begin{cases} -\alpha_o A\cos(k_t a - \varphi)e^{-\alpha_o(x-a)} & , \quad x > a \\ -k_t A\sin(k_t x - \varphi) \quad , & -a \le x \le a \quad . \\ \alpha_s A\cos(k_t a + \varphi)e^{\alpha_s(x+a)} & , \quad x < -a \end{cases} \tag{19}$$

From the conditions that dE_y / dx are continuous at $x = \pm a$, we obtain

$$k_t \tan(k_t a + \varphi) = \alpha_s \quad , \tag{20}$$

$$k_t \tan(k_t a - \varphi) = \alpha_o \quad . \tag{21}$$

Consequently, we obtain the eigenvalue equations as

$$k_t a = \frac{m\pi}{2} + \frac{1}{2}\tan^{-1}\left(\frac{\alpha_s}{k_t}\right) + \frac{1}{2}\tan^{-1}\left(\frac{\alpha_o}{k_t}\right) \quad , \tag{22}$$

$$\varphi = \frac{m\pi}{2} + \frac{1}{2}\tan^{-1}\left(\frac{\alpha_s}{k_t}\right) - \frac{1}{2}\tan^{-1}\left(\frac{\alpha_o}{k_t}\right) \quad . \tag{23}$$

In terms of the normalized frequency

$$v = ka\sqrt{(n_1^2 - n_s^2)} \tag{24}$$

and the parameters

$$b = \frac{k^2(n_1^2 - n_s^2) - k_t^2}{k^2(n_1^2 - n_s^2)} = \left(\frac{\alpha_s a}{v}\right)^2 \tag{25}$$

and

$$\gamma = \frac{n_s^2 - n_o^2}{n_1^2 - n_s^2} \quad , \tag{26}$$

the wavenumbers k_t , α_o , and α_s can be expressed as

$$k_t a = v\sqrt{1-b} \quad , \tag{27}$$

$$\alpha_s a = v\sqrt{b} \quad , \tag{28}$$

$$\alpha_o a = v\sqrt{b+\gamma} \; . \tag{29}$$

The parameter b is called the normalized propagation constant. The parameter γ is for a measure of the asymmetry of the cladding refractive indices. Substituting Eqs. (27)-(29) into Eq. (22), the eigenvalue (dispersion) equations can be rewritten as

$$v\sqrt{1-b} = \frac{m\pi}{2} + \frac{1}{2}\tan^{-1}\left(\sqrt{\frac{b}{1-b}}\right) + \frac{1}{2}\tan^{-1}\left(\sqrt{\frac{b+\gamma}{1-b}}\right) \quad , \tag{30}$$

$$\varphi = \frac{m\pi}{2} + \frac{1}{2}\tan^{-1}\left(\sqrt{\frac{b}{1-b}}\right) - \frac{1}{2}\tan^{-1}\left(\sqrt{\frac{b+\gamma}{1-b}}\right) \quad . \tag{31}$$

For the symmetrical waveguides with $n_o = n_s$, we have $\gamma = 0$ and the dispersion equations in (30) and (31) are reduced to

$$v\sqrt{1-b} = \frac{m\pi}{2} + \tan^{-1}\left(\sqrt{\frac{b}{1-b}}\right) \quad , \tag{32}$$

$$\varphi = \frac{m\pi}{2} \quad . \tag{33}$$

From Eq. (30), the normalized propagation constant *b* can be calculated for each normalized frequency *v* for the given *γ*-value. The *v*-*b* relationship is therefore called dispersion curve. With the calculated *b*-value, the transverse wavenumber k_t is then obtained by using $k_t a = v\sqrt{1-b}$ in Eq. (27) and the propagation constant k_z is found by

$$k_z = \sqrt{k^2 n_1^2 - k_t^2} = \sqrt{k^2 n_s^2 + b(v^2/a^2)} \quad . \tag{34}$$

The value of $v_c = \pi/2$ gives the cutoff normalized frequency for the $m=1$ mode. Generally the cutoff *v*-value for the TE_m mode is given by Eq. (30) with $b=0$ as

$$v_{c,TE} = \frac{m\pi}{2} + \frac{1}{2}\tan^{-1}\left(\sqrt{\gamma}\right) \quad . \tag{35}$$

Based on Eq. (11), the dispersion equation for the TM mode can be obtained in a similar manner to that of the TE mode. The magnetic field can be given by

$$H_y = \begin{cases} A\cos(k_t a - \varphi)e^{-\alpha_o(x-a)} & , \quad x > a \\ A\cos(k_t x - \varphi) \quad , & -a \le x \le a \\ A\cos(k_t a + \varphi)e^{\alpha_s(x+a)} & , \quad x < -a \end{cases} \quad . \tag{36}$$

Another boundary condition is that the magnetic field component E_z should be continuous at the boundaries. As given by Eq. (13), the boundary condition for E_z is equivalently treated by the continuity condition of $(dH_y/dx)/n^2$ as

$$\frac{1}{n^2}\frac{dH_y}{dx} = \begin{cases} -\frac{\alpha_o}{n_o^2}A\cos(k_t a - \varphi)e^{-\alpha_o(x-a)} & , \quad x > a \\ -\frac{k_t}{n_1^2}A\sin(k_t x - \varphi) \quad , & -a \le x \le a \\ \frac{\alpha_s}{n_s^2}A\cos(k_t a + \varphi)e^{\alpha_s(x+a)} & , \quad x < -a \end{cases} \quad . \tag{37}$$

From the conditions that $(dH_y/dx)/n^2$ are continuous at $x = \pm a$, we obtain

$$k_t \tan(k_t a + \varphi) = \alpha_s \frac{n_1^2}{n_s^2} \quad , \tag{38}$$

$$k_t \tan(k_t a - \varphi) = \alpha_o \frac{n_1^2}{n_o^2} \quad . \tag{39}$$

Consequently, we obtain the eigenvalue equations as

$$k_t a = \frac{m\pi}{2} + \frac{1}{2}\tan^{-1}\left(\frac{n_1^2}{n_s^2}\frac{\alpha_s}{k_t}\right) + \frac{1}{2}\tan^{-1}\left(\frac{n_1^2}{n_o^2}\frac{\alpha_o}{k_t}\right) \quad , \tag{40}$$

$$\varphi = \frac{m\pi}{2} + \frac{1}{2}\tan^{-1}\left(\frac{n_1^2}{n_s^2}\frac{\alpha_s}{k_t}\right) - \frac{1}{2}\tan^{-1}\left(\frac{n_1^2}{n_o^2}\frac{\alpha_o}{k_t}\right) \quad . \tag{41}$$

In terms of the normalized propagation constant b and the normalized frequency v, Eq. (40) can be reduced to

$$v\sqrt{1-b} = \frac{m\pi}{2} + \frac{1}{2}\tan^{-1}\left(\frac{n_1^2}{n_s^2}\sqrt{\frac{b}{1-b}}\right) + \frac{1}{2}\tan^{-1}\left(\frac{n_1^2}{n_o^2}\sqrt{\frac{b+\gamma}{1-b}}\right) \quad . \tag{42}$$

$\gamma := 0$

$B(m,v,b) := \mathrm{root}\left[\left(v\cdot\sqrt{1-b} - \frac{m\cdot\pi}{2} - \frac{1}{2}\cdot\mathrm{atan}\left(\sqrt{\frac{b}{1-b}}\right) - \frac{1}{2}\cdot\mathrm{atan}\left(\sqrt{\frac{b+\gamma}{1-b}}\right)\right), b\right]$

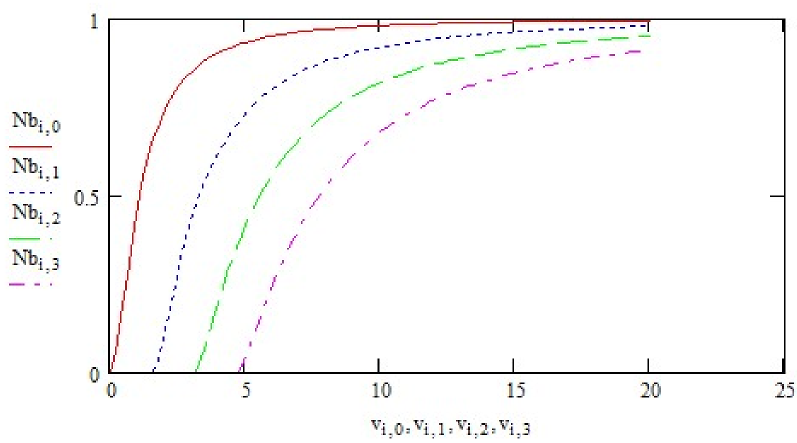
$m := 0..9$

$vc_m := m\cdot\frac{\pi}{2} + \frac{1}{2}\cdot\mathrm{atan}(\sqrt{\gamma})$

$i := 0..100$

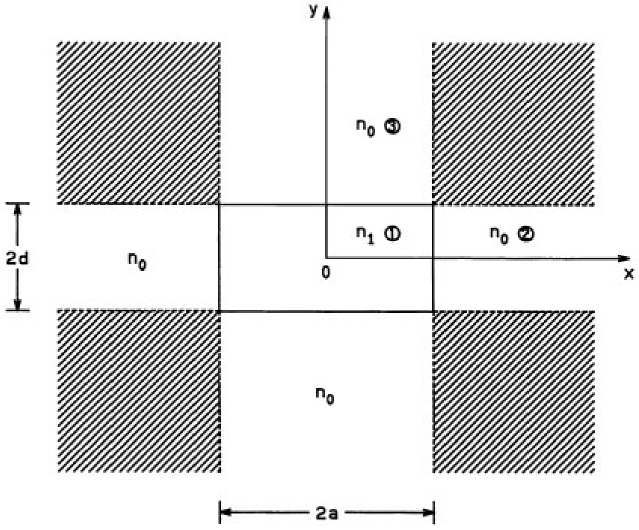
$v_{i,m} := vc_m + \frac{i+0.2}{100}\cdot(20 - vc_m)$

$Nb_{i,m} := B(m, v_{i,m}, 0.5)$



Rectangular waveguides

Here we use the analytical method proposed by Marcatili ("Dielectric rectangular waveguide and directional coupler for integrated optics", Bell Syst. Tech. J. 48, 2071-2102, 1969) to deal with the three-dimensional optical waveguide, as shown in Fig. . The important assumption of Marcatili's method is that the electromagnetic field in the shaded area can be neglected since the electromagnetic field of the well-guided mode decays quite rapidly in the cladding region. Then the boundary conditions for the electromagnetic field in the shaded area are not imposed.



We first consider the electromagnetic mode in which E_x and H_y are predominant. When we set $H_x = 0$ in Eqs. (5) and (6), the Maxwell's equations become

$$\begin{aligned} &\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = i\omega\mu_o H_z \\ &\frac{\partial E_z}{\partial y} - ik_z E_y = 0 \\ &ik_z E_x - \frac{\partial E_z}{\partial x} = i\omega\mu_o H_y \end{aligned} \tag{43}$$

and

$$\begin{aligned} &\frac{\partial H_y}{\partial x} - 0 = -i\omega\varepsilon_o n^2 E_z \\ &\frac{\partial H_z}{\partial y} - ik_z H_y = -i\omega\varepsilon_o n^2 E_x \quad . \\ &0 - \frac{\partial H_z}{\partial x} = -i\omega\varepsilon_o n^2 E_y \end{aligned} \tag{44}$$

With Eqs. (43) and (44), the electromagnetic field representation and the wave equation can be derived as

$$
\begin{aligned}
E_z &= \frac{i}{\omega\varepsilon_o n^2}\frac{\partial H_y}{\partial x} \\
E_y &= \frac{1}{ik_z}\frac{\partial E_z}{\partial y} = \frac{1}{k_z\omega\varepsilon_o n^2}\frac{\partial^2 H_y}{\partial y\partial x} \\
E_x &= \frac{\omega\mu_o}{k_z}H_y + \frac{1}{ik_z}\frac{\partial E_z}{\partial x} = \frac{\omega\mu_o}{k_z}H_y + \frac{1}{k_z\omega\varepsilon_o n^2}\frac{\partial^2 H_y}{\partial x^2} \\
H_z &= \frac{1}{i\omega\mu_o}\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right) = \frac{i}{k_z}\frac{\partial H_y}{\partial y}
\end{aligned}
\quad , \tag{45}
$$

and

$$
\frac{\partial H_z}{\partial y} - ik_z H_y = -i\omega\varepsilon_o n^2 E_x \quad \Rightarrow \quad \frac{\partial^2 H_y}{\partial x^2} + \frac{\partial^2 H_y}{\partial y^2} + \left(k^2 n^2 - k_z^2\right)H_y = 0 \quad . \tag{46}
$$

On the other hand, when the electromagnetic mode in which E_y and H_x are predominant, we set $H_x = 0$ in Eqs. (5) and (6) to simplify the Maxwell's equations as

$$
\begin{aligned}
&\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = i\omega\mu_o H_z \\
&\frac{\partial E_z}{\partial y} - ik_z E_y = i\omega\mu_o H_x \\
&ik_z E_x - \frac{\partial E_z}{\partial x} = 0
\end{aligned} \tag{47}
$$

and

$$
\begin{aligned}
&0 - \frac{\partial H_x}{\partial y} = -i\omega\varepsilon_o n^2 E_z \\
&\frac{\partial H_z}{\partial y} - 0 = -i\omega\varepsilon_o n^2 E_x \\
&ik_z H_x - \frac{\partial H_z}{\partial x} = -i\omega\varepsilon_o n^2 E_y
\end{aligned} \quad . \tag{48}
$$

With Eqs. (43) and (44), the electromagnetic field representation and the wave equation can be derived as

$$E_z = \frac{-i}{\omega\varepsilon_o n^2}\frac{\partial H_x}{\partial y}$$

$$E_x = \frac{1}{ik_z}\frac{\partial E_z}{\partial x} = \frac{-1}{k_z\omega\varepsilon_o n^2}\frac{\partial^2 H_x}{\partial y\,\partial x}$$

$$E_y = -\frac{\omega\mu_o}{k_z}H_x + \frac{1}{ik_z}\frac{\partial E_z}{\partial y} = -\frac{\omega\mu_o}{k_z}H_x - \frac{1}{k_z\omega\varepsilon_o n^2}\frac{\partial^2 H_x}{\partial y^2} \quad , \tag{49}$$

$$H_z = \frac{1}{i\omega\mu_o}\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right) = \frac{i}{k_z}\frac{\partial H_x}{\partial x}$$

and

$$ik_z H_x - \frac{\partial H_z}{\partial x} = -i\omega\varepsilon_o n^2 E_y \quad \Rightarrow \quad \frac{\partial^2 H_x}{\partial x^2} + \frac{\partial^2 H_x}{\partial y^2} + \left(k^2 n^2 - k_z^2\right)H_x = 0 \quad . \tag{50}$$

Since E_x and H_y are the predominant electromagnetic fields, the modes in Eqs. (45) and (46) are described as E^x_{pq}, where p and q are integers. On the other hand, the modes in Eqs. (49) and (50) are called E^y_{pq} because E_y and H_x are the predominant electromagnetic fields. Next, we use the separation of variables to find the solution of the wave equation. Considering the electromagnetic mode in which E_x and H_y are predominant, the solution in Eq. (46) can be expressed as

$$H_y(x, y) = X(x)Y(y) \quad . \tag{51}$$

Substituting Eq. (52) into Eq. (46), the wave equation in the core region can be expressed as two independent wave equations as

$$\frac{d^2X}{dx^2} + k_x^2 X = 0 \quad , \tag{52}$$

$$\frac{d^2Y}{dy^2} + k_y^2 Y = 0 \quad . \tag{53}$$

Here the transverse wavenumbers k_x and k_y satisfy

$$k_x^2 + k_y^2 = k^2 n_1^2 - k_z^2 \quad . \tag{54}$$

On the other hand, the wave equation outside the waveguide can be expressed as two independent wave equations as

$$\frac{d^2X}{dx^2} - \alpha_x^2 X = 0 \quad , \tag{55}$$

$$\frac{d^2Y}{d\,y^2} - \alpha_y^2\,Y = 0 \quad , \tag{56}$$

where the transverse wavenumbers α_x and α_y are given by

$$\alpha_x^2 = k^2(n_1^2 - n_o^2) - k_x^2 \quad , \tag{57}$$

$$\alpha_y^2 = k^2(n_1^2 - n_o^2) - k_y^2 \quad , \tag{58}$$

The solution fields of Eqs. (52) and (55) are given by

$$X(x) = \begin{cases} A\cos(k_x\,a + \varphi_x)e^{-\alpha_x(x-a)} & , \quad x > a \\ A\cos(k_x\,x + \varphi_x) \quad , & \quad -a \le x \le a \end{cases} \quad , \tag{59}$$

$$Y(y) = \begin{cases} B\cos(k_y\,d + \varphi_y)e^{-\alpha_y(y-d)} & , \quad y > d \\ B\cos(k_y\,y + \varphi_y) \quad , & \quad -d \le y \le d \end{cases} \quad , \tag{60}$$

where only the first quadrant is considered due to the symmetry of the waveguide. The optical phases φ_x and φ_y are expressed by

$$\varphi_x = p\pi/2 \quad , \quad p = 0, 1, 2, \cdots \tag{61}$$

$$\varphi_y = q\pi/2 \quad , \quad q = 0, 1, 2, \cdots \quad . \tag{62}$$

Using the boundary conditions that the electric field $E_z = (i/\omega\varepsilon_o n^2)(\partial H_y/\partial x)$ should be continuous at $x = a$ and the magnetic field $H_z = (i/k_z)(\partial H_y/\partial y)$ should be continuous at $y = d$, we obtain the following dispersion equations:

$$k_x\,a = \frac{p\pi}{2} + \tan^{-1}\left(\frac{n_1^2}{n_o^2}\frac{\alpha_x}{k_x}\right) \quad , \tag{63}$$

$$k_y\,d = \frac{q\pi}{2} + \tan^{-1}\left(\frac{\alpha_y}{k_y}\right) \quad . \tag{64}$$

In order to find the dispersion equation for the E_{pq}^y mode, the magnetic field H_x is given by the same equations in (51), (59) and (60). Applying the boundary conditions, we can obtain the following dispersion equations

$$k_x a = \frac{p\pi}{2} + \tan^{-1}\left(\frac{\alpha_x}{k_x}\right) \quad , \tag{65}$$

$$k_y d = \frac{q\pi}{2} + \tan^{-1}\left(\frac{n_1^2}{n_o^2}\frac{\alpha_y}{k_y}\right) \quad . \tag{66}$$

Radiation field from waveguide

The radiation field from an optical waveguide into free space propagates divergently. The radiation field is different from the field in the waveguide. Therefore it is important to know the profile of the radiation field for efficiently coupling the light between two waveguides or between a waveguide an optical fiber.

