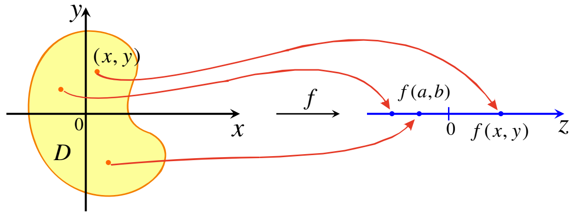
[1] Functions of Several Variables $f : R^n \to R$

Definition. domain, range, independent variable, dependent variable.

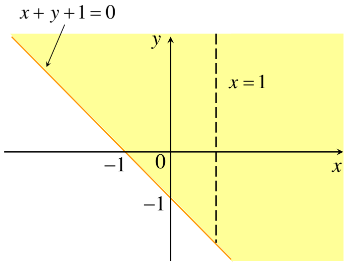


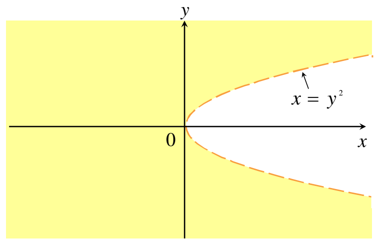
Examples

- Find the domains of the following functions and evaluate $f(3, 2)$.

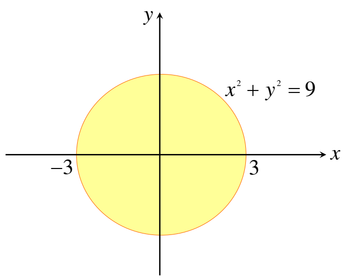
1. $f(x, y) = \dfrac{\sqrt{x+y+1}}{x-1}$.

2. $f(x, y) = x \ln(y^2 - x)$.





- Find the domain and range of $g(x, y) = \sqrt{9 - x^2 - y^2}$.

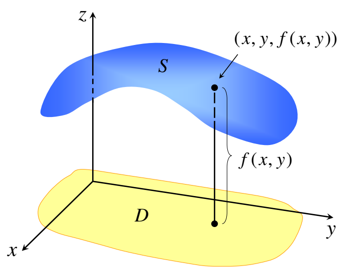


- $f(x_1, x_2, \cdots, x_n) = x_1 + 2x_2 + \cdots + nx_n$

[2] Graphs

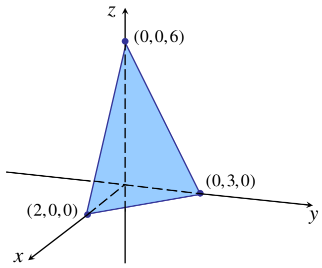
Definition. If f is a function of two variables with domain D, then the graph of f is the set of all points (x, y, z) in $\mathbb{R}^3$ such that $z = f(x, y)$ and $(x, y) \in D$.

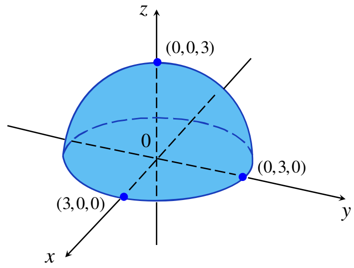
Graph$(f) = \{(x, y, z) \in R^3 | z = f(x, y), (x, y) \in D\}$.



Examples

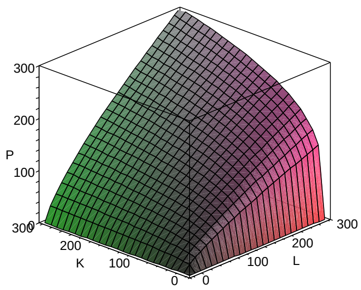
- Sketch the graph of the function $f(x, y) = 6 - 3x - 2y$.



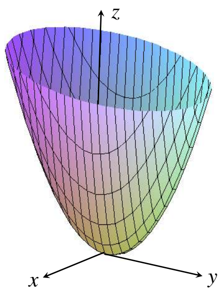


- Sketch the graph of $g(x, y) = \sqrt{9 - x^2 - y^2}$.
- Use a computer to draw the graph of the Cobb-Douglas production function

$$P(L, K) = 1.01L^{0.75}K^{0.25}.$$



- Find the domain and range and sketch the graph of $h(x, y) = 4x^2 + y^2$.