For the end face of a waveguide at $z = 0$, the electric field at the end face is denoted by $g(x', y', 0)$. By the Fresnel-Kirchhoff diffraction formula, the radiation field on the observation plane at a distance z is related to the field $g(x', y', 0)$ as

$$f(x, y, z) = \frac{kn}{2\pi i}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x', y', 0)\frac{1}{r}e^{iknr}dx'dy' \quad , \tag{1}$$

where n is the free-space refractive index, k is the free-space wavenumber, and the distance r between $(x', y', 0)$ and (x, y, z) is given by

$$r = \left[(x - x')^2 + (y - y')^2 + z^2\right]^{1/2} \quad . \tag{2}$$

When the distance of the observation plane z is very large compared with $|x - x'|$ and $|y - y'|$, Eq. (2) can be approximated by

$$\begin{aligned} r &= z\left[1 + \frac{(x - x')^2 + (y - y')^2}{z^2}\right]^{1/2} = z + \frac{(x - x')^2 + (y - y')^2}{2z} + \cdots \\ &= z + \frac{x^2 + y^2}{2z} - \frac{xx' + yy'}{z} + \frac{x'^2 + y'^2}{2z} + \cdots \end{aligned} \quad . \tag{3}$$

The number of expansion terms to approximate r accurately depends on the distance z between $(x', y', 0)$ and (x, y, z). Generally, the electromagnetic field in the waveguide is confined in a small area of the order of 10 μm. Therefore if z is larger than, for example, 1 mm, any term higher than the fourth term in the right-hand side of Eq. (3) can be neglected. Under this circumstance, the radiation field is called in the far-field region or Fraunhofer region. On the other hand, when z is not so large, the fourth term in Eq. (3) should be taken into account. Under this condition, the radiation field is called in the near-field region or

Fresnel region. However, it should be noted that even the Fresnel approximation is not satisfied in the region close to the end face of the waveguide. In general, the contribution to knr by the fourth term, $kn(x'^2 + y'^2/2z)$ determines whether the Fresnel or Fraunhofer approximation should be used. With $kn(x'^2 + y'^2/2z) = \pi$ as the measure for the optical filed confined in the rectangular region with square core area d^2, then we have the criteria

$$\begin{aligned} &z >> n\frac{d^2}{\lambda} \quad \text{Fraunhofer region} \\ &z < n\frac{d^2}{\lambda} \quad \text{Fresnel region} \end{aligned} \quad . \tag{4}$$

In the Fraunhofer region, Eq. (1) can be simplified as

$$f(x,y,z) = \frac{kn}{2\pi i z}\exp\left\{ikn\left[z + \frac{x^2+y^2}{2z}\right]\right\}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x',y',0)\exp\left\{-ikn\left[\frac{xx'+yy'}{z}\right]\right\}dx'dy' \ . \tag{5}$$

It is known that the radiation field $f(x,y,z)$ in the Fraunhofer region is a spatial Fourier transformation of the field profile $g(x',y',0)$ at the end face of the waveguide.

Gaussian Beam

Here we consider the propagation property of a Gaussian beam. The Gaussian profile for the field $g(x',y',0)$ can be given by

$$g(x',y',0) = A\exp\left[-\left(\frac{x'^2}{w_{ox}^2} + \frac{y'^2}{w_{oy}^2}\right)\right] \ , \tag{6}$$

where A is a constant and w_{ox} and w_{oy} are the spot sizes of the field along the x- and y- axis directions, respectively. Substituting Eqs. (3) and (6) into Eq. (1), we have

$$f(x,y,z) = \frac{kn}{2\pi i z}Ae^{iknz}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left[-\left(\frac{x'^2}{w_{ox}^2} + \frac{y'^2}{w_{oy}^2}\right)\right]\exp\left[ikn\left(\frac{(x-x')^2+(y-y')^2}{2z}\right)\right]dx'dy' \ , \tag{7}$$

where the Fresnel approximation to r has been used. Since the integral in Eq. (7) for x' and y' has the same form, detailed calculation only for x' is descried. The integral with respect to x' in Eq. (7) becomes

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\exp\left[-\frac{x'^2}{w_{ox}^2}+i\frac{kn}{2z}(x-x')^2\right]dx' \\
&=\exp\left(i\frac{kn}{2z}x^2\right)\int_{-\infty}^{\infty}\exp\left[-\left(\frac{1}{w_{ox}^2}-i\frac{kn}{2z}\right)x'^2-2i\frac{kn}{2z}xx'\right]dx' \\
&=\exp\left(i\frac{kn}{2z}x^2\right)\exp\left(\frac{-\left(\frac{kn}{2z}\right)^2x^2}{\left(\frac{1}{w_{ox}^2}-i\frac{kn}{2z}\right)}\right)\int_{-\infty}^{\infty}\exp\left[-\left(\frac{1}{w_{ox}^2}-i\frac{kn}{2z}\right)\left(x'+\frac{i\frac{kn}{2z}x}{\left(\frac{1}{w_{ox}^2}-i\frac{kn}{2z}\right)}\right)^2\right]dx' \\
&=\sqrt{\frac{\pi}{\left(\frac{1}{w_{ox}^2}-i\frac{kn}{2z}\right)}}\exp\left(i\frac{kn}{2z}x^2\right)\exp\left[\frac{-\left(\frac{kn}{2z}\right)^2x^2}{\left(\frac{1}{w_{ox}^2}-i\frac{kn}{2z}\right)}\right] \\
&=\sqrt{\frac{\pi}{\left(\frac{1}{w_{ox}^2}+\frac{kn}{2iz}\right)}}\exp\left(i\frac{kn}{2z}x^2\right)\exp\left[\frac{-\left(\frac{kn}{2z}\right)^2\left(\frac{1}{w_{ox}^2}+i\frac{kn}{2z}\right)x^2}{\frac{1}{w_{ox}^4}+\left(\frac{kn}{2z}\right)^2}\right] \\
&=\sqrt{\frac{1}{\frac{kn}{2\pi iz}}}\sqrt{\frac{1}{\left(1+i\frac{2z}{knw_{ox}^2}\right)}}\exp\left\{\frac{-x^2}{w_{xo}^2\left[1+\left(\frac{2z}{knw_{ox}^2}\right)^2\right]}\right\}\exp\left\{i\frac{kn}{2z}\frac{x^2}{\left[1+\left(\frac{knw_{ox}^2}{2z}\right)^2\right]}\right\}
\end{aligned}
\quad . \tag{8}
$$

We further introduce new variables:

$$z_{Rx}=\frac{knw_{ox}^2}{2} \quad , \tag{9}$$

$$w_x^2(z)=w_{xo}^2\left[1+\left(\frac{z}{z_{Rx}}\right)^2\right] \quad , \tag{10}$$

$$R_x(z)=z\left[1+\left(\frac{z_{Rx}}{z}\right)^2\right] \quad , \tag{11}$$

$$\Theta_x(z)=\tan^{-1}\left(\frac{z}{z_{Rx}}\right). \tag{12}$$

With Eqs. (9)-(12), Eq. (8) can be expressed as

$$\int_{-\infty}^{\infty} \exp\left[-\frac{x'^2}{w_{ox}^2} + i\frac{kn}{2z}(x-x')^2\right]dx'$$
$$= \sqrt{\frac{1}{\frac{kn}{2\pi i z}}}\sqrt{\frac{w_{ox}}{w_x(z)}}\exp\left[-i\frac{\Theta_x(z)}{2}\right]\exp\left[\frac{-x^2}{w_x^2(z)}\right]\exp\left[i\frac{knx^2}{2R_x(z)}\right] \qquad (13)$$

Similarly, the integral with respect to y' in Eq. (7) is given by

$$\int_{-\infty}^{\infty} \exp\left[-\frac{y'^2}{w_{oy}^2} + i\frac{kn}{2z}(y-y')^2\right]dx'$$
$$= \sqrt{\frac{1}{\frac{kn}{2\pi i z}}}\sqrt{\frac{w_{oy}}{w_y(z)}}\exp\left[-i\frac{\Theta_y(z)}{2}\right]\exp\left[\frac{-y^2}{w_y^2(z)}\right]\exp\left[i\frac{kny^2}{2R_y(z)}\right], \qquad (14)$$

where the parameters are defined as

$$z_{Ry} = \frac{knw_{oy}^2}{2}, \qquad (15)$$

$$w_y^2(z) = w_{oy}^2\left[1+\left(\frac{z}{z_{Ry}}\right)^2\right], \qquad (16)$$

$$R_y(z) = z\left[1+\left(\frac{z_{Ry}}{z}\right)^2\right], \qquad (17)$$

$$\Theta_y(z) = \tan^{-1}\left(\frac{z}{z_{Ry}}\right). \qquad (18)$$

Substituting Eqs. (13) and (14) into (7), the radiation pattern of the Gaussian beam is given by

$$f(x,y,z) = \sqrt{\frac{w_{ox}}{w_x(z)}\frac{w_{oy}}{w_y(z)}}\exp\left\{-i\frac{[\Theta_x(z)+\Theta_y(z)]}{2}\right\}$$
$$\times\exp\left\{-\left[\frac{x^2}{w_x^2(z)}+\frac{y^2}{w_y^2(z)}\right]\right\}\exp\left\{ikn\left[\frac{x^2}{2R_x(z)}+\frac{y^2}{2R_y(z)}+z\right]\right\}. \qquad (19)$$

Eq. (19) reveals that $w_x(z)$ and $w_y(z)$ represent the spot size of the Gaussian beam, and $R_x(z)$ and $R_y(z)$ represent the radii of curvature of the wavefronts, respectively. When $w_{ox} = w_{oy} = w_o$ for a symmetric Gaussian beam, Eq. (19) can be expressed as

$$f(x,y,z) = A\frac{w_o}{w(z)}\exp[-i\Theta(z)]\exp\left[-\frac{r^2}{w^2(z)}\right]\exp\left\{ikn\left[\frac{r^2}{2R(z)}+z\right]\right\} \quad , \tag{20}$$

where the parameters are defined as

$$z_R = \frac{knw_o^2}{2} \quad , \tag{21}$$

$$w^2(z) = w_o^2\left[1+\left(\frac{z}{z_R}\right)^2\right] \quad , \tag{22}$$

$$R(z) = z\left[1+\left(\frac{z_R}{z}\right)^2\right] \quad , \tag{23}$$

$$\Theta(z) = \tan^{-1}\left(\frac{z}{z_R}\right). \tag{24}$$

We can use the propagation properties of the Gaussian beam to analyze the mode size of the spherical resonators. In a stable linear spherical resonator, the Gaussian beam will adapt itself to the resonator mirror configuration. In other words, the equiphase contours $R(z)$ should match the mirror radii of curvature. For instance, if the resonator has a flat front mirror and a curved rear mirror, then the beam waist of the Gaussian beam (where $R(z) \to \infty$) will be at the flat mirror and the radius of beam curvature $R(L)$ at the rear mirror will be equal to the curvature of the rear mirror R_2. For a cavity with two curved mirrors, the Gaussian beam will adapt itself so that both mirror curvatures match the curvature of the Gaussian beam at the mirrors. This property makes it relatively straightforward to calculate the beam waists and curvatures inside the laser resonator. Based on this criteria, consider a laser with two curved mirrors, where the beam waist is defined at $z = 0$ with a distance of L_1 to mirror R_1 and L_2 to mirror R_2. Assume that both mirrors R_1 and R_2 have positive values for the curvatures and both distances L_1 and L_2 are positive such that $L_1 + L_2 = L$. Under this circumstance, the beam radii on the two mirrors are given by

$$R_1 = L_1\left[1+\left(\frac{z_R}{L_1}\right)^2\right] \quad , \tag{25}$$

$$R_2 = L_2\left[1+\left(\frac{z_R}{L_2}\right)^2\right] \quad . \tag{26}$$

Review

$$\int_{-\infty}^{\infty} \exp\left[-\frac{x'^2}{w_{ox}^2} + i\frac{kn}{2z}(x-x')^2\right]dx'$$
$$= \sqrt{\frac{1}{\frac{kn}{2\pi i z}}}\sqrt{\frac{w_{ox}}{w_x(z)}}\exp\left[-i\frac{\Theta_x(z)}{2}\right]\exp\left[\frac{-x^2}{w_x^2(z)}\right]\exp\left[i\frac{k\,n\,x^2}{2R_x(z)}\right]$$

Changing variables

$$\int_{-\infty}^{\infty} \exp\left[-\frac{x'^2}{w_{ox}^2} + i\frac{kn}{2z}(x-x')^2\right]dx' \quad \leftarrow \quad \frac{\sqrt{2}x'}{w_{ox}} = \xi', \quad \frac{\sqrt{2}x}{w_{ox}} = \xi, \quad -i\frac{k\,n\,w_{ox}^2}{2z} = -i\frac{z_{Rx}}{z} = a$$
$$= \frac{w_{ox}}{\sqrt{2}}\int_{-\infty}^{\infty}\exp\left[-\frac{\xi'^2}{2} - \frac{a}{2}(\xi'-\xi)^2\right]d\xi'$$
$$= \frac{w_{ox}}{\sqrt{2}}\left\{\int_{-\infty}^{\infty}\exp\left[-\frac{(a+1)}{2}\left(\xi' - \frac{a}{a+1}\xi\right)^2\right]d\xi'\right\}\exp\left[-\frac{a}{2(a+1)}\xi^2\right]$$
$$= \frac{w_{ox}}{\sqrt{2}}\sqrt{\frac{2\pi}{a+1}}\exp\left[-\frac{a}{2(a+1)}\xi^2\right]$$
$$= \frac{w_{ox}}{\sqrt{2}}\sqrt{\frac{2\pi}{a+1}}\quad\exp\left[-\frac{a^2}{2(a^2-1)}\xi^2\right]\exp\left[\frac{a}{2(a^2-1)}\xi^2\right]$$
$$= \sqrt{\frac{2\pi i z}{nk}}\frac{1}{\sqrt{1+(z/z_{Rx})^2}}\exp\left[-\frac{i}{2}\tan^{-1}\left(\frac{z}{z_{Rx}}\right)\right]\exp\left[-\frac{x^2}{w_x^2(z)}\right]\exp\left[i\frac{kn\,x^2}{2R_x(z)}\right]$$

Schrodinger harmonic oscillator

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2x^2\right)\Psi_n(x) = E_n\,\Psi_n(x) \quad \leftarrow \quad \sqrt{\frac{m\omega}{\hbar}}x = \xi, \quad \frac{2E_n}{\hbar\omega} = \lambda_n$$
$$\left(\frac{d^2}{d\xi^2} - \xi^2 + \lambda_n\right)\psi_n(\xi) = 0 \quad \leftarrow \quad \psi_n(\xi) = H_n(\xi)e^{-\xi^2/2}$$
$$\frac{d^2H_n}{d\xi^2} - 2\xi\frac{dH_n}{d\xi} + (\lambda_n - 1)H_n = 0$$

Generating function for Hermite polynomials

$$\exp\left(-s^2+2s\xi\right)=\sum_{n=0}^{\infty}\frac{s^n}{n!}H_n(\xi)$$

$$\frac{d}{d\xi}\exp\left(-s^2+2s\xi\right)=2s\sum_{n=0}^{\infty}\frac{s^n}{n!}H_n(\xi)=\sum_{n=0}^{\infty}\frac{s^n}{n!}\frac{dH_n}{d\xi}\quad\Rightarrow 2n\,H_{n-1}(\xi)=\frac{dH_n}{d\xi}$$

$$\frac{d}{ds}\exp\left(-s^2+2s\xi\right)=2(\xi-s)\sum_{n=0}^{\infty}\frac{s^n}{n!}H_n(\xi)=\sum_{n=0}^{\infty}\frac{s^{n-1}}{(n-1)!}H_n(\xi)$$

$$\Rightarrow\quad 2\xi\,H_n(\xi)-2nH_{n-1}(\xi)=H_{n+1}(\xi)$$

$$\Rightarrow\quad 2\xi\,H_n(\xi)-\frac{dH_n}{d\xi}=H_{n+1}(\xi)\quad\Rightarrow\quad 2\,H_n(\xi)+2\xi\frac{dH_n}{d\xi}-\frac{d^2H_n}{d\xi^2}=2(n+1)H_n(\xi)$$

$$\Rightarrow\quad \frac{d^2H_n}{d\xi^2}-2\xi\frac{dH_n}{d\xi}+2n\,H_n=0$$

$$\frac{d^2H_n}{d\xi^2}-2\xi\frac{dH_n}{d\xi}+(\lambda_n-1)H_n=0\quad\Rightarrow\quad\lambda_n=2n+1$$

$$H_0(\xi)=1,\quad H_1(\xi)=2\xi,\quad H_2(\xi)=4\xi^2-2$$

Exploiting the off-axis Gaussian beam to derive the propagation of high-order Hermite Gaussian modes

$$\int_{-\infty}^{\infty}\psi_n(\xi')\exp\left[-\frac{a}{2}(\xi'-\xi)^2\right]d\xi'$$

$$\exp\left(-s^2+2s\xi\right)e^{-\xi^2/2}=\sum_{n=0}^{\infty}\frac{s^n}{n!}H_n(\xi)\,e^{-\xi^2/2}=\sum_{n=0}^{\infty}\frac{s^n}{n!}\psi_n(\xi)$$

$$e^{s^2}\exp\left[-\frac{1}{2}\left(\xi-2s\right)^2\right]=\sum_{n=0}^{\infty}\frac{s^n}{n!}\psi_n(\xi)$$

$$e^{s^2}\int_{-\infty}^{\infty}\exp\left[-\frac{1}{2}\left(\xi'-2s\right)^2-\frac{a}{2}(\xi'-\xi)^2\right]d\xi'=\sum_{n=0}^{\infty}\frac{s^n}{n!}\int_{-\infty}^{\infty}\psi_n(\xi')\exp\left[-\frac{a}{2}(\xi'-\xi)^2\right]d\xi'$$

Using generating function again to identify each term to correspond to the beam propagation of the specific high-order Hermite Gaussian mode

$$e^{s^2}\int_{-\infty}^{\infty}\exp\left[-\frac{1}{2}(\xi'-2s)^2-\frac{a}{2}(\xi'-\xi)^2\right]d\xi' = e^{s^2}\int_{-\infty}^{\infty}\exp\left\{-\frac{1}{2}(\xi')^2-\frac{a}{2}[\xi'-(\xi-2s)]^2\right\}d\xi'$$

$$= e^{s^2}\left\{\int_{-\infty}^{\infty}\exp\left[-\frac{(a+1)}{2}\left(\xi'-\frac{a}{a+1}(\xi-2s)\right)^2\right]d\xi'\right\}\exp\left[-\frac{a}{2(a+1)}(\xi-2s)^2\right]$$

$$= \sqrt{\frac{2\pi}{a+1}}\; e^{s^2}\;\exp\left[-\frac{a}{2(a+1)}(\xi-2s)^2\right]$$

$$= \sqrt{\frac{2\pi}{a+1}}\;\exp\left[-\frac{(a-1)}{(a+1)}s^2+2\frac{a}{(a+1)}\xi\, s\right]\exp\left[-\frac{a}{2(a+1)}\xi^2\right]$$

$$= \sqrt{\frac{2\pi}{a+1}}\;\exp\left[-\frac{(a-1)}{(a+1)}s^2+2\left(\frac{a}{\sqrt{a^2-1)}}\xi\right)\sqrt{\frac{(a-1)}{(a+1)}}s\right]\exp\left[-\frac{a}{2(a+1)}\xi^2\right]$$

$$= \sqrt{\frac{2\pi}{a+1}}\sum_{n=0}^{\infty}\frac{s^n}{n!}\left(\frac{a-1}{a+1}\right)^{n/2}H_n\left(\frac{a}{\sqrt{a^2-1)}}\xi\right)\exp\left[-\frac{a^2}{2(a^2-1)}\xi^2\right]\exp\left[\frac{a}{2(a^2-1)}\xi^2\right]$$

$$\Rightarrow \int_{-\infty}^{\infty}\psi_n(\xi')\exp\left[-\frac{a}{2}(\xi'-\xi)^2\right]d\xi' = \sqrt{\frac{2\pi}{a+1}}\left(\frac{a-1}{a+1}\right)^{n/2}\psi_n\left(\frac{a}{\sqrt{a^2-1)}}\xi\right)\exp\left[\frac{a}{2(a^2-1)}\xi^2\right]$$

$$\sqrt{\frac{2\pi i z}{nk}}\frac{1}{\sqrt{1+(z/z_{Rx})^2}}e^{-i\left(n+\frac{1}{2}\right)\tan^{-1}\left(\frac{z}{z_{Rx}}\right)}H_n\left[\frac{\sqrt{2}x}{w_x(z)}\right]\exp\left[-\frac{x^2}{w_x^2(z)}\right]\exp\left[i\frac{knx^2}{2R_x(z)}\right]$$

The eigenmodes for a spherical cavity with the front mirror of a radius of curvature R_1 at $z=z_1$ and the rear mirror of a radius of curvature R_2 at $z=z_2$ are given by

$$\Phi_{n,m,l}^{(HG)}(x,y,z,t) = C_{n,m}\frac{\sqrt{2}}{w(z)}\psi_n\left[\frac{\sqrt{2}x}{w(z)}\right]\psi_m\left[\frac{\sqrt{2}y}{w(z)}\right]\sin\left[\zeta_{n,m,l}(x,y,z)-(m+n+1)\theta_G(z)\right]e^{-i\omega_{n,m,l}\,t}$$

$$\zeta_{n,m,l}(x,y,z) = k_{n,m,l}(z-z_1)\;\left[1+(x^2+y^2)/2(z^2+z_R^2)\right]$$

$$\theta_G(z) = \tan^{-1}(z/z_R)-\tan^{-1}(z_1/z_R),\;\; \omega_{n,m,l} = ck_{n,m,l},\;\; C_{n,m} = \frac{1}{\sqrt{\pi\, 2^{m+n}\, n!m!}}$$

The overall cavity length is given by $L = z_2 - z_1$. Note that the coordinates of the cavity setup are $z_2 > z_1$. The Rayleigh range z_R is determined with the following equations:

$$R_1 = -z_1\left[1+\left(\frac{z_R}{z_1}\right)^2\right]\;\;,\qquad R_2 = z_2\left[1+\left(\frac{z_R}{z_2}\right)^2\right]\;.$$

With the above two equations and $L = z_2 - z_1$, we have

$$R_1 - L = -z_2 - \frac{z_R^2}{z_1}\;\;,\qquad R_2 - L = z_1 + \frac{z_R^2}{z_2}\;.$$

The ratio and the sum of the above two equations leads to

$$\frac{R_1 - L}{R_2 - L} = -\frac{z_2}{z_1} \quad ,$$

$$z_R^2 = -\frac{(R_1 + R_2 - L)}{L} z_1 z_2$$

After simple algebra, we have

$$z_1 = -\frac{L(R_2 - L)}{R_1 + R_2 - 2L} \quad , \qquad z_2 = \frac{L(R_1 - L)}{R_1 + R_2 - 2L}$$

and

$$z_R^2 = \frac{L(R_1 + R_2 - L)(R_1 - L)(R_2 - L)}{(R_1 + R_2 - 2L)^2} \quad .$$

Consequently, the mode sizes on the waist, the front mirror, and the rear mirror are

$$w_o^2 = \frac{\lambda}{\pi} z_R = \frac{\lambda}{\pi} \sqrt{\frac{L(R_1 + R_2 - L)(R_1 - L)(R_2 - L)}{(R_1 + R_2 - 2L)^2}}$$

$$w_1^2 = w_o^2 \left[1 + \left(\frac{z_1}{z_R} \right)^2 \right] = \frac{\lambda}{\pi} \sqrt{\frac{L(R_1 + R_2 - L)(R_1 - L)(R_2 - L)}{(R_1 + R_2 - 2L)^2}} \left[1 + \frac{L(R_2 - L)}{(R_1 + R_2 - L)(R_1 - L)} \right]$$

$$w_2^2 = w_o^2 \left[1 + \left(\frac{z_2}{z_R} \right)^2 \right] = \frac{\lambda}{\pi} \sqrt{\frac{L(R_1 + R_2 - L)(R_1 - L)(R_2 - L)}{(R_1 + R_2 - 2L)^2}} \left[1 + \frac{L(R_1 - L)}{(R_1 + R_2 - L)(R_2 - L)} \right]$$

The expression of the wave function $\Phi_{n,m,l}^{(HG)}(x,y,z,t)$ satisfies one of the boundary conditions at $z = z_1$, i.e. $\Phi_{n,m,l}^{(HG)}(x,y,z_1,t) = 0$. To determine the eigenvalues $k_{n,m,l}$, we use the periodic condition that

$$\zeta_{n,m,l}(0,0,z_2) - (m+n+1)\,\theta_G(z_2) = k_{n,m,l}\,L - (m+n+1)\left[\tan^{-1}(z_2/z_R) - \tan^{-1}(z_1/z_R)\right] = l\pi \, .$$

After simple algebra, we have

$$k_{n,m,l} = \frac{\pi}{L}\left\{ l + \frac{(m+n+1)}{\pi}\left[\tan^{-1}(z_2/z_R) - \tan^{-1}(z_1/z_R)\right] \right\} \quad .$$

Now we simplify the term in the square brackets. First of all, the addition formulas of trigonometric functions can be used to show

$$\tan^{-1}(z_2/z_R) - \tan^{-1}(z_1/z_R) = \cos^{-1}\left(\frac{z_R^2 + z_1 z_2}{\sqrt{z_1^2 + z_R^2}\sqrt{z_2^2 + z_R^2}} \right) \quad .$$

Then, we can use $R_1 = -z_1\left[1+\left(\frac{z_R}{z_1}\right)^2\right]$ and $R_2 = z_2\left[1+\left(\frac{z_R}{z_2}\right)^2\right]$ to obtain

$$\sqrt{z_1^2+z_R^2}\sqrt{z_2^2+z_R^2} = \sqrt{-R_1R_2z_1z_2} \quad .$$

We further use $R_1 - L = -z_2 - \frac{z_R^2}{z_1}$ and $R_2 - L = z_1 + \frac{z_R^2}{z_2}$ to obtain

$$z_R^2 + z_1z_2 = \sqrt{-(R_1-L)(R_2-L)z_1z_2} \quad .$$

As a result, we have

$$\tan^{-1}(z_2/z_R) - \tan^{-1}(z_1/z_R) = \cos^{-1}\left(\frac{z_R^2+z_1z_2}{\sqrt{z_1^2+z_R^2}\sqrt{z_2^2+z_R^2}}\right) = \cos^{-1}\left[\sqrt{\left(1-\frac{L}{R_1}\right)\left(1-\frac{L}{R_2}\right)}\right].$$

In terms of the g-parameters,

$$g_1 = 1-\frac{L}{R_1} \quad , \quad g_2 = 1-\frac{L}{R_2} \quad ,$$

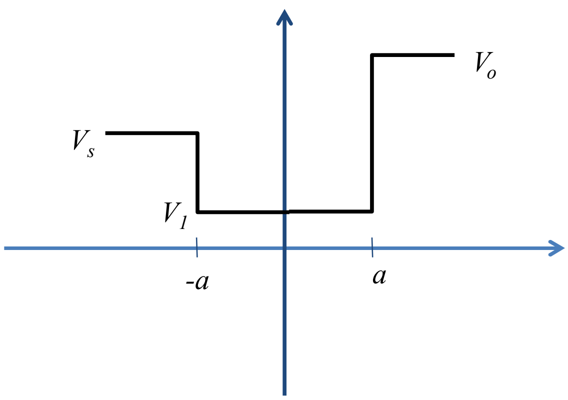
The eigenvalues $k_{n,m,l}$ can be given by

$$k_{n,m,l} = (\pi/L)\left[\, l + (n+m+1)(\Omega_T/\Omega_L)\right]$$

with the ratio of the transverse to longitudinal mode spacing as $\frac{\Omega_T}{\Omega_L} = \frac{1}{\pi}\cos^{-1}\left(\sqrt{g_1g_2}\right)$.

Analogy with quantum mechanics

Considering a 1D quantum system with potential as shown in Fig., the eigenvalues and eigenfunctions can be solved with the Schrodinger equation.



With the Schrodinger equation, the wave function is expressed as

$$\psi(x) = \begin{cases} A\cos(ka+\varphi)e^{-\alpha_o(x-a)} & , \quad x > a \\ A\cos(kx+\varphi) \quad , & -a \le x \le a \quad , \\ A\cos(ka-\varphi)e^{\alpha_s(x+a)} & , \quad x < -a \end{cases} \tag{1}$$

where k , α_o , and α_s are wavenumbers along the *x*-axis in the well and outside regions and are given by

$$k = \sqrt{\frac{2m}{\hbar^2}(E - V_1)} \quad , \tag{2}$$

$$\alpha_o = \sqrt{\frac{2m}{\hbar^2}(V_o - E)} = \sqrt{\frac{2m}{\hbar^2}(V_o - V_1) - k^2} \quad , \tag{3}$$

$$\alpha_s = \sqrt{\frac{2m}{\hbar^2}(V_s - E)} = \sqrt{\frac{2m}{\hbar^2}(V_s - V_1) - k^2} \quad . \tag{4}$$

The wave function $\psi(x)$ in Eq. (1) is continuous at the boundaries of well interfaces $x = \pm a$. Another boundary condition is that the derivative of the wave function $d\psi / dx$ should be continuous at the boundaries:

$$\frac{d\psi}{dx} = \begin{cases} -\alpha_o A\cos(ka+\varphi)e^{-\alpha_o(x-a)} & , \quad x > a \\ -k A\sin(kx+\varphi) \quad , & -a \le x \le a \quad . \\ \alpha_s A\cos(ka-\varphi)e^{\alpha_s(x+a)} & , \quad x < -a \end{cases} \tag{5}$$

From the conditions that $d\psi / dx$ are continuous at $x = \pm a$, we obtain

$$k\tan(ka - \varphi) = \alpha_s \quad , \tag{6}$$

$$k\tan(ka + \varphi) = \alpha_o \quad . \tag{7}$$

Consequently, we obtain the eigenvalue equations as

$$ka = \frac{m\pi}{2} + \frac{1}{2}\tan^{-1}\left(\frac{\alpha_s}{k}\right) + \frac{1}{2}\tan^{-1}\left(\frac{\alpha_o}{k}\right) \quad , \tag{8}$$

$$\varphi = \frac{m\pi}{2} - \frac{1}{2}\tan^{-1}\left(\frac{\alpha_s}{k}\right) + \frac{1}{2}\tan^{-1}\left(\frac{\alpha_o}{k}\right) \quad . \tag{9}$$

In terms of the normalized parameter

$$v = a\sqrt{\frac{2m}{\hbar^2}(V_s - V_1)} \tag{10}$$

and the parameters

$$b = 1 - \frac{\hbar^2 k^2}{2m(V_s - V_1)} = \left(\frac{\alpha_s a}{v}\right)^2 \tag{11}$$

and

$$\gamma = \frac{V_o - V_s}{V_s - V_1} \ , \tag{12}$$

the wavenumbers k_t , α_o , and α_s can be expressed as

$$k\,a = v\sqrt{1-b} \ \ , \tag{13}$$

$$\alpha_s a = v\sqrt{b} \ \ , \tag{14}$$

$$\alpha_o a = v\sqrt{b+\gamma} \ . \tag{15}$$

Substituting Eqs. (13)-(15) into Eqs. (8) and (9), the eigenvalue (dispersion) equations can be rewritten as

$$v\sqrt{1-b} = \frac{m\pi}{2} + \frac{1}{2}\tan^{-1}\left(\sqrt{\frac{b}{1-b}}\right) + \frac{1}{2}\tan^{-1}\left(\sqrt{\frac{b+\gamma}{1-b}}\right) \ , \tag{16}$$

$$\varphi = \frac{m\pi}{2} - \frac{1}{2}\tan^{-1}\left(\sqrt{\frac{b}{1-b}}\right) + \frac{1}{2}\tan^{-1}\left(\sqrt{\frac{b+\gamma}{1-b}}\right) \ . \tag{17}$$

For the symmetrical well with $V_o = V_s$, we have $\gamma = 0$ and the dispersion equations in (16) and (17) are reduced to

$$v\sqrt{1-b} = \frac{m\pi}{2} + \tan^{-1}\left(\sqrt{\frac{b}{1-b}}\right) \ , \tag{18}$$

$$\varphi = \frac{m\pi}{2} \ . \tag{19}$$

From Eq. (16), the normalized confined energy *b* can be calculated for each normalized parameter *v* for a given *γ*-value. The *v*-*b* relationship is therefore called dispersion curve. With the calculated *b*-value, the wavenumber k is then obtained by using $k\,a = v\sqrt{1-b}$ in Eq. (13). The value of $v_c = \pi/2$ gives the cutoff normalized frequency for the $m=1$ mode. Generally the cutoff *v*-value for the eigenmode is given by Eq. (16) with $b=0$ as

$$v_{c,TE} = \frac{m\pi}{2} + \frac{1}{2}\tan^{-1}\left(\sqrt{\gamma}\right) \ . \tag{20}$$

The eigenmodes for a plano-concave spherical cavity between z =0 and $z = L$ are given by

$$\Phi_{n,m,l}^{(HG)}(x,y,z,t) = C_{n,m} \frac{\sqrt{2}}{w(z)} \psi_n \left[\frac{\sqrt{2}x}{w(z)} \right] \psi_m \left[\frac{\sqrt{2}y}{w(z)} \right] \sin\left[\zeta_{n,m,l}(x,y,z) - (m+n+1)\theta_G(z) \right] e^{-i\omega_{n,m,l}\,t}$$

$$\zeta_{n,m,l}(x,y,z) = k_{n,m,l}\, z \left[1 + (x^2+y^2)/2(z^2+z_R^2) \right]$$

$$\theta_G(z) = \tan^{-1}(z/z_R), \quad \omega_{n,m,l} = ck_{n,m,l}, \quad C_{n,m} = \frac{1}{\sqrt{\pi\, 2^{m+n}\, n!\, m!}}$$

$$k_{n,m,l} = (\pi/L)\left[\, l + (n+m+1)(\Omega_T/\Omega_L) \right]$$

Chapter Ten: Electromagnetic Radiation

The inhomogeneous wave equation

With the Lorenz gauge condition, the equations for Φ and **A** can be written

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} = -\frac{\rho}{\varepsilon_o}$$

$$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_o\mathbf{J} \ .$$

In order to relate the radiation fields to the sources, it is usually needed to solve the inhomogeneous wave equations for the potentials. The two equations are not independent because ρ and **J** are related by the continuity equation. The electric and magnetic fields may be obtained from the solutions of the equations as

$$\mathbf{E} = -\nabla\Phi - \frac{\partial \mathbf{A}}{\partial t} \quad ,$$

$$\mathbf{B} = \nabla \times \mathbf{A} \ .$$

In practice it is sufficient to evaluate **A** since **B** may be found directly as its curl, and outside the source, **E** may be obtained from **B**.

Both wave equations have the same form as

$$\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} = -f(\mathbf{r},t) \quad .$$

One commonly useful method of solving the wave equation is based on the Fourier analysis to deal with only one frequency component. After the single-frequency solution has been found, the time dependent solution can be found by summing the frequency components. With the Fourier transform, the source function $f(\mathbf{r},t)$ and the frequency spectrum $F(\mathbf{r},\omega)$ can be related by

$$f(\mathbf{r},t) = \frac{1}{\sqrt{2\pi}}\int_0^\infty F(\mathbf{r},\omega)e^{-i\omega t}d\omega \quad ,$$

$$F(\mathbf{r},\omega) = \frac{1}{\sqrt{2\pi}}\int_0^\infty f(\mathbf{r},t)e^{i\omega t}dt \quad .$$

In the same way, the solution ψ can be expressed as

$$\psi(\mathbf{r},t) = \frac{1}{\sqrt{2\pi}}\int_0^\infty \Psi(\mathbf{r},\omega)e^{-i\omega t}d\omega \quad ,$$

$$\Psi(\mathbf{r},\omega) = \frac{1}{\sqrt{2\pi}}\int_0^\infty \psi(\mathbf{r},t)e^{i\omega t}dt \quad .$$

Substituting the Fourier components into the wave equation, we obtain

$$(\nabla^2 + k^2)\Psi(\mathbf{r},\omega) = -F(\mathbf{r},\omega) \quad ,$$

where $k = \omega/c$. The solution of this equation can be synthesized with the Green's function that satisfies the equation

$$(\nabla^2 + k^2)G_k(\mathbf{r},\mathbf{r}') = -4\pi\delta(\mathbf{r}-\mathbf{r}') \quad .$$

It is clear that the Green's function that is the solution of the unit point source. The frequency component $\Psi(\mathbf{r},\omega)$ of the total solution of the source function $f(\mathbf{r},t)$ can be found by integrating all point source solutions with the appropriate weight $F(\mathbf{r},\omega)$:

$$\Psi(\mathbf{r},\omega) = \frac{1}{4\pi}\int F(\mathbf{r}',\omega)G_k(\mathbf{r},\mathbf{r}')d^3r' \quad .$$

The Green's function $G_k(\mathbf{r},\mathbf{r}')$ can be conveniently found by using the property of the spherical symmetry about $\mathbf{r}'$. Denoting the distance from the source by $R = |\mathbf{r}-\mathbf{r}'|$, the Green's function at points other than $R = 0$ must satisfy

$$\frac{1}{R}\frac{d^2}{dR^2}(RG_k) + k^2G_k = 0 \quad .$$

It is well known that the solution of this equation is given by

$$RG_k^{(\pm)} = Ce^{\pm ikR} \quad .$$

Thus the general solution for the Green function can be expressed as

$$G_k(R) = AG_k^{(+)}(R) + BG_k^{(-)} \quad .$$

where

$$G_k^{(\pm)} = \frac{e^{\pm ikR}}{R} \quad .$$

In the limit of $R \to 0$, the wave equation reduces to the Poisson equation, since $kR << 1$. With the help of $\nabla^2(1/R) = -4\pi\delta(R)$, the coefficients of A and B need to satisfy $A + B = 1$. We further evaluate $\Psi(\mathbf{r},\omega)$

$$\Psi^{(\pm)}(\mathbf{r},\omega) = \frac{1}{4\pi}\int F(\mathbf{r}',\omega)\frac{e^{\pm ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}d^3r' .$$

The time-dependent solution $\psi(\mathbf{r},t)$ can be obtained by taking the inverse Fourier transform

$$\psi^{(\pm)}(\mathbf{r},t) = \frac{1}{\sqrt{2\pi}}\int_0^\infty \left[\frac{1}{4\pi}\int F(\mathbf{r}',\omega)\frac{e^{\pm ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}d^3r'\right]e^{-i\omega t}d\omega \quad .$$

With the definition of $t' = t \mp \frac{|\mathbf{r}-\mathbf{r}'|}{c}$, the solution is rewritten as

$$\psi^{(\pm)}(\mathbf{r},t) = \frac{1}{4\pi}\int_0^\infty \frac{1}{|\mathbf{r}-\mathbf{r}'|}\left[\frac{1}{\sqrt{2\pi}}\int_0^\infty F(\mathbf{r}',\omega)e^{-i\omega t'}d\omega\right]d^3r'$$
$$= \frac{1}{4\pi}\int_0^\infty \frac{f(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|}d^3r' .$$

Physically, the term with the + sign (the retarded solution) states that the present potential at **r** was caused by the source a travel time *R*/c earlier. The term with the – sign (the advanced solution) means that the current potential depends on the behavior of the source in the future at $t' = t + R/c$.