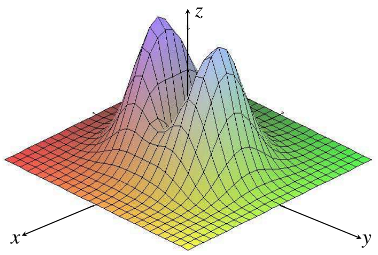


- $f(x, y, z) = x^2 + y^2 + z^2$, Graph$(f)$ =?

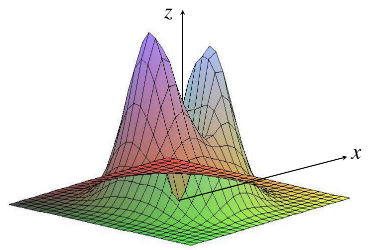
[3] Level Curves (Contour Curves) and Level Surfaces

- The level curve of a function f of two variables are the curves with equation $f(x, y) = k$, where k is a constant.

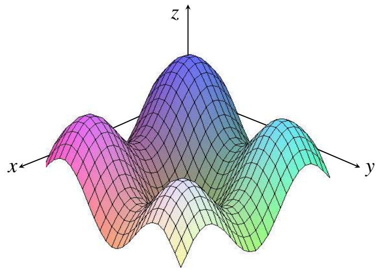
Examples



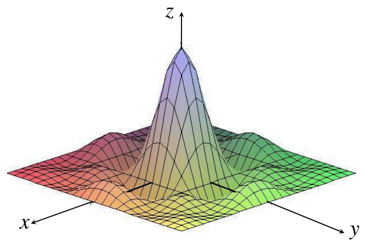
(a) $(x^2+3y^2)e^{-x^2-y^2}$



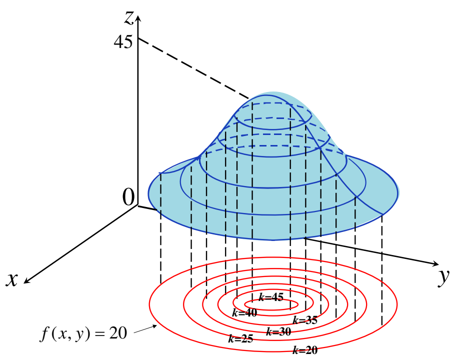
(b) $(x^2+3y^2)e^{-x^2-y^2}$

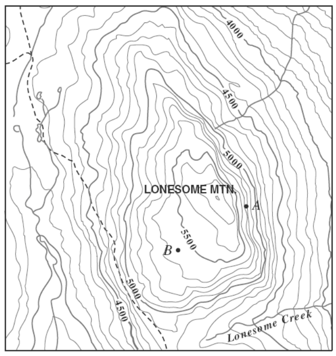


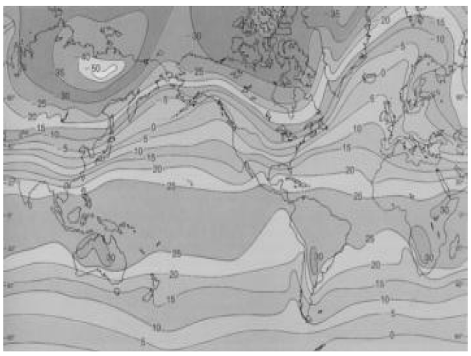
(c) $(\sin x+\sin y)e^{-x^2-y^2}$



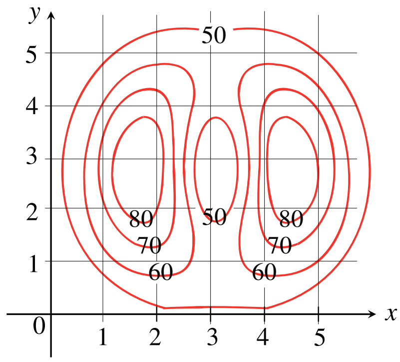
(d) $\frac{\sin x \sin y}{xy}$

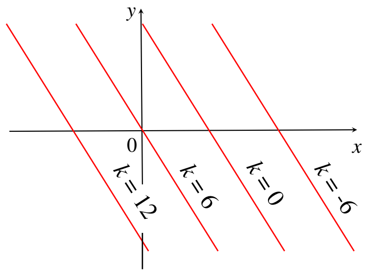


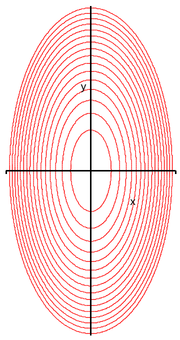


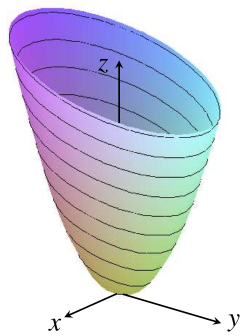


- A contour map for a function f is shown below. Use it to estimate the values of $f(1,3)$ and $f(4,5)$.
- Sketch the level curves of the function $f(x,y)=6-3x-2y$ for the values $k=-6,0,6,12$.
- Sketch the level curves of the function $g(x,y)=\sqrt{9-x^2-y^2}$ for $k=0,1,2,3$.
- Sketch some level curves of the function $h(x,y)=4x^2+y^2$.





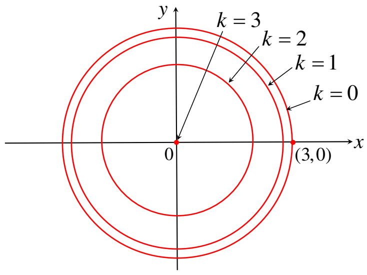


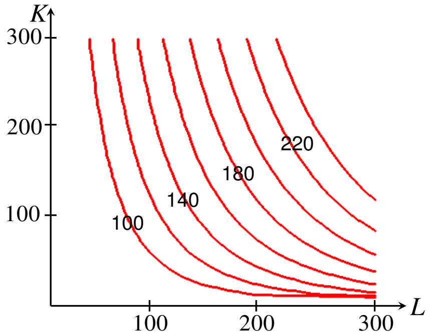


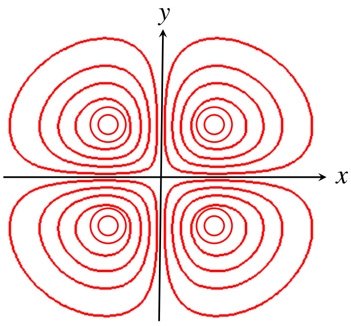
- Plot level curves for the Cobb-Douglas production function $P(L, K) = 1.01L^{0.75}K^{0.25}$.

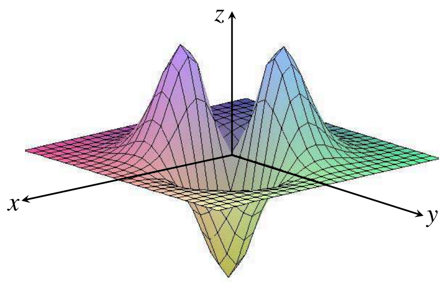
Exercises

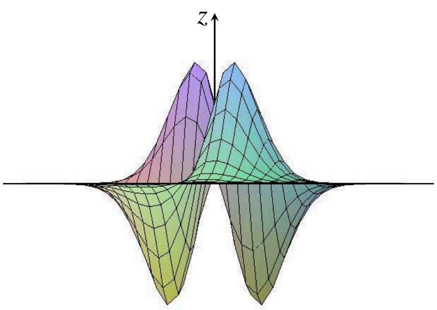
- Find the domain of f if $f(x, y, z) = \ln(z - y) + xy \sin z$.