Spherical wave solutions of the scalar wave equation

Spherical harmonic expansions for the solutions of the Laplace or Poisson equations were employed in potential problems with spherical boundaries or to develop multipole expansions of charge densities and their fields. We extend spherical harmonic expansions to the development of spherical wave solutions of the scalar wave equation for radiating sources.

A scalar field $\psi(\mathbf{r},t)$ satisfying the source-free wave equation is given by

$$\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = 0 \quad .$$

With the Fourier transform, the solution ψ can be expressed as

$$\psi(\mathbf{r},t) = \frac{1}{\sqrt{2\pi}}\int_0^\infty \Psi(\mathbf{r},\omega)e^{-i\omega t}d\omega \quad ,$$

$$\Psi(\mathbf{r},\omega) = \frac{1}{\sqrt{2\pi}}\int_0^\infty \psi(\mathbf{r},t)e^{i\omega t}dt \quad .$$

Substituting the Fourier components into the wave equation, we obtain

$$(\nabla^2 + k^2)\Psi(\mathbf{r},k) = 0 \quad ,$$

where $k = \omega/c$. For problems possessing symmetry properties about some origin, it is convenient to have fundamental solutions appropriate to spherical coordinates. The separation of the angular and radial variables follows the expansion

$$\Psi(\mathbf{r},k) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} A_{l,m} f_l(r) Y_{l,m}(\theta,\phi) \quad .$$

The radial functions $f_l(r)$ satisfy the radial equation, independent of *m*,

$$\left[\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr} - \frac{l(l+1)}{r^2} + k^2\right]f_l(r) = 0 \quad .$$

With the substitution,

$$f_l(r) = r^{-1/2}u_l(r) \quad ,$$

Eq. () is transformed into

$$\left[\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{(l+1/2)^2}{r^2}+k^2\right]u_l(r)=0 \quad .$$

This equation is just the Bessel equation with $m=l+1/2$. Thus the solution for $f_l(r)$ are

$$f_l(r)=A_l\, r^{-1/2}J_{l+1/2}(kr)+B_l\, r^{-1/2}N_{l+1/2}(kr) \quad .$$

The solutions are customarily expressed as spherical Bessel and Hankel functions, denoted by $j_l(x)$, $n_l(x)$, $h_l^{(1,2)}(x)$, as follows:

$$j_l(x)=\left(\frac{\pi}{2x}\right)^{1/2}J_{l+1/2}(x) \quad ,$$

$$n_l(x)=\left(\frac{\pi}{2x}\right)^{1/2}N_{l+1/2}(x) \quad ,$$

$$h_l^{(1,2)}(x)=\left(\frac{\pi}{2x}\right)^{1/2}\left[J_{l+1/2}(x)\pm iN_{l+1/2}(x)\right] \quad .$$

It can be further shown that

$$j_l(x)=(-x)^l\left(\frac{1}{x}\frac{d}{dx}\right)^l\left(\frac{\sin x}{x}\right) \quad ,$$

$$n_l(x)=-(-x)^l\left(\frac{1}{x}\frac{d}{dx}\right)^l\left(\frac{\cos x}{x}\right) \quad .$$

For the first few values of l the explicit forms are given by

$$\begin{cases} j_0(x)=\dfrac{\sin x}{x} \\ n_0(x)=-\dfrac{\cos x}{x} \end{cases} \Rightarrow \quad h_0^{(1)}(x)=-i\frac{e^{ix}}{x} \quad ,$$

$$\begin{cases} j_1(x)=\dfrac{\sin x}{x^2}-\dfrac{\cos x}{x} \\ n_1(x)=-\dfrac{\cos x}{x^2}-\dfrac{\sin x}{x} \end{cases} \Rightarrow \quad h_1^{(1)}(x)=-\frac{e^{ix}}{x}\left(1+\frac{i}{x}\right) \quad ,$$

$$\begin{cases} j_2(x)=\left(\dfrac{3}{x^3}-\dfrac{1}{x}\right)\sin x-\dfrac{3}{x^2}\cos x \\ n_2(x)=-\left(\dfrac{3}{x^3}-\dfrac{1}{x}\right)\cos x-\dfrac{3}{x^2}\sin x \end{cases} \Rightarrow \quad h_2^{(1)}(x)=i\frac{e^{ix}}{x}\left(1+\frac{3i}{x}-\frac{3}{x^2}\right) \quad .$$

From the series

$$J_m(x)=\sum_{j=0}^{\infty}\frac{(-1)^j}{\Gamma(j+m+1)\,j!}\left(\frac{x}{2}\right)^{2j+m} \qquad (7)$$

$$J_{-m}(x)=\sum_{j=0}^{\infty}\frac{(-1)^j}{\Gamma(j-m+1)j!}\left(\frac{x}{2}\right)^{2j-m} , \tag{8}$$

the small argument limits are given by

$$j_l(x)\to\frac{x^l}{(2l+1)!!}\left(1-\frac{x^2}{2(2l+3)}+\cdots\right) ,$$

$$n_l(x)\to-\frac{(2l-1)!!}{x^{l+1}}\left(1-\frac{x^2}{2(1-2l)}+\cdots\right) ,$$

$$h_l^{(1)}(x)\to in_l(x)\to-i\frac{(2l-1)!!}{x^{l+1}}\left(1-\frac{x^2}{2(1-2l)}+\cdots\right).$$

Similarly the large argument limits are

$$j_l(x)\to\frac{1}{x}\sin\left(x-\frac{l\pi}{2}\right) ,$$

$$n_l(x)\to-\frac{1}{x}\cos\left(x-\frac{l\pi}{2}\right)$$

$$h_l^{(1)}(x)\to(-i)^{l+1}\frac{e^{ix}}{x} .$$

The spherical Bessel functions satisfy the recursion formuals,

$$z_{l-1}(x)+z_{l+1}(x)=\frac{2l+1}{x}z_l(x) ,$$

$$lz_{l-1}(x)-(l+1)z_{l+1}(x)=(2l+1)z_l'(x) .$$

The Wronskians of the various pairs are

$$W(j_l,n_l)=\frac{1}{i}W(j_l,h_l^{(1)})=-W(n_l,h_l^{(1)})=\frac{1}{x^2} .$$

The general solution for the Helmholtz equation

$$(\nabla^2+k^2)\Psi(\mathbf{r},k)=0$$

in spherical coordinates can be written

$$\Psi(\mathbf{r},k)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left[A_{l,m}h_l^{(1)}(r)+B_{l,m}h_l^{(2)}(r)\right]Y_{l,m}(\theta,\phi) .$$

where the coefficients $A_{l,m}$ and $B_{l,m}$ are determined by the boundary conditions.

The outgoing wave Green function $G_k(\mathbf{r},\mathbf{r}')$, which is appropriate to the equation,

$$(\nabla^2+k^2)G_k(\mathbf{r},\mathbf{r}')=-4\pi\delta(\mathbf{r}-\mathbf{r}') ,$$

is given by

$$G_k(\mathbf{r},\mathbf{r}') = \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \quad .$$

The spherical wave expansion for $G_k(\mathbf{r},\mathbf{r}')$ can be obtained in exactly the same way as was done for the Poisson equation. Thus the expansion of the Green function is

$$G_k(\mathbf{r},\mathbf{r}') = \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} = 4\pi ik \sum_{l=0}^{\infty} j_l(kr_<)\, h_l^{(1)}(kr_>) \sum_{m=-l}^{l} Y_{lm}^*(\theta',\phi')\, Y_{lm}(\theta,\phi)$$

Radiation from a localized oscillating source

With a system whose charges and currents varying in time, we can deal with each Fourier component separately. Therefore we lose no generality by considering potentials, fields, and radiation from localized systems to be sinusoidal in time. We take care

$$\rho(\mathbf{r}',t) = \rho(\mathbf{r}')e^{-i\omega t}$$

and

$$\mathbf{J}(\mathbf{r}',t) = \mathbf{J}(\mathbf{r}')e^{-i\omega t}$$

where $\mathbf{J}$ and ρ are required to satisfy the continuity equation

$$\nabla' \cdot \mathbf{J}(\mathbf{r}') = i\omega\rho(\mathbf{r}') \quad .$$

The vector potential arising from the time harmonic source is then

$$\mathbf{A}(\mathbf{r})e^{-i\omega t} = e^{-i\omega t} \frac{\mu_o}{4\pi} \int \mathbf{J}(\mathbf{r}') \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} d^3r'$$

or

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_o}{4\pi} \int \mathbf{J}(\mathbf{r}') \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} d^3r' \quad .$$

With the expansion

$$\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} = 4\pi ik \sum_{l=0}^{\infty} j_l(kr_<)\, h_l^{(1)}(kr_>) \sum_{m=-l}^{l} Y_{lm}^*(\theta',\phi')\, Y_{lm}(\theta,\phi),$$

the general expression for the vector potential when $r > r'$ becomes

$$\mathbf{A}(\mathbf{r}) = \mu_o ik \sum_{l=0}^{\infty} h_l^{(1)}(kr) \sum_{m=-l}^{l} Y_{lm}(\theta,\phi) \int \mathbf{J}(\mathbf{r}') j_l(kr') Y_{lm}^*(\theta',\phi') d^3r' \quad .$$

Alternatively, substituting the identity

$$\sum_{m=-l}^{l} Y_{lm}(\theta,\phi) Y_{lm}^*(\theta',\phi') = \frac{2l+1}{4\pi} P_l(\cos\gamma) = \frac{2l+1}{4\pi} P_l(\mathbf{a}_r \cdot \mathbf{a}_{r'})$$

into the general expression for the vector potential, we can write Eq. () as

$$\mathbf{A}(\mathbf{r}) = \mu_o ik \sum_{l=0}^{\infty} h_l^{(1)}(kr) \frac{2l+1}{4\pi} \int \mathbf{J}(\mathbf{r}') j_l(kr') P_l(\mathbf{a}_r \cdot \mathbf{a}_{r'}) d^3r' \quad .$$

If the source dimensions are small compared to the wavelength ($kr' << 1$), we can use

$$j_l(x) \rightarrow \frac{x^l}{(2l+1)!!}\left(1-\frac{x^2}{2(2l+3)}+\cdots\right)$$

to approximate $j_l(x)$ as $x^l/(2l+1)!!$ and the vector potential can be written

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_o ik}{4\pi}\sum_{l=0}^{\infty} h_l^{(1)}(kr)\int \mathbf{J}(\mathbf{r}')\frac{(kr')^l}{(2l-1)!!}P_l(\mathbf{a}_r \cdot \mathbf{a}_{r'})d^3r' .$$

Note that the condition $kr' << 1$ includes the important example of an atom of typical dimensions 0.1 nm radiating visible light of wavelength 500 nm. In the near zone (also called the induction zone) with $r > r'$, we can use

$$h_l^{(1)}(x) \rightarrow in_l(x) \rightarrow -i\frac{(2l-1)!!}{x^{l+1}}\left(1-\frac{x^2}{2(1-2l)}+\cdots\right)$$

to show that

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_o}{4\pi r^{l+1}}\sum_{l=0}^{\infty} \int \mathbf{J}(\mathbf{r}')r'^l P_l(\mathbf{a}_r \cdot \mathbf{a}_{r'})d^3r' \quad .$$

The form is just the expansion of the static solution.

Next we can discuss the first terms of $\mathbf{A}(\mathbf{r})$ for the localized source in detail. For $P_0(x) = 1$ and $h_0^{(1)}(x) = -ie^{ix}/x$, the $l = 0$ term yields

$$\mathbf{A}_0(\mathbf{r}) = \frac{\mu_o}{4\pi}\frac{e^{ikr}}{r}\int \mathbf{J}(\mathbf{r}')d^3r' .$$

It can be shown that this term arises from the dipole component of the charge distribution. For $P_1(x) = x$ and $h_1^{(1)}(x) = -e^{ix}(1+i/x)/x$, the $l = 1$ term yields

$$\begin{aligned}\mathbf{A}_1(\mathbf{r}) &= \frac{\mu_o ik^2}{4\pi}h_1^{(1)}(kr)\int \mathbf{J}(\mathbf{r}')r'(\mathbf{a}_r \cdot \mathbf{a}_{r'})d^3r' \\ &= \frac{-\mu_o ik}{4\pi r^2}e^{ikr}\left(1-\frac{1}{ikr}\right)\int \mathbf{J}(\mathbf{r}')(\mathbf{r}\cdot\mathbf{r}')d^3r'\end{aligned}\; .$$

Again we can relate the expression more directly to moments of the charge and current distributions.

For a localized oscillating source, the vector potential is given by

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_o}{4\pi}\int_V \frac{\mathbf{J}\left(\mathbf{r}', t-\frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)}{|\mathbf{r}-\mathbf{r}'|}d^3r'$$

Using Taylor series expansion, we can express the term in the integrand as

$$\frac{\mathbf{J}\left(\mathbf{r}',t-\frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)}{|\mathbf{r}-\mathbf{r}'|}=\frac{\mathbf{J}\left(\mathbf{r}',t-\frac{r}{c}\right)}{r}+(-\mathbf{r}')\cdot\nabla\left[\frac{\mathbf{J}\left(\mathbf{r}',t-\frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)}{|\mathbf{r}-\mathbf{r}'|}\right]_{\mathbf{r}-\mathbf{r}'\to\mathbf{r}}+\cdots$$

$$=\frac{\mathbf{J}\left(\mathbf{r}',t-\frac{r}{c}\right)}{r}+\frac{(\mathbf{r}'\cdot\mathbf{r})}{r^2}\left[\frac{\mathbf{J}\left(\mathbf{r}',t-\frac{r}{c}\right)}{r}+\frac{1}{c}\frac{\partial\mathbf{J}\left(\mathbf{r}',t-\frac{r}{c}\right)}{\partial t}\right]+\cdots$$

As a consequence, the vector potential can be given by

$$\mathbf{A}(\mathbf{r},t)=\frac{\mu_o}{4\pi}\int_V\frac{\mathbf{J}\left(\mathbf{r}',t-\frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)}{|\mathbf{r}-\mathbf{r}'|}d^3r'$$

$$=\frac{\mu_o}{4\pi}\int_V\left\{\frac{\mathbf{J}\left(\mathbf{r}',t-\frac{r}{c}\right)}{r}+\frac{(\mathbf{r}'\cdot\mathbf{r})}{r^2}\left[\frac{\mathbf{J}\left(\mathbf{r}',t-\frac{r}{c}\right)}{r}+\frac{1}{c}\frac{\partial\mathbf{J}\left(\mathbf{r}',t-\frac{r}{c}\right)}{\partial t}\right]+\cdots\right\}d^3r'$$

The first-order and second-order terms are respectively expressed as

$$\mathbf{A}^{(0)}(\mathbf{r},t)=\frac{\mu_o}{4\pi r}\int_V\mathbf{J}\left(\mathbf{r}',t-\frac{r}{c}\right)d^3r'$$

$$\mathbf{A}^{(1)}(\mathbf{r},t)=\frac{\mu_o}{4\pi r^3}\int_V(\mathbf{r}'\cdot\mathbf{r})\left[\mathbf{J}\left(\mathbf{r}',t-\frac{r}{c}\right)+\frac{r}{c}\frac{\partial\mathbf{J}\left(\mathbf{r}',t-\frac{r}{c}\right)}{\partial t}\right]d^3r'$$

Radiation from an oscillating electric dipole

The first-order term $\mathbf{A}^{(0)}(\mathbf{r},t)$ represents the radiation from an oscillating electric dipole and it can be further derived as

$$\mathbf{A}^{(0)}(\mathbf{r},t)=\frac{\mu_o}{4\pi r}\int_V\mathbf{J}\left(\mathbf{r}',t-\frac{r}{c}\right)d^3r'=\frac{-\mu_o}{4\pi r}\int_V\mathbf{r}'\left[\nabla'\cdot\mathbf{J}\left(\mathbf{r}',t-\frac{r}{c}\right)\right]d^3r'=\frac{\mu_o}{4\pi r}\int_V\mathbf{r}'\frac{\partial\rho\left(\mathbf{r}',t-\frac{r}{c}\right)}{\partial t}d^3r'$$

Define

$$\dot{\mathbf{p}}\left(t-\frac{r}{c}\right)=\frac{d\,\mathbf{p}\left(t-\frac{r}{c}\right)}{dt}=\int_V\mathbf{r}'\frac{\partial\rho\left(\mathbf{r}',t-\frac{r}{c}\right)}{\partial t}d^3r'\quad.$$

The first-order term is then given by

$$\mathbf{A}^{(0)}(\mathbf{r},t) = \frac{\mu_o}{4\pi r}\dot{\mathbf{p}}\left(t-\frac{r}{c}\right) = \frac{1}{4\pi\varepsilon_o c^2 r}\dot{\mathbf{p}}\left(t-\frac{r}{c}\right)$$

Using the condition of Lorenz gauge transformation, $\nabla\cdot\mathbf{A}+\frac{1}{c^2}\frac{\partial\Phi}{\partial t}=0$, the time derivative of the scalar potential is given by

$$\frac{\partial\Phi}{\partial t} = -c^2(\nabla\cdot\mathbf{A}) = -\frac{1}{4\pi\varepsilon_o}\left[\nabla\cdot\frac{\dot{\mathbf{p}}\left(t-\frac{r}{c}\right)}{r}\right] = \frac{1}{4\pi\varepsilon_o}\frac{\dot{\mathbf{p}}\left(t-\frac{r}{c}\right)\cdot\mathbf{r}}{r^3} + \frac{1}{4\pi\varepsilon_o c}\frac{\ddot{\mathbf{p}}\left(t-\frac{r}{c}\right)\cdot\mathbf{r}}{r^2}$$

Consequently, the scalar potential can be expressed as

$$\Phi = \frac{1}{4\pi\varepsilon_o}\frac{\mathbf{p}\left(t-\frac{r}{c}\right)\cdot\mathbf{r}}{r^3} + \frac{1}{4\pi\varepsilon_o c}\frac{\dot{\mathbf{p}}\left(t-\frac{r}{c}\right)\cdot\mathbf{r}}{r^2}.$$

Using the vector potential, the magnetic induction field is given by

$$\mathbf{B}(\mathbf{r},t) = \nabla\times\mathbf{A}^{(0)}(\mathbf{r},t) = \frac{\mu_o}{4\pi}\left[\nabla\times\frac{\dot{\mathbf{p}}\left(t-\frac{r}{c}\right)}{\mathrm{r}}\right] = -\frac{\mu_o}{4\pi}\frac{1}{r^2}\left(\mathbf{n}\times\dot{\mathbf{p}}\left(t-\frac{r}{c}\right)\right) - \frac{\mu_o}{4\pi}\frac{1}{cr}\left(\mathbf{n}\times\ddot{\mathbf{p}}\left(t-\frac{r}{c}\right)\right).$$