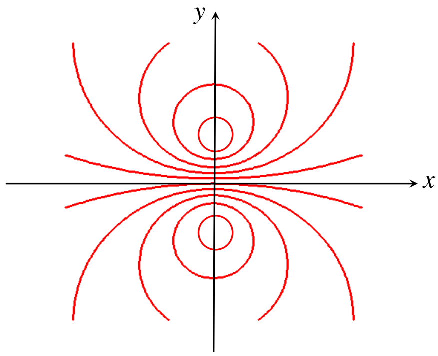




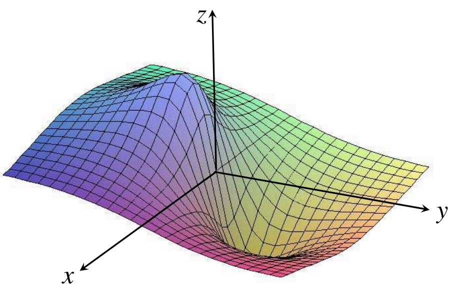






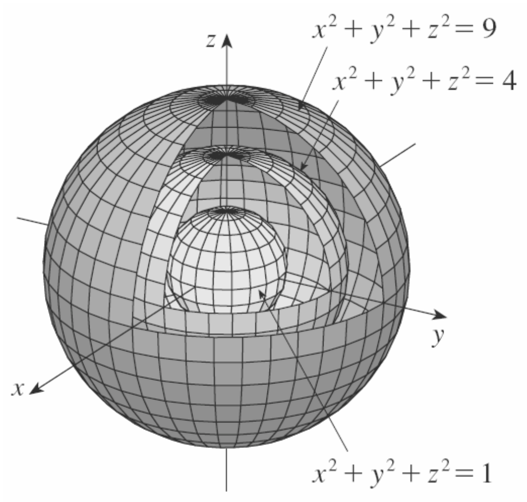


(j) $f(x, y) = -xye^{x^2-y^2}$



(l) $\dfrac{-3y}{x^2 + y^2 + 1}$

- Find the level surfaces of $f(x,y,z) = x^2 + y^2 + z^2$.



- Let $f(x,y) = \ln(x + y - 1)$.
  1. Evaluate $f(1,1)$.
  2. Evaluate $f(e,1)$.
  3. Find and sketch the domain of f.
  4. Find the range of f.

- Find and sketch the domain of the function.
  (a) $f(x,y) = \dfrac{x - 3y}{x + 3y}$.
  (b) $f(x,y) = \sqrt{x^2 + y^2 - 1} + \ln(4 - x^2 - y^2)$.

- Sketch the graph of the function $f(x,y) = \sqrt{16 - x^2 - 16y^2}$.

- Draw a contour map of the function showing several level curves.
  (a) $f(x,y) = xy$.
  (b) $f(x,y) = \frac{y}{x^2+y^2}$.

- Describe the level surfaces of the function $f(x,y,z) = x^2 - y^2$.

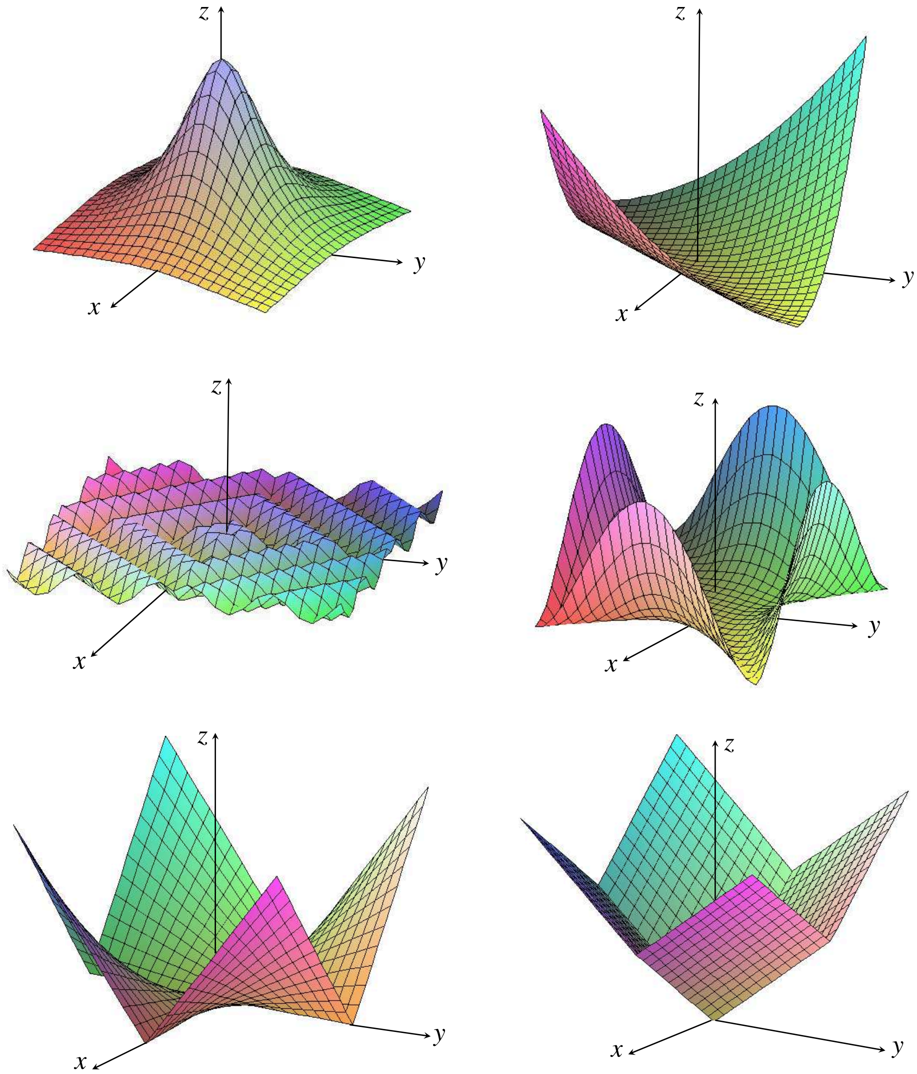
- Graph the functions

  1. $\sqrt{x^2 + y^2}$.
  2. $e^{\sqrt{x^2+y^2}}$.
  3. $\ln \sqrt{x^2 + y^2}$.
  4. $\sin\left(\sqrt{x^2 + y^2}\right)$.
  5. $\frac{1}{\sqrt{x^2+y^2}}$.

  In general, if g is a function of one variable, how is the graph of $f(x,y) = g(\sqrt{x^2 + y^2})$ obtained from the graph of g?

- Match the function with its graph

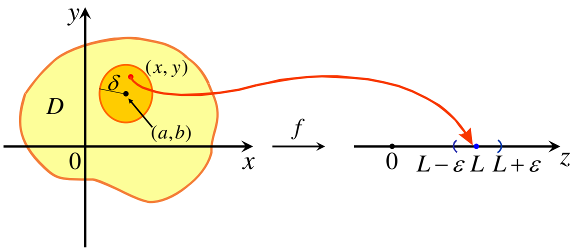
1. $|x| + |y|$.
2. $|xy|$.
3. $\dfrac{1}{1 + x^2 + y^2}$.
4. $(x^2 - y^2)^2$.
5. $(x - y)^2$.
6. $\sin(|x| + |y|)$.

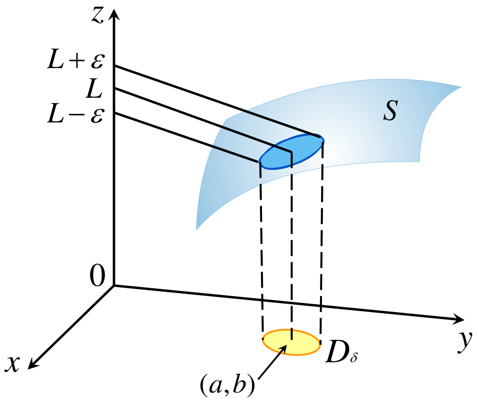


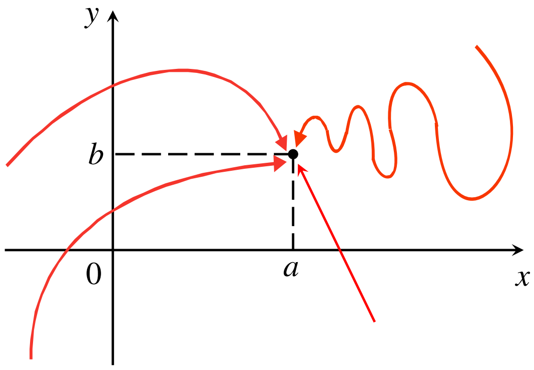
[1]Limits

The idea of points being "closed" in R^2 or R^3.