Or

$$\mathbf{H}(\mathbf{r},t) = -\frac{1}{4\pi}\frac{1}{r^2}\left(\mathbf{n}\times\dot{\mathbf{p}}\left(t-\frac{r}{c}\right)\right) - \frac{1}{4\pi}\frac{1}{cr}\left(\mathbf{n}\times\ddot{\mathbf{p}}\left(t-\frac{r}{c}\right)\right)$$

To evaluate **E**, we first derive the following term

$$-\nabla\Phi = \frac{-1}{4\pi\varepsilon_o}\nabla\left[\frac{\mathbf{p}\left(t-\frac{r}{c}\right)\cdot\mathbf{r}}{r^3} + \frac{1}{c}\frac{\dot{\mathbf{p}}\left(t-\frac{r}{c}\right)\cdot\mathbf{r}}{r^2}\right]$$

$$\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{B}\cdot\nabla)\mathbf{A} + (\mathbf{A}\cdot\nabla)\mathbf{B}$$

$$-\nabla\left[\frac{\mathbf{p}\left(t-\frac{r}{c}\right)\cdot\mathbf{r}}{r^3}\right] = \frac{3\left(\mathbf{p}\left(t-\frac{r}{c}\right)\cdot\mathbf{r}\right)\mathbf{r}}{r^5} - \frac{1}{r^3}\nabla\left[\mathbf{p}\left(t-\frac{r}{c}\right)\cdot\mathbf{r}\right]$$

$$-\nabla\left[\mathbf{p}\left(t-\frac{r}{c}\right)\cdot\mathbf{r}\right] = 0 + \frac{1}{cr}\mathbf{r}\times\left(\mathbf{r}\times\dot{\mathbf{p}}\left(t-\frac{r}{c}\right)\right) + \frac{r}{c}\dot{\mathbf{p}}\left(t-\frac{r}{c}\right) - \mathbf{p}\left(t-\frac{r}{c}\right)$$

Using $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$,

$$\frac{1}{cr}\mathbf{r}\times\left(\mathbf{r}\times\dot{\mathbf{p}}\left(t-\frac{r}{c}\right)\right)=\frac{1}{cr}\mathbf{r}\left(\mathbf{r}\cdot\dot{\mathbf{p}}\left(t-\frac{r}{c}\right)\right)-\frac{r}{c}\dot{\mathbf{p}}\left(t-\frac{r}{c}\right)$$

$$-\nabla\left[\frac{\mathbf{p}\left(t-\frac{r}{c}\right)\cdot\mathbf{r}}{r^3}\right]=\frac{3\mathbf{n}\left(\mathbf{n}\cdot\mathbf{p}\left(t-\frac{r}{c}\right)\right)-\mathbf{p}\left(t-\frac{r}{c}\right)}{r^3}+\frac{\mathbf{n}\left(\mathbf{n}\cdot\dot{\mathbf{p}}\left(t-\frac{r}{c}\right)\right)}{cr^2}$$

$$-\nabla\left[\frac{\dot{\mathbf{p}}\left(t-\frac{r}{c}\right)\cdot\mathbf{r}}{cr^2}\right]=\frac{2\left(\dot{\mathbf{p}}\left(t-\frac{r}{c}\right)\cdot\mathbf{r}\right)\mathbf{r}}{cr^4}-\frac{1}{cr^2}\nabla\left[\dot{\mathbf{p}}\left(t-\frac{r}{c}\right)\cdot\mathbf{r}\right]$$

$$-\nabla\left[\dot{\mathbf{p}}\left(t-\frac{r}{c}\right)\cdot\mathbf{r}\right]=0+\frac{1}{cr}\mathbf{r}\times\left(\mathbf{r}\times\ddot{\mathbf{p}}\left(t-\frac{r}{c}\right)\right)+\frac{r}{c}\ddot{\mathbf{p}}\left(t-\frac{r}{c}\right)-\dot{\mathbf{p}}\left(t-\frac{r}{c}\right)$$

$$-\nabla\left[\frac{\dot{\mathbf{p}}\left(t-\frac{r}{c}\right)\cdot\mathbf{r}}{cr^2}\right]=\frac{2\mathbf{n}\left(\mathbf{n}\cdot\dot{\mathbf{p}}\left(t-\frac{r}{c}\right)\right)-\dot{\mathbf{p}}\left(t-\frac{r}{c}\right)}{cr^2}+\frac{1}{c^2r^3}\mathbf{r}\times\left(\mathbf{r}\times\ddot{\mathbf{p}}\left(t-\frac{r}{c}\right)\right)+\frac{1}{c^2r}\ddot{\mathbf{p}}\left(t-\frac{r}{c}\right)$$

$$-\frac{\partial\mathbf{A}^{(0)}(\mathbf{r},t)}{\partial t}=\frac{-1}{4\pi\varepsilon_o c^2 r}\ddot{\mathbf{p}}\left(t-\frac{r}{c}\right)$$

Using $\mathbf{E}=-\nabla\Phi-\frac{\partial\mathbf{A}}{\partial t}$, we can obtain

$$\mathbf{E}=\frac{1}{4\pi\varepsilon_o}\left[\frac{3\mathbf{n}\left(\mathbf{n}\cdot\mathbf{p}\left(t-\frac{r}{c}\right)\right)-\mathbf{p}\left(t-\frac{r}{c}\right)}{r^3}+\frac{3\mathbf{n}\left(\mathbf{n}\cdot\dot{\mathbf{p}}\left(t-\frac{r}{c}\right)\right)-\dot{\mathbf{p}}\left(t-\frac{r}{c}\right)}{cr^2}+\frac{\mathbf{n}\times\left(\mathbf{n}\times\ddot{\mathbf{p}}\left(t-\frac{r}{c}\right)\right)}{c^2r}\right]$$

To be brief, we obtain the electric field and magnetic fields:

$$\mathbf{E}=\frac{1}{4\pi\varepsilon_o}\left[\frac{3\mathbf{n}\left(\mathbf{n}\cdot\mathbf{p}\left(t-\frac{r}{c}\right)\right)-\mathbf{p}\left(t-\frac{r}{c}\right)}{r^3}+\frac{3\mathbf{n}\left(\mathbf{n}\cdot\dot{\mathbf{p}}\left(t-\frac{r}{c}\right)\right)-\dot{\mathbf{p}}\left(t-\frac{r}{c}\right)}{cr^2}-\frac{\left(\mathbf{n}\times\ddot{\mathbf{p}}\left(t-\frac{r}{c}\right)\right)\times\mathbf{n}}{c^2r}\right]$$

$$\mathbf{H}=-\frac{1}{4\pi}\frac{1}{r^2}\left(\mathbf{n}\times\dot{\mathbf{p}}\left(t-\frac{r}{c}\right)\right)-\frac{1}{4\pi}\frac{1}{cr}\left(\mathbf{n}\times\ddot{\mathbf{p}}\left(t-\frac{r}{c}\right)\right)$$

Restricting ourselves to the two limiting cases then we obtain for the near zone ($r\ll\lambda$), because the higher powers of r dominate in the denominator and the radiation term drops out:

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_o}\left[\frac{3\mathbf{n}\left(\mathbf{n}\cdot\mathbf{p}\left(t-\frac{r}{c}\right)\right)-\mathbf{p}\left(t-\frac{r}{c}\right)}{r^3}\right]$$

$$\mathbf{H} = -\frac{1}{4\pi}\frac{1}{r^2}\left(\mathbf{n}\times\dot{\mathbf{p}}\left(t-\frac{r}{c}\right)\right)$$

The electric field is just the static electric dipole field. The magnetic field is significantly smaller than the electric field in the near-field region. In other words, the fields in the near zone are dominantly electric in nature. In the far field ($r>>\lambda$), all terms with the higher powers of r can be neglected so that there remains only:

$$\mathbf{E} = \frac{-1}{4\pi\varepsilon_o}\frac{\left(\mathbf{n}\times\ddot{\mathbf{p}}\left(t-\frac{r}{c}\right)\right)\times\mathbf{n}}{c^2 r}$$

$$\mathbf{H} = -\frac{1}{4\pi}\frac{1}{cr}\left(\mathbf{n}\times\ddot{\mathbf{p}}\left(t-\frac{r}{c}\right)\right).$$

It follows directly that in the far zone (**n**, **E**, **H**) form an orthogonal system and **E** and **H** can be expressed as

$$\mathbf{E} = Z_o\mathbf{H}\times\mathbf{n}\,.$$

Therefore, in the far zone **E** and **H** are mutually orthogonal outgoing spherical waves, that is, propagating in **r**-direction.

The Poynting vector S of the dipole field

Substituting $\mathbf{E} = Z_o\mathbf{H}\times\mathbf{n}$ into the Poynting vector, we obtain for the far zone:

$$\mathbf{S} = \mathbf{E}\times\mathbf{H} = Z_o\mathbf{H}\times\mathbf{n}\times\mathbf{H} = Z_o\mathbf{n}(\mathbf{H}\cdot\mathbf{H}) - Z_o\mathbf{H}(\mathbf{H}\cdot\mathbf{n})\,.$$

Since $\mathbf{H}\perp\mathbf{n}$, $\mathbf{H}\cdot\mathbf{n} = 0$. So, in the far field the energy-flux density is given by

$$\mathbf{S} = \frac{Z_o}{16\pi^2}\frac{1}{c^2r^2}\left|\mathbf{n}\times\ddot{\mathbf{p}}\left(t-\frac{r}{c}\right)\right|^2 = \frac{Z_o}{16\pi^2}\frac{\left|\ddot{\mathbf{p}}\left(t-\frac{r}{c}\right)\right|^2}{c^2r^2}\sin^2\theta\,,$$

where θ is the angle between the axis of oscillation and the position vector **r**. Therefore, in the far zone the energy flux flows in the radial direction, for increasing distance r from the dipole it decreases proportional to $1/r^2$. This is important for energy conservation. Furthermore, a $\sin^2\theta$-dependence folds in the far field. Obviously, for $\theta=0$, $\sin\theta=0$ and S=0, that is, in the far field the dipole does not radiate in the direction of oscillation. Further, energy is irradiated only if there is an acceleration. The bremsstrahlung of accelerated charges is based on this fact.

These processes play an important role in heavy-ion physics and also in the acceleration of particles by large accelerators. The power radiated per unit solid angle by the oscillating dipole moment **p** is

$$\frac{dP}{d\Omega} = r^2 S = \frac{Z_o}{16\pi^2}\frac{1}{c^2}\left|\mathbf{n}\times\ddot{\mathbf{p}}\left(t-\frac{r}{c}\right)\right|^2 = \frac{Z_o}{16\pi^2}\frac{\left|\ddot{\mathbf{p}}\left(t-\frac{r}{c}\right)\right|^2}{c^2}\sin^2\theta$$

The total power radiated is

$$P = \int\frac{dP}{d\Omega}d\Omega = \frac{Z_o}{16\pi^2}\frac{\left|\ddot{\mathbf{p}}\left(t-\frac{r}{c}\right)\right|^2}{c^2}\int\sin^2\theta\, d\Omega = \frac{Z_o}{6\pi}\frac{\left|\ddot{\mathbf{p}}\left(t-\frac{r}{c}\right)\right|^2}{c^2}$$

Assume the electric dipole to be a harmonic oscillation,

$$\mathbf{p}\left(t-\frac{r}{c}\right) = \mathbf{p}_o\sin\left[\omega\left(t-\frac{r}{c}\right)\right]$$

Hence, $\ddot{\mathbf{p}}\left(t-\frac{r}{c}\right) = -\omega^2\mathbf{p}_o\sin\left[\omega\left(t-\frac{r}{c}\right)\right]$. Substituting into the equation for the total power, we obtain

$$P = \frac{Z_o\omega^4}{6\pi}\frac{|\mathbf{p}_o|^2}{c^2}\sin^2\left[\omega\left(t-\frac{r}{c}\right)\right].$$

The time-averaged total power is given by

$$\langle P\rangle = \frac{Z_o\omega^4}{12\pi}\frac{|\mathbf{p}_o|^2}{c^2} = \frac{c^2 Z_o k^4}{12\pi}|\ \mathbf{p}_o\ |^2$$

Example: Scattering of light from a polarizable molecule

Calculate the total cross section for the scattering of light from a polarizable molecule. How does the scattering cross section depend on the wavelength λ and the polarizability α?

Solution: The dipole moment induced in the molecule by the electric $\mathbf{E}(t) = \mathbf{E}_o e^{-i\omega t}$ of the light is

$$\mathbf{p}(t) = \alpha\,\mathbf{E}(t) = \alpha\mathbf{E}_o e^{-i\omega t} = \mathbf{p}_o e^{-i\omega t}$$

$$\alpha = 4\pi\varepsilon_o\left(\frac{\varepsilon_r - 1}{\varepsilon_r + 2}\right)a^3$$

$$\sigma = \frac{\langle P \rangle}{\frac{1}{2Z_o}|\mathbf{E}_o|^2} = \frac{c^2 Z_o^2 k^4}{6\pi} \frac{|\mathbf{p}_o|^2}{|\mathbf{E}_o|^2} = \frac{c^2 Z_o^2 k^4}{6\pi} \left| 4\pi\varepsilon_o \left(\frac{\varepsilon_r - 1}{\varepsilon_r + 2} \right) a^3 \right|^2 = \frac{8\pi k^4 a^6}{3} \left| \frac{\varepsilon_r - 1}{\varepsilon_r + 2} \right|^2$$

This scattering cross section was derived by Lord Rayleigh who used it to explain the blueness of the sky and the redness of the sunrise and the sunset. To understand this somewaht better we notice first that the polarizability of a nitrogen molecule (the main constitute of the air) can be regarded to be approximately equal to that of a conducting sphere of radius 0.12 nm. For λ_{red} = 650 nm

$$\sigma_{red} = \frac{8\pi k^4 a^6}{3} = \frac{4(2\pi)^5 (1.2\times 10^{-8})^6}{(6.5\times 10^{-5})^4} = 2.1\times 10^{-27} \quad \text{cm}^2$$

Under normal atmospheric conditions at sea level, the number of molecules per cm^3 is

$$n = \frac{6\times 10^{23}}{22.4\times 10^3} = 2.7\times 10^{19} \quad \text{cm}^{-3}$$

Therefore, the mean free path of the red light (that is, the mean path it travels without being scattered)

$$L_{red} = \frac{1}{n\sigma} = \frac{1}{2.7\times 10^{19}\cdot 2.1\times 10^{-27}} = 1.8\times 10^7 \quad \text{cm} = 180\,\text{km}$$

For λ_{blue} = 470 nm

$$L_{blue} = \frac{1}{n\sigma} = 180\,\text{km}\left(\frac{470}{650}\right)^4 = 49\,\text{km}$$

The sky is blue because the light coming directly from the sun is scattered as soon as it enters the atmosphere. The small mean free path of the blue light component compared to the red one indicates that the scattering process for the blue light is much more effective than for the red one. The redness of the sunset or the sunrise can be explained similarly: At sunrise and sunset, the light has to cover a longer path through the atmosphere (in particular, through the dense zone). As demonstrated by the mean free paths the blue light is scattered much more strongly than the red one. Therefore, the red light remains. These estimations give the principles of the process of scattering of light in the atmosphere. Note that statistical fluctuations and air pollution may play an important role.

Radiation from magnetic dipole and electric quadrupole fields

The second-order term for the vector potential is given by

$$\mathbf{A}^{(1)}(\mathbf{r},t) = \frac{\mu_o}{4\pi\, r^3}\int_V (\mathbf{r}'\cdot\mathbf{r})\left[\mathbf{J}\left(\mathbf{r}',t-\frac{r}{c}\right)+\frac{r}{c}\frac{\partial \mathbf{J}\left(\mathbf{r}',t-\frac{r}{c}\right)}{\partial t}\right]d^3r' \quad .$$

Using the identity $(\mathbf{A}\times\mathbf{B})\times\mathbf{C} = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{A}(\mathbf{B}\cdot\mathbf{C})$, we can write

$$\mathbf{B}(\mathbf{A}\cdot\mathbf{C}) = \frac{1}{2}(\mathbf{A}\times\mathbf{B})\times\mathbf{C} + \frac{1}{2}\left[\mathbf{B}(\mathbf{A}\cdot\mathbf{C}) + \mathbf{A}(\mathbf{B}\cdot\mathbf{C})\right].$$

Therefore, we obtain

$$\mathbf{J}\left(\mathbf{r}',t-\frac{r}{c}\right)(\mathbf{r}'\cdot\mathbf{r}) = \frac{1}{2}\left[\mathbf{r}'\times\mathbf{J}\left(\mathbf{r}',t-\frac{r}{c}\right)\right]\times\mathbf{r} + \frac{1}{2}\left\{\mathbf{J}\left(\mathbf{r}',t-\frac{r}{c}\right)(\mathbf{r}'\cdot\mathbf{r}) + \mathbf{r}'\left[\mathbf{J}\left(\mathbf{r}',t-\frac{r}{c}\right)\cdot\mathbf{r}\right]\right\}$$

and

$$\dot{\mathbf{J}}\left(\mathbf{r}',t-\frac{r}{c}\right)(\mathbf{r}'\cdot\mathbf{r}) = \frac{1}{2}\left[\mathbf{r}'\times\dot{\mathbf{J}}\left(\mathbf{r}',t-\frac{r}{c}\right)\right]\times\mathbf{r} + \frac{1}{2}\left\{\dot{\mathbf{J}}\left(\mathbf{r}',t-\frac{r}{c}\right)(\mathbf{r}'\cdot\mathbf{r}) + \mathbf{r}'\left[\dot{\mathbf{J}}\left(\mathbf{r}',t-\frac{r}{c}\right)\cdot\mathbf{r}\right]\right\}$$

The first anti-symmetric part $\frac{1}{2}\left[\mathbf{r}'\times\mathbf{J}\left(\mathbf{r}',t-\frac{r}{c}\right)\right]\times\mathbf{r}$ and $\frac{1}{2}\left[\mathbf{r}'\times\dot{\mathbf{J}}\left(\mathbf{r}',t-\frac{r}{c}\right)\right]\times\mathbf{r}$ is recognizable as the magnetization due to the current $\mathbf{J}$:

$$\mathcal{M} = \frac{1}{2}\left[\mathbf{r}'\times\mathbf{J}\left(\mathbf{r}',t-\frac{r}{c}\right)\right].$$

The second, symmetric term $\frac{1}{2}\left\{\mathbf{J}\left(\mathbf{r}',t-\frac{r}{c}\right)(\mathbf{r}'\cdot\mathbf{r}) + \mathbf{r}'\left[\mathbf{J}\left(\mathbf{r}',t-\frac{r}{c}\right)\cdot\mathbf{r}\right]\right\}$ and $\frac{1}{2}\left\{\dot{\mathbf{J}}\left(\mathbf{r}',t-\frac{r}{c}\right)(\mathbf{r}'\cdot\mathbf{r}) + \mathbf{r}'\left[\dot{\mathbf{J}}\left(\mathbf{r}',t-\frac{r}{c}\right)\cdot\mathbf{r}\right]\right\}$ will be shown to be related to the electric quadrupole moment density.