Definition. Let $f : D \to \mathbb{R}$. Then $\lim_{(x,y)\to(a,b)} f(x,y) = L$ if $\forall\epsilon > 0$, $\exists\delta > 0$ such that $|f(x,y) - L| < \epsilon$ whenever $(x,y) \in D$ and $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$.



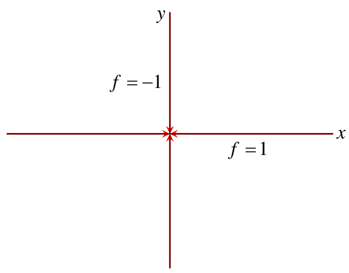




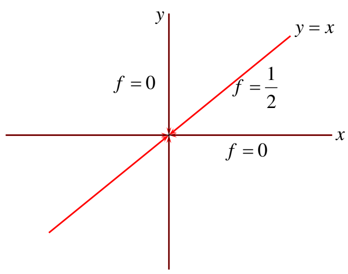
Examples

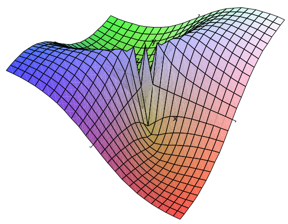
- Show that $\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2} = 1.$

- Show that $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ does not exist.

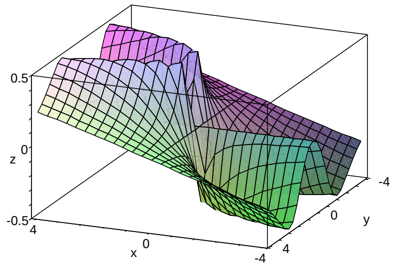


- If $f(x,y) = \frac{xy}{x^2+y^2}$, does $\lim_{(x,y)\to(0,0)} f(x,y)$ exist?





- If $f(x,y)=\dfrac{xy^2}{x^2+y^4}$, does $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ exist?



- Find $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2}$ if it exists.

Exercises

- $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y^3}{x^2+y^2}$.
- $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2+y^3}{x^2+y^2}$.
- $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2+\sin^2 y}{x^2+y^2}$.
- $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x+y^3}{x^2+y^2}$.
- $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2}$.

(Limit does not exist) If $f\to L_1$ as $(x,y)\to(a,b)$ along C_1, $f\to L_2$ as $(x,y)\to(a,b)$ along C_2, $L_1\neq L_2$, then $\displaystyle\lim_{(x,y)\to(a,b)} f(x,y)$ does not exist.

Examples

$\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$.

$\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{xyz}{x^2+y^2+z^2}$.

$\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y^3}{x^2+y^2}$.

$\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$.

$\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{xy+yz^2+xz^2}{x^2+y^2+z^4}$.

[2] Continuity

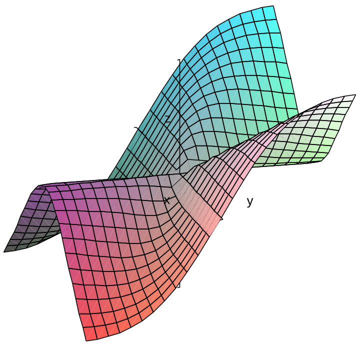
Definition. f is continuous at (a,b) if $\displaystyle\lim_{(x,y)\to(a,b)} f(x,y)=f(a,b)$.

the four fundamental operations of arithmetic, polynomials, rational functions.

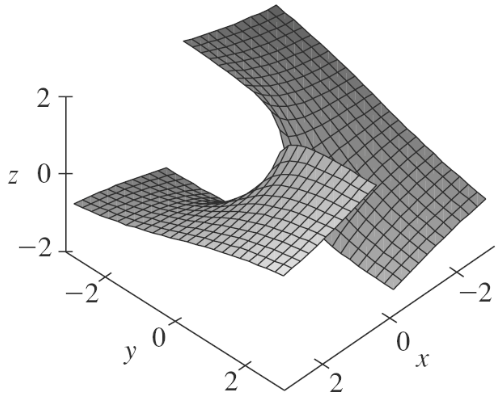
Examples

- Evaluate $\displaystyle\lim_{(x,y)\to(1,2)} (x^2y^3-x^3y^2+3x+2y)$.