Radiation form magnetic dipole

Considering only the magnetization term, we have the vector potential

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_o}{4\pi\, r^3}\left[\mathbf{m}\left(t-\frac{r}{c}\right)\times\mathbf{r} + \frac{r}{c}\dot{\mathbf{m}}\left(t-\frac{r}{c}\right)\times\mathbf{r}\right]$$

where

$$\mathbf{m}\left(t-\frac{r}{c}\right) = \frac{1}{2}\int_V \left[\mathbf{r}'\times\mathbf{J}\left(\mathbf{r}',t-\frac{r}{c}\right)\right]d^3r' \quad ; \quad \dot{\mathbf{m}}\left(t-\frac{r}{c}\right) = \frac{1}{2}\int_V \left[\mathbf{r}'\times\dot{\mathbf{J}}\left(\mathbf{r}',t-\frac{r}{c}\right)\right]d^3r'$$

Then, we can use $\mathbf{B} = \nabla\times\mathbf{A}$ and $\mathbf{H} = \mathbf{B}/\mu_o$ to evaluate the magnetic field:

$$\nabla\times\mathbf{A} = \frac{\mu_o}{4\pi}\nabla\times\left[\frac{\mathbf{m}\left(t-\frac{r}{c}\right)\times\mathbf{r}}{r^3} + \frac{\dot{\mathbf{m}}\left(t-\frac{r}{c}\right)\times\mathbf{r}}{cr^2}\right]$$

The first term in the bracket can be derived as

$$\nabla\times\left[\frac{\mathbf{m}\left(t-\frac{r}{c}\right)\times\mathbf{r}}{r^3}\right]=\nabla\left(\frac{1}{r^3}\right)\times\mathbf{m}\left(t-\frac{r}{c}\right)\times\mathbf{r}+\frac{1}{r^3}\nabla\times\left[\mathbf{m}\left(t-\frac{r}{c}\right)\times\mathbf{r}\right]$$

$$\nabla\left(\frac{1}{r^3}\right)\times\mathbf{m}\left(t-\frac{r}{c}\right)\times\mathbf{r}=\frac{-3}{r^5}\mathbf{r}\times\left[\mathbf{m}\left(t-\frac{r}{c}\right)\times\mathbf{r}\right]=\frac{3\mathbf{n}\left[\mathbf{n}\cdot\mathbf{m}\left(t-\frac{r}{c}\right)\right]-3\mathbf{m}\left(t-\frac{r}{c}\right)}{r^3}$$

Using $\nabla\times(\mathbf{A}\times\mathbf{B})=(\mathbf{B}\cdot\nabla)\mathbf{A}+(\nabla\cdot\mathbf{B})\mathbf{A}-(\mathbf{A}\cdot\nabla)\mathbf{B}-(\nabla\cdot\mathbf{A})\mathbf{B}$, we find

$$\frac{1}{r^3}\nabla\times\left[\mathbf{m}\left(t-\frac{r}{c}\right)\times\mathbf{r}\right]=\frac{1}{r^3}\left\{(\mathbf{r}\cdot\nabla)\mathbf{m}\left(t-\frac{r}{c}\right)+(\nabla\cdot\mathbf{r})\mathbf{m}\left(t-\frac{r}{c}\right)-\left[\mathbf{m}\left(t-\frac{r}{c}\right)\cdot\nabla\right]\mathbf{r}-\left[\nabla\cdot\mathbf{m}\left(t-\frac{r}{c}\right)\right]\mathbf{r}\right\}$$

$$=\frac{1}{r^3}\left\{-\frac{(\mathbf{r}\cdot\mathbf{n})}{c}\dot{\mathbf{m}}\left(t-\frac{r}{c}\right)+3\mathbf{m}\left(t-\frac{r}{c}\right)-\mathbf{m}\left(t-\frac{r}{c}\right)+\left[\frac{\mathbf{n}}{c}\cdot\dot{\mathbf{m}}\left(t-\frac{r}{c}\right)\right]\mathbf{r}\right\}$$

Summing the last two equations, we obtain

$$\nabla\times\left[\frac{\mathbf{m}\left(t-\frac{r}{c}\right)\times\mathbf{r}}{r^3}\right]=\frac{3\mathbf{n}\left[\mathbf{n}\cdot\mathbf{m}\left(t-\frac{r}{c}\right)\right]-\mathbf{m}\left(t-\frac{r}{c}\right)}{r^3}+\frac{\mathbf{n}\left[\mathbf{n}\cdot\dot{\mathbf{m}}\left(t-\frac{r}{c}\right)\right]-\dot{\mathbf{m}}\left(t-\frac{r}{c}\right)}{cr^2}$$

The second term in the bracket of Eq. () can be derived as

$$\nabla\times\left[\frac{\dot{\mathbf{m}}\left(t-\frac{r}{c}\right)\times\mathbf{r}}{cr^2}\right]=\nabla\left(\frac{1}{cr^2}\right)\times\dot{\mathbf{m}}\left(t-\frac{r}{c}\right)\times\mathbf{r}+\frac{1}{cr^2}\nabla\times\left[\dot{\mathbf{m}}\left(t-\frac{r}{c}\right)\times\mathbf{r}\right]$$

$$\nabla\left(\frac{1}{cr^2}\right)\times\dot{\mathbf{m}}\left(t-\frac{r}{c}\right)\times\mathbf{r}=\frac{-2}{cr^4}\mathbf{r}\times\left[\dot{\mathbf{m}}\left(t-\frac{r}{c}\right)\times\mathbf{r}\right]=\frac{2\mathbf{n}\left[\mathbf{n}\cdot\dot{\mathbf{m}}\left(t-\frac{r}{c}\right)\right]-2\dot{\mathbf{m}}\left(t-\frac{r}{c}\right)}{cr^2}$$

Similarly, using $\nabla\times(\mathbf{A}\times\mathbf{B})=(\mathbf{B}\cdot\nabla)\mathbf{A}+(\nabla\cdot\mathbf{B})\mathbf{A}-(\mathbf{A}\cdot\nabla)\mathbf{B}-(\nabla\cdot\mathbf{A})\mathbf{B}$

$$\frac{1}{cr^2}\nabla\times\left[\dot{\mathbf{m}}\left(t-\frac{r}{c}\right)\times\mathbf{r}\right]=\frac{1}{cr^2}\left\{-\frac{(\mathbf{r}\cdot\mathbf{n})}{c}\ddot{\mathbf{m}}\left(t-\frac{r}{c}\right)+3\dot{\mathbf{m}}\left(t-\frac{r}{c}\right)-\dot{\mathbf{m}}\left(t-\frac{r}{c}\right)+\left[\frac{\mathbf{n}}{c}\cdot\ddot{\mathbf{m}}\left(t-\frac{r}{c}\right)\right]\mathbf{r}\right\}$$

$$\nabla\times\left[\frac{\dot{\mathbf{m}}\left(t-\frac{r}{c}\right)\times\mathbf{r}}{cr^2}\right]=\frac{2\mathbf{n}\left[\mathbf{n}\cdot\dot{\mathbf{m}}\left(t-\frac{r}{c}\right)\right]}{cr^2}+\frac{\left[\mathbf{n}\cdot\ddot{\mathbf{m}}\left(t-\frac{r}{c}\right)\right]\mathbf{n}-\ddot{\mathbf{m}}\left(t-\frac{r}{c}\right)}{c^2r}$$

$$=\frac{2\mathbf{n}\left[\mathbf{n}\cdot\dot{\mathbf{m}}\left(t-\frac{r}{c}\right)\right]}{cr^2}-\frac{\left[\mathbf{n}\times\ddot{\mathbf{m}}\left(t-\frac{r}{c}\right)\right]\times\mathbf{n}}{c^2r}$$

Therefore, the magnetic field due to a magnetic dipole source is given by

$$\mathbf{H}=\frac{1}{4\pi}\left\{\frac{3\mathbf{n}\left[\mathbf{n}\cdot\mathbf{m}\left(t-\frac{r}{c}\right)\right]-\mathbf{m}\left(t-\frac{r}{c}\right)}{r^3}+\frac{3\mathbf{n}\left[\mathbf{n}\cdot\dot{\mathbf{m}}\left(t-\frac{r}{c}\right)\right]-\dot{\mathbf{m}}\left(t-\frac{r}{c}\right)}{cr^2}-\frac{\left[\mathbf{n}\times\ddot{\mathbf{m}}\left(t-\frac{r}{c}\right)\right]\times\mathbf{n}}{c^2r}\right\}$$

Comparing the result for an electric dipole source, the electric field for a magnetic dipole source is the negative of Z_o times the magnetic field for an electric dipole with $\mathbf{p}\to\mathbf{m}/c$. Therefore, we can obtain

$$\mathbf{E}=\frac{Z_o}{4\pi}\left[\frac{1}{cr^2}\left(\mathbf{n}\times\dot{\mathbf{m}}\left(t-\frac{r}{c}\right)\right)+\frac{1}{c^2r}\left(\mathbf{n}\times\ddot{\mathbf{m}}\left(t-\frac{r}{c}\right)\right)\right]$$

All the arguments concerning the behavior of the fields in the near and far zones are the same as for the electric dipole source, with the interchange $\mathbf{E}\to Z_o\mathbf{H}$, $Z_o\mathbf{H}\to-\mathbf{E}$, $\mathbf{p}\to\mathbf{m}/c$. Similarly the radiation pattern and total power radiated are the same for the two kinds of dipole. The only difference in the radiation fields is in the polarization. For an electric dipole the electric vector lies in the plane defined by **n** and **p**, while for a magnetic dipole it is perpendicular to the plane defined by **n** and **m**.

Radiation form electric quadrupole

The integral of the symmetric term in the following equations

$$\mathbf{J}\left(\mathbf{r}',t-\frac{r}{c}\right)(\mathbf{r}'\cdot\mathbf{r})=\frac{1}{2}\left[\mathbf{r}'\times\mathbf{J}\left(\mathbf{r}',t-\frac{r}{c}\right)\right]\times\mathbf{r}+\frac{1}{2}\left\{\mathbf{J}\left(\mathbf{r}',t-\frac{r}{c}\right)(\mathbf{r}'\cdot\mathbf{r})+\mathbf{r}'\left[\mathbf{J}\left(\mathbf{r}',t-\frac{r}{c}\right)\cdot\mathbf{r}\right]\right\}$$

and

$$\dot{\mathbf{J}}\left(\mathbf{r}',t-\frac{r}{c}\right)(\mathbf{r}'\cdot\mathbf{r})=\frac{1}{2}\left[\mathbf{r}'\times\dot{\mathbf{J}}\left(\mathbf{r}',t-\frac{r}{c}\right)\right]\times\mathbf{r}+\frac{1}{2}\left\{\dot{\mathbf{J}}\left(\mathbf{r}',t-\frac{r}{c}\right)(\mathbf{r}'\cdot\mathbf{r})+\mathbf{r}'\left[\dot{\mathbf{J}}\left(\mathbf{r}',t-\frac{r}{c}\right)\cdot\mathbf{r}\right]\right\}$$

can be transformed by an integration by parts and some rearrangement:

$$\begin{aligned}\frac{1}{2}\int\left\{\mathbf{J}\left(\mathbf{r}',t-\frac{r}{c}\right)(\mathbf{r}'\cdot\mathbf{r})+\mathbf{r}'\left[\mathbf{J}\left(\mathbf{r}',t-\frac{r}{c}\right)\cdot\mathbf{r}\right]\right\}d^3r'&=\frac{-r}{2}\int\mathbf{r}'(\mathbf{r}'\cdot\mathbf{n})\nabla\cdot\mathbf{J}\left(\mathbf{r}',t-\frac{r}{c}\right)d^3r'\\&=\frac{r}{2}\frac{\partial}{\partial t}\int\mathbf{r}'(\mathbf{r}'\cdot\mathbf{n})\rho\left(\mathbf{r}',t-\frac{r}{c}\right)d^3r'\end{aligned}$$

The continuity equation has been used to replace $\nabla\cdot\mathbf{J}\left(\mathbf{r}',t-\frac{r}{c}\right)$ by $-\frac{\partial}{\partial t}\rho\left(\mathbf{r}',t-\frac{r}{c}\right)$. Since the integral involves second moments of the charge density, this symmetric part corresponds to an electric quadrupole source. The vector potential is then given by

$$\mathbf{A}(\mathbf{r},t)=\frac{\mu_o}{8\pi r^2}\left[\frac{\partial}{\partial t}\int\mathbf{r}'(\mathbf{r}'\cdot\mathbf{n})\rho\left(\mathbf{r}',t-\frac{r}{c}\right)d^3r'+\frac{r}{c}\frac{\partial^2}{\partial t^2}\int\mathbf{r}'(\mathbf{r}'\cdot\mathbf{n})\rho\left(\mathbf{r}',t-\frac{r}{c}\right)d^3r'\right]$$

The complete fields are somewhat complicated to write down. We just consider the fields in the radiation zone. The vector potential is then given by

$$\mathbf{A}(\mathbf{r},t)=\frac{\mu_o}{8\pi}\frac{1}{r}\frac{1}{c}\frac{\partial^2}{\partial t^2}\int \mathbf{r}'(\mathbf{r}'\cdot\mathbf{n})\rho\left(\mathbf{r}',t-\frac{r}{c}\right)d^3r'$$

Consequently the magnetic field in the radiation zone is

$$\mathbf{H}(\mathbf{r},t)=\frac{1}{\mu_o}\nabla\times\mathbf{A}\approx\frac{-1}{8\pi}\frac{1}{r}\frac{1}{c^2}\frac{\partial^3}{\partial t^3}\int (\mathbf{n}\times\mathbf{r}')(\mathbf{r}'\cdot\mathbf{n})\rho\left(\mathbf{r}',t-\frac{r}{c}\right)d^3r' \quad .$$

The electric field can be determined by $\mathbf{E}=Z_o\mathbf{H}\times\mathbf{n}$. With the definition for the quadrupole moment tensor,

$$Q_{\alpha\beta}=\int(3x_\alpha\, x_\beta-r^2\delta_{\alpha\beta})\rho(\mathbf{r})\,d^3r \ ,$$

the integral in the form of the magnetic field can be written

$$\int (\mathbf{n}\times\mathbf{r}')(\mathbf{r}'\cdot\mathbf{n})\rho\left(\mathbf{r}',t-\frac{r}{c}\right)d^3r'=\frac{1}{3}\mathbf{n}\times\mathbf{Q(n)}$$

The vector $\mathbf{Q(n)}$ is defined as having components

$$Q_\alpha=\sum_\beta Q_{\alpha\beta}n_\beta \ .$$

Note that it depends in magnitude and direction on the direction of observation as well as on the properties of the source. Therefore, the magnetic field in the radiation zone is given by

$$\mathbf{H}(\mathbf{r},t)=\frac{-1}{24\pi\, r}\frac{1}{c^2}\mathbf{n}\times\dddot{\mathbf{Q}}\mathbf{(n)} \quad ,$$

and the power radiated per unit solid angle is given by

$$\frac{dP}{d\Omega}=r^2S=\frac{Z_o}{576\pi^2}\frac{1}{c^4}\left|\ \mathbf{n}\times\dddot{\mathbf{Q}}(\mathbf{n})\times\mathbf{n}\ \right|^2 .$$

Radiation from extended sources

When the dimensions of the radiating system are not small compared to the wavelength of radiation, the multiple expansion of the potential is not valid and the integrals for the potential must be evaluated directly. To discuss more concretely, we consider the radiation arising from the thin, linear, center-fed antenna. We choose the z axis to lie along the antenna. The antenna of length d is split by a small gap at its midpoint where each half is supplied by current $\pm I_o e^{-i\omega t}$. To deduce the magnitude of the current on any point of the antenna, we neglect radiation damping. The current must be symmetric about the gap in the middle, and further, it must vanish at the ends. If the antenna were short, we would expect the entire right side to be uniformly charges to one polarity while the other side would be uniformly charged with the opposite polarity. For the one-dimensional problem, the continuity equation then states $\partial J/\partial z = -\partial \rho/\partial t = \text{constant}$. Thus **J** would be of the form $(\text{constant})\times\left(\mathrm{d}/2-|z|\right)\delta(x)\delta(y)$.

Therefore we take

$$\mathbf{J}(\mathbf{r},t) = \hat{z}\, I \sin\left(\frac{kd}{2} - k|z|\right)\delta(x)\delta(y)\, e^{-i\omega t} \qquad \text{for } |z| \le \frac{d}{2}$$

The vector potential due to an oscillating current is given by

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_o}{4\pi}\int\int \frac{\mathbf{J}(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|}\delta\left[t' - \left(t - \frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)\right] dt'\, d^3r'$$

$$= \frac{\mu_o e^{-i\omega t}}{4\pi}\hat{z}\, I \int \sin\left(\frac{kd}{2} - k|z'|\right)\delta(x')\delta(y')\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\, d^3r'$$

In the radiation zone, we may approximate

$$\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \approx \frac{e^{ikr}}{r}e^{-ik(\mathbf{r}\cdot\mathbf{r}')/r} = \frac{e^{ikr}}{r}e^{-ikz'\cos\theta}$$

$$\mathbf{A}(\mathbf{r},t) = \hat{z}\frac{\mu_o e^{-i\omega t}}{4\pi} I \int \sin\left(\frac{kd}{2} - k|z'|\right)\delta(x')\delta(y')\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\, d^3r'$$

$$= \hat{z}\frac{\mu_o e^{-i\omega t}e^{ikr}}{4\pi r} I \int_{-d/2}^{d/2} \sin\left(kd/2 - k|z'|\right) e^{-ikz'\cos\theta}\, dz'$$

$$= \hat{z}\frac{\mu_o e^{-i\omega t}e^{ikr}}{4\pi r} 2I\left[\frac{\cos\left(\frac{kd}{2}\cos\theta\right) - \cos\left(\frac{kd}{2}\right)}{\sin^2\theta}\right]$$

$$\mathbf{A}(\mathbf{r},t) = \hat{z}\frac{\mu_o e^{-i\omega t} e^{ikr}}{4\pi r} I \int_{-d/2}^{d/2} \sin\left(kd/2 - k|z'|\right) e^{-ikz'\cos\theta}\, dz'$$

$$= \hat{z}\frac{\mu_o e^{-i\omega t} e^{ikr}}{4\pi r} I \left[\int_0^{d/2} \sin\left(kd/2 - kz'\right) e^{-ikz'\cos\theta}\, dz' + \int_0^{d/2} \sin\left(kd/2 - kz'\right) e^{ikz'\cos\theta}\, dz'\right]$$

$$= \hat{z}\frac{\mu_o e^{-i\omega t} e^{ikr}}{4\pi r} 2I \int_0^{d/2} \sin\left(kd/2 - kz'\right) \cos(kz'\cos\theta)\, dz'$$

$$= \hat{z}\frac{\mu_o e^{-i\omega t} e^{ikr}}{4\pi r} I \int_0^{d/2} \sin\left(kd/2 - kz' + kz'\cos\theta\right) + \sin\left(kd/2 - kz' - kz'\cos\theta\right) dz'$$

$$= \hat{z}\frac{\mu_o e^{-i\omega t} e^{ikr}}{4\pi r} I \left\{ \frac{\left[\cos\left(\frac{kd}{2}\cos\theta\right) - \cos\left(\frac{kd}{2}\right)\right]}{k(1-\cos\theta)} + \frac{\left[\cos\left(\frac{kd}{2}\cos\theta\right) - \cos\left(\frac{kd}{2}\right)\right]}{k(1+\cos\theta)} \right\}$$