- Where is the function $f(x,y) = \dfrac{x^2 - y^2}{x^2 + y^2}$ continuous?
- Let $g(x,y) = \begin{cases} \dfrac{x^2 - y^2}{x^2 + y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$ g is discontinuous at 0.
- Let $f(x,y) = \begin{cases} \dfrac{3x^2 y}{x^2 + y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$ f is continuous on $\mathbb{R}^2$.



- Where is the function $h(x,y) = \tan^{-1}(\frac{y}{x})$ continuous?



- $\lim_{(x_1,x_2,\cdots,x_n)\to(a_1,a_2,\cdots,a_n)} \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$.
- Find the limit.

1. $\displaystyle\lim_{(x,y)\to(5,-2)} xy\cos(x-2y)$.
2. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{6x^3y}{2x^4+y^4}$.
3. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2\sin^2 y}{x^2+2y^2}$.
4. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^4}{x^2+y^8}$.
5. $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{x^2+2y^2+3z^2}{x^2+y^2+z^2}$.
6. $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{xy+yz+zx}{x^2+y^2+z^2}$.

- Explain why the limit $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^6}$ does not exist.
- Find $h(x,y) = g(f(x,y))$ and the set on which h is continuous.

$$g(t) = \frac{\sqrt{t}-1}{\sqrt{t}+1}, \quad f(x,y) = x^2 - y.$$

- Determine the set of points at which the function is continuous.

1. $\displaystyle\frac{x-y}{1+x^2+y^2}$.
2. $\ln(x^2+y^2-4)$.
3. $\displaystyle\frac{\sqrt{y}}{x^2-y^2+z^2}$.
4. $\begin{cases} \frac{xy}{x^2+xy+y^2} & \text{if } (x,y) \neq (0,0). \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$

- Find the limit.

1. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2}$.
2. $\displaystyle\lim_{(x,y)\to(0,0)} (x^2+y^2)\ln(x^2+y^2)$.
3. $f(x,y) = \displaystyle\frac{\sin(x^2+y^2)}{x^2+y^2}$.

- Show that the function f given by $f(x) = |x|$ is continuous on $\mathbb{R}$.

Remarks.

- If $\lim_{(x,y)\to(a,b)} f(x,y)$ and $\lim_{x\to a}\lim_{y\to b} f(x,y)$ both exist, then $\lim_{(x,y)\to(a,b)} f(x,y) = \lim_{x\to a}\lim_{y\to b} f(x,y)$
- If $\lim_{x\to a}\lim_{y\to b} f(x,y) \neq \lim_{y\to b}\lim_{x\to a} f(x,y)$, then $\lim_{(x,y)\to(a,b)} f(x,y)$ does not exist.

- (cts in each single variable, discts in two variables) $f(x,y) = \frac{xy}{x^2+y^2}; f(0,0) = 0$.
- (along lines, along $y = x^2$) $f(x,y) = \frac{x^2y}{x^4+y^2}; f(0,0) = 0$.
- (along $y = cx^{\frac{m}{n}}$, along $y = e^{-\frac{1}{x^2}}$ $f(x,y) = e^{-\frac{1}{x^2}}y; f(0,0) = 0$. $f(x,y) = e^{-\frac{2}{x^2}} + y^2; f(0,0) = 0$. (f_x, f_y exist, but not cts)
- Compare $\lim_{(x,y)\to(a,b)} f(x,y)$, $\lim_{x\to a}\lim_{y\to b} f(x,y)$ and $\lim_{y\to b}\lim_{x\to a} f(x,y)$.
  (exact 2 exist) $\frac{xy}{x^2+y^2}$, $y + x\sin\frac{1}{y}$ and $x + y\sin\frac{1}{x}$.
  (exact 1 exist) combine $\frac{xy}{x^2+y^2}$, $x\sin\frac{1}{y}$ and $y\sin\frac{1}{x}$.