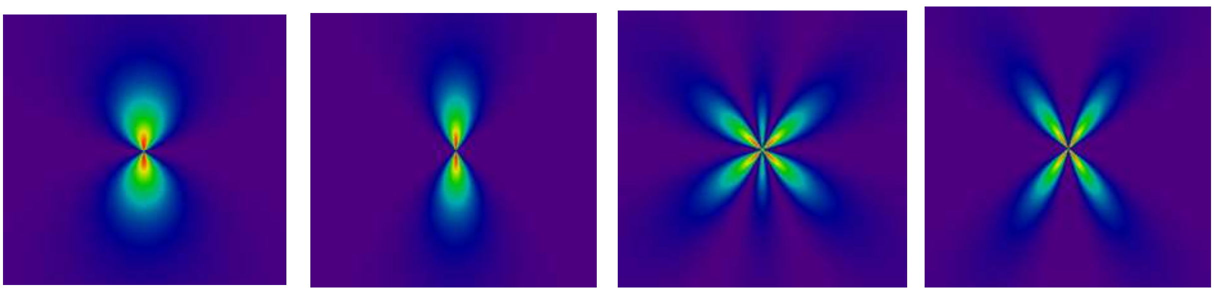
$$= \hat{z}\frac{\mu_o e^{-i\omega t} e^{ikr}}{4\pi k r} 2I \left[\frac{\cos\left(\frac{kd}{2}\cos\theta\right) - \cos\left(\frac{kd}{2}\right)}{\sin^2\theta}\right]$$

Since $\mathbf{H} = ik(\mathbf{n}\times\mathbf{A})/\mu_o$ in the radiation zone, it magnitude is

$$|\,\mathbf{H}\,| = \frac{k\sin\theta}{\mu_o}\frac{\mu_o e^{-i\omega t} e^{ikr}}{4\pi k r} 2I \left[\frac{\cos\left(\frac{kd}{2}\cos\theta\right) - \cos\left(\frac{kd}{2}\right)}{\sin^2\theta}\right]$$

Thus the time-averaged power radiated per unit solid angle is

$$\frac{dP}{d\Omega} = Z_o \frac{I^2}{8\pi^2} \left|\frac{\cos\left(\frac{kd}{2}\cos\theta\right) - \cos\left(\frac{kd}{2}\right)}{\sin\theta}\right|^2$$



m=1 m=2 m=3 m=4

Chapter Eleven: Radiation by moving charge

Lienard-Wiechert Potentials

Considering a point charge q moving with a velocity $\mathbf{v} = \dot{\mathbf{r}}_q(t')$, the scalar and vector potentials are given by

$$\Phi(\mathbf{r},t) = \frac{1}{4\pi\varepsilon_o}\int\int \frac{\rho(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|}\delta\left(t' - \left(t - \frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)\right) d^3r'dt'$$

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_o}{4\pi}\int\int \frac{\mathbf{J}(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|}\delta\left(t' - \left(t - \frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)\right) d^3r'dt'$$

For appoint charge q the charge and current densities are

$$\rho(\mathbf{r}',t') = q\delta(\mathbf{r}' - \mathbf{r}_q(t'))$$

$$\mathbf{J}(\mathbf{r}',t') = q\dot{\mathbf{r}}_q(t')\delta(\mathbf{r}' - \mathbf{r}_q(t'))$$

Note that $\mathbf{r}_q(t')$ is the position of charge at t'. We next have

$$\Phi(\mathbf{r},t) = \frac{q}{4\pi\varepsilon_o}\int \frac{1}{|\mathbf{r}-\mathbf{r}_q(t')|}\delta\left(t' - \left(t - \frac{|\mathbf{r}-\mathbf{r}_q(t')|}{c}\right)\right) dt'$$

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_o q}{4\pi}\int \frac{\dot{\mathbf{r}}_q(t')}{|\mathbf{r}-\mathbf{r}_q(t')|}\delta\left(t' - \left(t - \frac{|\mathbf{r}-\mathbf{r}_q(t')|}{c}\right)\right) dt'$$

The further evaluation can be done with the property of delta function

$$\int g(x)\delta(f(x)-y)dx = \left.\frac{g(x)}{df/dx}\right|_{y=f(x)}$$

Here $y = t$, $f(t') = t' + |\mathbf{r}-\mathbf{r}_q(t')|/c$, and

$$\frac{df(t')}{dt'} = 1 + \frac{1}{c}\frac{d|\mathbf{r}-\mathbf{r}_q(t')|}{dt'}$$

$$|\mathbf{r}-\mathbf{r}_q(t')| = \sqrt{(x-x_q(t'))^2 + (y-y_q(t'))^2 + (z-z_q(t'))^2}$$

$$\frac{d|\mathbf{r}-\mathbf{r}_q(t')|}{dt'} = -\frac{(x-x_q(t'))\frac{dx_q(t')}{dt'} + (y-y_q(t'))\frac{dy_q(t')}{dt'} + (z-z_q(t'))\frac{dz_q(t')}{dt'}}{\sqrt{(x-x_q(t'))^2 + (y-y_q(t'))^2 + (z-z_q(t'))^2}}$$

With $R(t') = \left| \mathbf{r} - \mathbf{r}_q(t') \right|$, we obtain

$$\frac{dR(t')}{dt'} = -\mathbf{n} \cdot \mathbf{v}(t'),$$

where $\mathbf{n} = \left(\mathbf{r} - \mathbf{r}_q(t')\right) / \left|\mathbf{r} - \mathbf{r}_q(t')\right|$ is an instantaneous unit vector. We can have

$$\frac{df(t')}{dt'} = 1 + \frac{1}{c}\frac{dR(t')}{dt'} = 1 - \boldsymbol{\beta} \cdot \mathbf{n}.$$

Therefore,

$$\Phi(\mathbf{r}, t) = \frac{q}{4\pi\varepsilon_o} \frac{1}{(1 - \boldsymbol{\beta} \cdot \mathbf{n}) \left| \mathbf{r} - \mathbf{r}_q(t') \right|}$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_o q}{4\pi} \frac{\mathbf{v}(t')}{(1 - \boldsymbol{\beta} \cdot \mathbf{n}) \left| \mathbf{r} - \mathbf{r}_q(t') \right|}$$

By means of these so-called Lienard-Wiechert potentials, we can now calculate the electric and magnetic fields due to a moving charge particle. Although our usual concern is with situations where $v << c$, we will for the moment carry along all orders in v/c so that in the future we can deal comfortably with charge particles that are moving at relativistic velocities. The work becomes one of differentiating these potentials. That is

$$\mathbf{B}(\mathbf{r}, t) = \nabla \times \frac{\mu_o q}{4\pi} \frac{\mathbf{v}(t')}{(1 - \boldsymbol{\beta} \cdot \mathbf{n}) \left| \mathbf{r} - \mathbf{r}_q(t') \right|}$$

$$\mathbf{E}(\mathbf{r}, t) = -\nabla \frac{q}{4\pi\varepsilon_o} \frac{1}{(1 - \boldsymbol{\beta} \cdot \mathbf{n}) \left| \mathbf{r} - \mathbf{r}_q(t') \right|} - \frac{\partial}{\partial t}\left[\frac{\mu_o q}{4\pi} \frac{\mathbf{v}(t')}{(1 - \boldsymbol{\beta} \cdot \mathbf{n}) \left| \mathbf{r} - \mathbf{r}_q(t') \right|}\right]$$

Differentiating the Lienard-Wiechert potentials for the Radiation fields

The differentiations of the Lienard-Wiechert potentials are laborious because the retarded time mixes the space and time. As a result, when the gradient and the time derivative are to be evaluated with present space and time, the process needs to be very careful. For this derivation, we have to prepare with a few preliminary derivatives. Let $\xi = (1 - \boldsymbol{\beta} \cdot \mathbf{n})$ and , $R = \left| \mathbf{r} - \mathbf{r}_q(t') \right|$, we can show that

$$\nabla R\big|_{t'} = \left(\nabla \left| \mathbf{r} - \mathbf{r}_q(t') \right|\right)_{t'} = \frac{\mathbf{r} - \mathbf{r}_q(t')}{\left| \mathbf{r} - \mathbf{r}_q(t') \right|} = \mathbf{n}$$

$$\nabla\xi\big|_{t'} = \nabla(1-\boldsymbol{\beta}\cdot\mathbf{n})\big|_{t'} = -\nabla(\boldsymbol{\beta}\cdot\mathbf{n})\big|_{t'} = -(\boldsymbol{\beta}\cdot\nabla)\,\mathbf{n} = -\frac{\boldsymbol{\beta}}{R} + \frac{(\boldsymbol{\beta}\cdot\mathbf{n})\,\mathbf{n}}{R}$$

$$\frac{\partial R}{\partial t'} = \frac{\partial\left|\mathbf{r}-\mathbf{r}_q(t')\right|}{\partial t'} = \frac{-\mathbf{v}\cdot\left(\mathbf{r}-\mathbf{r}_q(t')\right)}{\left|\mathbf{r}-\mathbf{r}_q(t')\right|} = -\mathbf{v}\cdot\mathbf{n}$$

$$\frac{\partial\xi}{\partial t'} = \frac{\partial(1-\boldsymbol{\beta}\cdot\mathbf{n})}{\partial t'} = -\frac{\mathbf{a}\cdot\mathbf{n}}{c} - \boldsymbol{\beta}\cdot\frac{\partial}{\partial t'}\left[\frac{\mathbf{r}-\mathbf{r}_q(t')}{\left|\mathbf{r}-\mathbf{r}_q(t')\right|}\right]$$

$$= -\frac{\mathbf{a}\cdot\mathbf{n}}{c} - \boldsymbol{\beta}\cdot\left(\frac{(\mathbf{v}\cdot\mathbf{n})\mathbf{n}-\mathbf{v}}{R}\right) = -\frac{\mathbf{a}\cdot\mathbf{n}}{c} - \frac{c\boldsymbol{\beta}}{R}\cdot\left((\boldsymbol{\beta}\cdot\mathbf{n})\mathbf{n}-\boldsymbol{\beta}\right)$$

$$\nabla t' = \nabla\left(t - \frac{\left|\mathbf{r}-\mathbf{r}_q(t')\right|}{c}\right) = -\frac{\left(\nabla\left|\mathbf{r}-\mathbf{r}_q(t')\right|\right)_{t'}}{c} - \frac{\partial}{\partial t'}\frac{\left|\mathbf{r}-\mathbf{r}_q(t')\right|}{c}\nabla t' = -\frac{\mathbf{n}}{c} + \frac{1}{c}\frac{\mathbf{v}\cdot\left(\mathbf{r}-\mathbf{r}_q(t')\right)}{\left|\mathbf{r}-\mathbf{r}_q(t')\right|}\nabla t'$$

$$\Rightarrow \quad \nabla t' = \frac{-\mathbf{n}}{c(1-\boldsymbol{\beta}\cdot\mathbf{n})} = -\frac{\mathbf{n}}{c\xi}$$

$$-\frac{\partial}{\partial t'}\left(\frac{1}{\xi R}\right) = \frac{1}{\xi^2 R^2}\left(\xi\frac{\partial R}{\partial t'} + R\frac{\partial\xi}{\partial t'}\right) = \frac{-c}{\xi^2 R^2}\left[\boldsymbol{\beta}\cdot(\mathbf{n}-\boldsymbol{\beta}) + \frac{R}{c^2}(\mathbf{a}\cdot\mathbf{n})\right]$$

$$\frac{\partial t'}{\partial t} = \frac{\partial}{\partial t}\left(t - \frac{\left|\mathbf{r}-\mathbf{r}_q(t')\right|}{c}\right) = 1 - \frac{\partial}{\partial t'}\frac{\left|\mathbf{r}-\mathbf{r}_q(t')\right|}{c}\frac{\partial t'}{\partial t} = 1 + (\boldsymbol{\beta}\cdot\mathbf{n})\frac{\partial t'}{\partial t}$$

$$\Rightarrow \quad \frac{\partial t'}{\partial t} = \frac{1}{1-\boldsymbol{\beta}\cdot\mathbf{n}} = \frac{1}{\xi}$$

$$-\nabla\Phi = -\nabla\Phi\big|_{t'} - \frac{\partial\Phi}{\partial t'}\nabla t'$$

$$-\nabla\Phi\big|_{t'} = \frac{q}{4\pi\varepsilon_o}\left[\frac{\xi\nabla R\big|_{t'} + R\nabla\xi\big|_{t'}}{\xi^2 R^2}\right]$$

$$-\nabla\Phi\big|_{t'} = \frac{q}{4\pi\varepsilon_o}\left[\frac{(1-\boldsymbol{\beta}\cdot\mathbf{n})\mathbf{n}+(\boldsymbol{\beta}\cdot\mathbf{n})\mathbf{n}-\boldsymbol{\beta}}{\xi^2R^2}\right] = \frac{q}{4\pi\varepsilon_o}\left[\frac{\mathbf{n}-\boldsymbol{\beta}}{\xi^2R^2}\right]$$

$$-\frac{\partial\Phi}{\partial t'}\nabla t' = -\frac{q}{4\pi\varepsilon_o}\frac{\partial}{\partial t'}\left(\frac{1}{\xi\ R}\right)\nabla t' = \frac{q}{4\pi\varepsilon_o}\frac{1}{\xi^2\ R^2}\left(\xi\frac{\partial R}{\partial t'}+R\frac{\partial\xi}{\partial t'}\right)\nabla t'$$

$$\begin{aligned}-\frac{\partial\Phi}{\partial t'}\nabla t' &= \frac{q}{4\pi\varepsilon_o}\frac{1}{\xi^2\ R^2}\left(\xi\frac{\partial R}{\partial t'}+R\frac{\partial\xi}{\partial t'}\right)\nabla t' \\ &= \frac{q}{4\pi\varepsilon_o}\frac{c}{\xi^2\ R^2}\left[\left(1-\boldsymbol{\beta}\cdot\mathbf{n}\right)(-\boldsymbol{\beta}\cdot\mathbf{n})+\frac{R}{c^2}(-\mathbf{a}\cdot\mathbf{n})-\boldsymbol{\beta}\cdot\left((\boldsymbol{\beta}\cdot\mathbf{n})\mathbf{n}-\boldsymbol{\beta}\right)\right]\left(-\frac{\mathbf{n}}{c\xi}\right) \\ &= \frac{q}{4\pi\varepsilon_o}\frac{1}{\xi^3\ R^2}\left[\boldsymbol{\beta}\cdot\left(\mathbf{n}-\boldsymbol{\beta}\right)\mathbf{n}+\frac{R}{c^2}(\mathbf{a}\cdot\mathbf{n})\mathbf{n}\right]\end{aligned}$$

$$\begin{aligned}-\nabla\Phi &= -\nabla\Phi\big|_{t'} - \frac{\partial\Phi}{\partial t'}\nabla t' \\ &= \frac{q}{4\pi\varepsilon_o\xi^2R^2}\left(\mathbf{n}-\boldsymbol{\beta}\right)+\frac{q}{4\pi\varepsilon_o}\frac{1}{\xi^3\ R^2}\left[\boldsymbol{\beta}\cdot\left(\mathbf{n}-\boldsymbol{\beta}\right)\mathbf{n}+\frac{R}{c^2}(\mathbf{a}\cdot\mathbf{n})\mathbf{n}\right] \\ &= \frac{q}{4\pi\varepsilon_o\xi^3R^2}\left[(1-\boldsymbol{\beta}\cdot\mathbf{n})\left(\mathbf{n}-\boldsymbol{\beta}\right)+\boldsymbol{\beta}\cdot\left(\mathbf{n}-\boldsymbol{\beta}\right)\mathbf{n}+\frac{R}{c^2}(\mathbf{a}\cdot\mathbf{n})\mathbf{n}\right] \\ &= \frac{q}{4\pi\varepsilon_o\xi^3R^2}\left[(1-\beta^2)\left(\mathbf{n}-\boldsymbol{\beta}\right)+\boldsymbol{\beta}\cdot\left(\mathbf{n}-\boldsymbol{\beta}\right)\boldsymbol{\beta}+\frac{R}{c^2}(\mathbf{a}\cdot\mathbf{n})\mathbf{n}\right]\end{aligned}$$

$$-\frac{\partial\mathbf{A}}{\partial t} = -\frac{\partial\mathbf{A}}{\partial t'}\frac{\partial t'}{\partial t} = -\frac{q}{4\pi\varepsilon_o c}\frac{\partial}{\partial t'}\left(\frac{\boldsymbol{\beta}}{\xi R}\right)\frac{\partial t'}{\partial t} = \frac{q}{4\pi\varepsilon_o c}\left[-\boldsymbol{\beta}\frac{\partial}{\partial t'}\left(\frac{1}{\xi R}\right)-\frac{1}{\xi R}\frac{\partial\,\boldsymbol{\beta}}{\partial t'}\right]\frac{\partial t'}{\partial t}$$

$$\begin{aligned}-\frac{\partial\mathbf{A}}{\partial t} &= \frac{q}{4\pi\varepsilon_o c}\left[-\boldsymbol{\beta}\frac{\partial}{\partial t'}\left(\frac{1}{\xi R}\right)-\frac{1}{\xi R}\frac{\partial\boldsymbol{\beta}}{\partial t'}\right]\frac{\partial t'}{\partial t} = \frac{q}{4\pi\varepsilon_o c}\left[\frac{-c}{\xi^2\ R^2}\boldsymbol{\beta}\left(\boldsymbol{\beta}\cdot\left(\mathbf{n}-\boldsymbol{\beta}\right)+\frac{R}{c^2}(\mathbf{a}\cdot\mathbf{n})\right)-\frac{1}{\xi Rc}\mathbf{a}\right]\frac{1}{\xi} \\ &= \frac{q}{4\pi\varepsilon_o\xi^3\ R^2}\left[-\boldsymbol{\beta}\left(\boldsymbol{\beta}\cdot\left(\mathbf{n}-\boldsymbol{\beta}\right)+\frac{R}{c^2}(\mathbf{a}\cdot\mathbf{n})\right)-\frac{R}{c^2}\left(1-\boldsymbol{\beta}\cdot\mathbf{n}\right)\mathbf{a}\right]\end{aligned}$$

$$\mathbf{E} = -\nabla\Phi - \frac{\partial \mathbf{A}}{\partial t}$$

$$= \frac{q}{4\pi\varepsilon_o \xi^3 R^2}\left[(1-\beta^2)(\mathbf{n}-\boldsymbol{\beta}) + \boldsymbol{\beta}\cdot(\mathbf{n}-\boldsymbol{\beta})\boldsymbol{\beta} + \frac{R}{c^2}(\mathbf{a}\cdot\mathbf{n})\mathbf{n} - \boldsymbol{\beta}\left(\boldsymbol{\beta}\cdot(\mathbf{n}-\boldsymbol{\beta}) + \frac{R}{c^2}(\mathbf{a}\cdot\mathbf{n})\right) - \frac{R}{c^2}(1-\boldsymbol{\beta}\cdot\mathbf{n})\mathbf{a}\right]$$

$$= \frac{q}{4\pi\varepsilon_o \xi^3 R^2}\left[(1-\beta^2)(\mathbf{n}-\boldsymbol{\beta}) + \frac{R}{c^2}(\mathbf{a}\cdot\mathbf{n})(\mathbf{n}-\boldsymbol{\beta}) - \frac{R}{c^2}(\mathbf{n}\cdot\mathbf{n}-\boldsymbol{\beta}\cdot\mathbf{n})\mathbf{a}\right]$$

$$= \frac{q}{4\pi\varepsilon_o \xi^3 R^2}\left[(1-\beta^2)(\mathbf{n}-\boldsymbol{\beta}) + \frac{R}{c^2}\mathbf{n}\times[(\mathbf{n}-\boldsymbol{\beta})\times\mathbf{a}]\right]$$

With the expression derived above, the following can be obtained

$$\mathbf{B} = \nabla\times\mathbf{A} = \nabla\times\mathbf{A}\big|_{t'} - \frac{\partial \mathbf{A}}{\partial t'}\times\nabla t' = \frac{\mathbf{n}}{c}\times\mathbf{E}$$

Radiation from slowly moving charges

When the velocity of the charges is small compared to *c*, the term involving b will be negligible and ξ~1. The radiation fields become

$$\mathbf{E} = \frac{q}{4\pi\varepsilon_o c^2 R}[\mathbf{n}\times(\mathbf{n}\times\mathbf{a})]$$

$$\mathbf{B} = \frac{\mathbf{n}}{c}\times\mathbf{E}$$

The Poynting vector is given by

$$S = \frac{\mathbf{E}\times\mathbf{B}}{\mu_o} = \frac{q^2\mathbf{n}}{(4\pi\varepsilon_o)^2\mu_o c^5 R^2}a^2\sin^2\theta = \frac{q^2\mathbf{n}}{(4\pi)^2\varepsilon_o c^3 R^2}a^2\sin^2\theta$$

where θ is the angle between the acceleration **a** and **n**. The angular distribution of radiation from the accelerated charge may be written

$$\frac{dP}{d\Omega} = \frac{q^2}{(4\pi)^2\varepsilon_o c^3}a^2\sin^2\theta$$

The total power radiated is found by integrating over the solid angle to be

$$P = \int\frac{dP}{d\Omega}d\Omega = \frac{q^2}{6\pi\varepsilon_o c^3}a^2 = \frac{1}{4\pi\varepsilon_o}\frac{2q^2}{3c^3}a^2 = \frac{1}{4\pi\varepsilon_o}\frac{2q^2}{3m^2c^3}F^2$$

where $F = ma$ is the applied force (the rate of change of momentum). This is the familiar Larmor result for a nonrelativistic, accelerated charge. The radiated power depends inversely on the square of the mass of the particles involved. Consequently these radiative effects are largest for electrons.

Radiation from relativistic charges

Considering the radiation fields only, the fields are given by

$$\mathbf{E} = \left[\frac{1}{4\pi\varepsilon_o}\right]\frac{q}{c\xi^3 R}\left\{\mathbf{n}\times\left[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\right]\right\}_{ret}$$

$$\mathbf{B} = \left[\frac{\mu_o c}{4\pi}\right]\frac{q}{c\xi^3 R}\left\{\mathbf{n}\times\mathbf{n}\times\left[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\right]\right\}_{ret}$$

The differential power radiated in $d\Omega$ at retarded time t' is

$$\frac{dP}{d\Omega} = R^2\mathbf{S}\cdot\mathbf{n}\frac{dt}{dt'} = \left[\frac{1}{4\pi\varepsilon_o}\right]\frac{c}{4\pi}\frac{q^2}{c^2\xi^6}\left|\,\mathbf{n}\times\left[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\right]\right|^2\frac{dt}{dt'}$$

$$= \left[\frac{1}{4\pi\varepsilon_o}\right]\frac{q^2}{4\pi c\xi^5}\left|\,\mathbf{n}\times\left[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\right]\right|^2 .$$

Henceforth we drop the suffix "ret", keeping in mind that it is unavoidable. Note that $R^2\mathbf{S}\cdot\mathbf{n}$ is the power radiated per unit solid angle detected at an observation point at time t of radiation emitted by the charge at time t'. It can be seen that $R^2\mathbf{S}\cdot\mathbf{n}(dt/dt')$ is the power radiated per unit solid angle in terms of the charge's own time. This formula has the correct non-relativistic reduction to Larmor's formula. It is evident that there are two types of relativistic effect present. One is the effect of the specific spatial relationship between $\boldsymbol{\beta}$ and $\dot{\boldsymbol{\beta}}$, which will determine the detailed angular distribution. The other is a general, relativistic effect arising from the transformation from the rest frame of the particle to the observer's frame and manifesting itself by the presence of the factor $(1-\boldsymbol{\beta}\cdot\mathbf{n})$ in the denominator. Now we discuss two important special cases:

1. $\boldsymbol{\beta}$ and $\dot{\boldsymbol{\beta}}$ are parallel, i.e. a charge particle moves in a straight line but is constantly accelerated. This situation occurs in x-ray tubes where a beam of high energy electrons impinge on a copper target (or any other target like tungsten) and loses its energy constantly. Then two kinds of radiations emerge. One is the characteristic line radiation of the target resulting from the atomic transitions. The other is continuous radiation because of deceleration of the electron as given by the equation above. Both of the radiations are superimposed. The continuous radiation in this case is called Bremsstrahlung. Since $\boldsymbol{\beta}$ and $\dot{\boldsymbol{\beta}}$ are parallel, the angular distribution of the power radiation is given by

$$\frac{dP}{d\Omega} = \left[\frac{1}{4\pi\varepsilon_o}\right]\frac{q^2}{4\pi c\xi^5}\left|\,\mathbf{n}\times\left[\mathbf{n}\times\dot{\boldsymbol{\beta}}\right]\right|^2$$

If θ is the angle between the acceleration **β** and **n**, then

$$\frac{dP}{d\Omega}=\left[\frac{1}{4\pi\varepsilon_o}\right]\frac{q^2a^2}{4\pi c^3}\frac{\sin^2\theta}{(1-\beta\cos\theta)^5}$$

Note that without the ξ^5 term in the denominator, the power pattern would be $\sin^2\theta$ type, same as Larmor type. However, because β~1 the denominator dominates the power pattern and more and more power is thrown in the forward direction as shown in **Fig.** The angle at which the maximum power is radiated can be gotten by

$$\frac{d}{d\theta}\left(\frac{\sin^2\theta}{(1-\beta\cos\theta)^5}\right)=0$$

From

$$\frac{d}{d\theta}\left(\frac{\sin^2\theta}{(1-\beta\cos\theta)^5}\right)=\frac{2\sin\theta\cos\theta}{(1-\beta\cos\theta)^5}-\frac{5\beta\sin^3\theta}{(1-\beta\cos\theta)^6}=0,$$

we can obtain the equation, $2(1-\beta\cos\theta)\cos\theta-5\beta\sin^2\theta=0$. Solving the equation can lead to

$$\cos\theta_{\max}=\frac{1}{3\beta}\left(\sqrt{1+15\beta^2}-1\right)$$

For β~0.5, corresponding to electrons of ~80 keV, θ_{max}~38.2°. In ultra-relativistic limit, $\beta\to 1$, θ_{max} is very small and we can show

$$\theta_{\max}\approx\frac{1}{2\gamma}.$$

where $\gamma=1/\sqrt{1-\beta^2}$. To show this result, we use the asymptotic forms of $\beta=1-1/(2\gamma^2)$ for $\beta\to 1$ and $\cos\theta_{\max}\approx 1-\theta_{\max}^2/2$ for $\theta\to 0$. With these asymptotic forms, we can obtain

$$1-\frac{\theta_{\max}^2}{2}=\frac{\left(\sqrt{1+15\left(1-\frac{1}{\gamma^2}\right)}-1\right)}{3\left(1-\frac{1}{2\gamma^2}\right)}\approx\frac{\left(4\left(1-\frac{15}{32\gamma^2}\right)-1\right)}{3\left(1-\frac{1}{2\gamma^2}\right)}\approx\frac{\left(1-\frac{5}{8\gamma^2}\right)}{\left(1-\frac{1}{2\gamma^2}\right)}\approx 1-\frac{1}{8\gamma^2}.$$

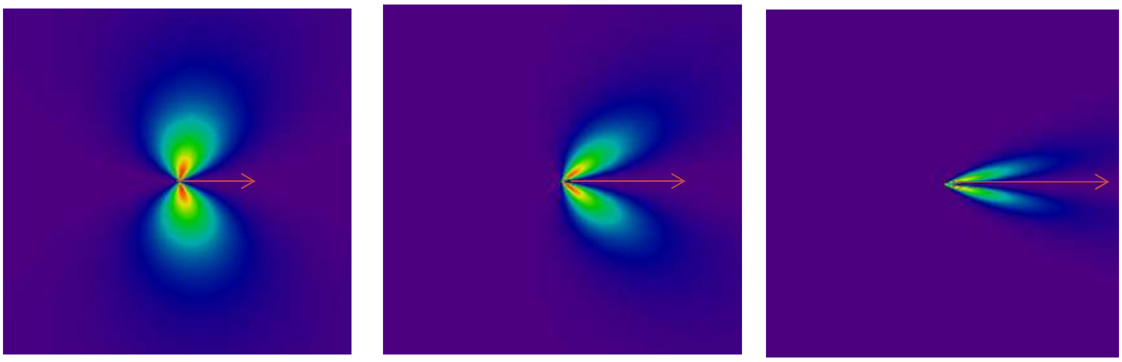
Thus the angular distribution is confined to a very narrow cone in the direction of motion. For a typical particle like electron, with β~0.9, θ_{max}~12°. Even For relativity, $E=\gamma\, m_oc^2$, $\theta_{\max}\approx m_oc^2/2E$ for β~1. Under the circumstances, $\sin\theta\approx\theta$, $\cos\theta\approx 1-\theta^2/2$ and

$$\frac{dP}{d\Omega} \approx \left[\frac{1}{4\pi\varepsilon_o}\right]\frac{q^2a^2}{4\pi c^3}\frac{\theta^2}{(1-\beta+\beta\theta^2/2)^5}$$

$$\approx \left[\frac{1}{4\pi\varepsilon_o}\right]\frac{q^2a^2}{4\pi c^3}\frac{\theta^2}{[(1/2\gamma^2)+(\theta^2/2)]^5} \approx \left[\frac{1}{4\pi\varepsilon_o}\right]\frac{q^2a^2}{4\pi c^3}\frac{32\gamma^8(\gamma\theta)^2}{[1+(\gamma\theta)^2]^5}$$

In this same limit the peak intensity is proportional to γ^8. The natural angular unit is evidently γ^{-1}. The angular distribution is shown in **Fig.** with angles measured in these units. The peak occurs at $\gamma\theta = 1/2$, and the half-power points at $\gamma\theta = 0.23$ and $\gamma\theta = 0.91$. The root mean square angle of emission of radiation in the relativistic limit is

$$\langle\theta^2\rangle^{1/2} = \frac{1}{\gamma} = m_oc^2/E.$$



β = 0.1 β = 0.5 β = 0.9

The total power radiated can be found by

$$P = \int\frac{dP}{d\Omega}d\Omega = \left[\frac{1}{4\pi\varepsilon_o}\right]\frac{q^2a^2}{4\pi c^3}2\pi\int_0^\pi \frac{\sin^2\theta}{(1-\beta\cos\theta)^5}\sin\theta\, d\theta = \left[\frac{1}{4\pi\varepsilon_o}\right]\frac{2q^2a^2}{3c^3}\gamma^6$$

Here we use

$$\int_0^\pi \frac{\sin^2\theta\sin\theta}{(1-\beta\cos\theta)^5}d\theta = \int_{-1}^1 \frac{1-x^2}{(1-\beta x)^5}dx = \frac{x}{6\beta^2(1-\beta x)^3}\Bigg|_{-1}^{1} - \frac{1}{6\beta^2}\int_{-1}^1 \frac{1}{(1-\beta x)^3}dx$$

$$= \frac{4}{3}\frac{1}{(1-\beta^2)^3}$$

This result is a relativistic generalization of the Larmor's formula for the straight line motion. In terms of the applied force, we can use

$$\mathbf{F} = \frac{d}{dt}\frac{m_ov}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{m_o\dot{\mathbf{v}}}{\left[1-\frac{v^2}{c^2}\right]^{3/2}} = \gamma^3 m_o\dot{\mathbf{v}}$$

for the case of $\boldsymbol{\beta}$ and $\dot{\boldsymbol{\beta}}$ to be parallel to write the total power radiated as

$$P = \left[\frac{1}{4\pi\varepsilon_o}\right]\frac{2q^2a^2}{3c^3}\gamma^6 = \left[\frac{1}{4\pi\varepsilon_o}\right]\frac{2q^2}{3m_o^2c^3}|\mathbf{F}|^2.$$

2. Another example of angular distribution of radiation is that for a charge in instantaneously circular motion with its acceleration $\dot{\boldsymbol{\beta}}$ perpendicular to its velocity $\boldsymbol{\beta}$. We choose a coordinate system in which, at t', $\boldsymbol{\beta}$ lies along the z axis and $\dot{\boldsymbol{\beta}}$ lies along the x axis. In Cartesian coordinates, when the observer has angular coordinates θ and ϕ, $\boldsymbol{\beta} = (0,0,\beta)$, $\dot{\boldsymbol{\beta}} = (\dot{\beta},0,0)$, and $\mathbf{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$. The term $\mathbf{n}\times[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}]$ then becomes

$$\mathbf{n}\times[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}] = (\mathbf{n}\cdot\dot{\boldsymbol{\beta}})(\mathbf{n}-\boldsymbol{\beta}) - [\mathbf{n}\cdot(\mathbf{n}-\boldsymbol{\beta})]\,\dot{\boldsymbol{\beta}} = \dot{\beta}\sin\theta\cos\phi(\mathbf{n}-\boldsymbol{\beta}) - (1-\beta\cos\theta)\,\dot{\boldsymbol{\beta}}$$

Which, when squares, gives

$$\left|\mathbf{n}\times[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\,]\right|^2$$
$$= (\dot{\beta}\sin\theta\cos\phi)^2(1-2\beta\cos\theta+\beta^2) + (1-\beta\cos\theta)^2\,\dot{\beta}^2 - 2(\dot{\beta}\sin\theta\cos\phi)\dot{\beta}(1-\beta\cos\theta)\sin\theta\cos\phi$$
$$= \dot{\beta}^2(1-\beta\cos\theta)^2\left[1+\frac{(\sin\theta\cos\phi)^2(1-2\beta\cos\theta+\beta^2-2+2\beta\cos\theta)}{(1-\beta\cos\theta)^2}\right]$$
$$= \dot{\beta}^2(1-\beta\cos\theta)^2\left[1-\frac{(\sin\theta\cos\phi)^2}{\gamma^2(1-\beta\cos\theta)^2}\right]$$

The angular power distribution becomes

$$\frac{dP}{d\Omega} = \left[\frac{1}{4\pi\varepsilon_o}\right]\frac{q^2}{4\pi c\xi^5}\left|\mathbf{n}\times\left[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\right]\right|^2 = \left[\frac{1}{4\pi\varepsilon_o}\right]\frac{q^2\dot{\beta}^2}{4\pi c(1-\beta\cos\theta)^3}\left[1-\frac{(\sin\theta\cos\phi)^2}{\gamma^2(1-\beta\cos\theta)^2}\right]$$

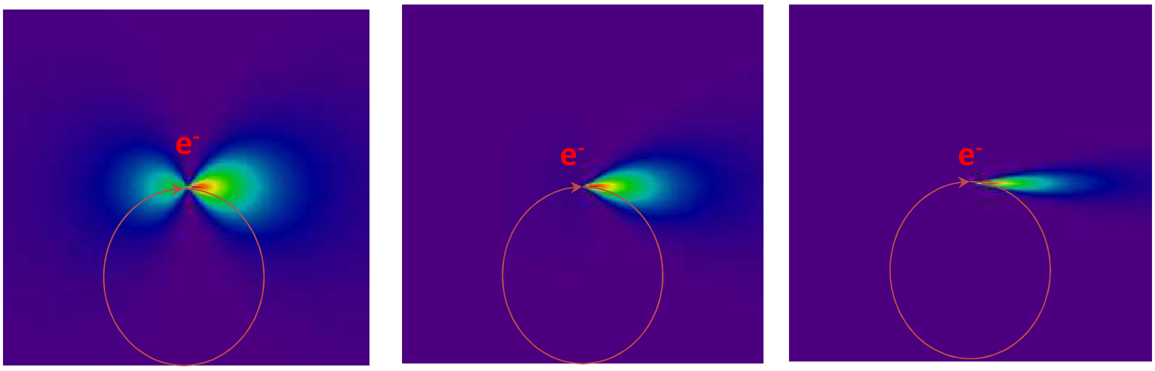
From this angular pattern it is clear that the radiation is peaked in the direction of $\boldsymbol{\beta}$ (**Fig.**). There is a small cashew like lobe which also carries a small energy. In the extreme relativistic limit, θ is very small. With the approximation similar to the previous case, we get

$$\frac{1}{(1-\beta\cos\theta)} \approx \frac{1}{(1-\beta+\beta\theta^2/2)} = \frac{2\gamma^2}{1+(\gamma\theta)^2}$$

and

$$\frac{dP}{d\Omega} = \left[\frac{1}{4\pi\varepsilon_o}\right]\frac{2q^2\dot{\beta}^2\gamma^6}{\pi c\,(1+\gamma^2\theta^2)^3}\left[1-\frac{4\gamma^2\theta^2\cos^2\phi}{(1+\gamma^2\theta^2)^2}\right]$$

This formula indicates that the width of the radiation pattern is 1/γ.



β = 0.1 β = 0.5 β = 0.9

The total power radiated is given by

$$P = \int \frac{dP}{d\Omega}\, d\Omega = \left[\frac{1}{4\pi\varepsilon_o}\right] \frac{2q^2a^2}{3c^3}\gamma^4$$

Thus Larmor's formula gets modified by γ^4 factor. In terms of the applied force, we can use

$$\mathbf{F} = \frac{d}{dt}\frac{m_o v}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{m_o \dot{\mathbf{v}}}{\sqrt{1-\frac{v^2}{c^2}}} = \gamma\, m_o \dot{\mathbf{v}}$$

for the case of $\boldsymbol{\beta}$ and $\dot{\boldsymbol{\beta}}$ to be perpendicular to write the total power radiated as

$$P = \left[\frac{1}{4\pi\varepsilon_o}\right] \frac{2q^2a^2}{3c^3}\gamma^4 = \left[\frac{1}{4\pi\varepsilon_o}\right] \frac{2q^2\gamma^2}{3m_o^2c^3} |\,\mathbf{F}\,|^2.$$

When this is compared to the corresponding result for rectilinear motion, we find that for a given magnitude of applied force the radiation emitted with a transverse acceleration is a factor of γ^2 larger than with a parallel acceleration.